N. PISKUNOV

DIFFERENTIAL and INTEGRAL CALCULUS

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Н. С. Пискунов

ДИФФЕРЕНЦИАЛЬНОЕ И ИНТЕГРАЛЬНОЕ ИСЧИСЛЕНИЯ

На английском языке

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PREFACE

This text is designed as a course of mathematics for higher technical schools. It contains many worked examples that illustrate the theoretical material and serve as models for solving problems.

The first two chapters "Number. Variable. Function" and "Limit. Continuity of a Function" have been made as short as possible. Some of the questions that are usually discussed in these chapters have been put in the third and subsequent chapters without loss of continuity. This has made it possible to take up very early the basic concept of differential calculus—the derivative—which is required in the study of technical subjects. Experience has shown this arrangement of the material to be the best and most convenient for the student.

A large number of problems have been included, many of which illustrate the interrelationships of mathematics and other disciplines. The problems are specially selected (and in sufficient number) for each section of the course thus helping the student to master the theoretical material. To a large extent, this makes the use of a separate book of problems unnecessary and extends the usefulness of this text as a course of mathematics for self-instruction.

N. S. Piskunov

CHAPTER I

NUMBER. VARIABLE. FUNCTION

SEC. 1. REAL NUMBERS. REAL NUMBERS AS POINTS ON A NUMBER SCALE

Number is one of the basic concepts of mathematics. It originated in ancient times and has undergone expansion and generalisation over the centuries.

Whole numbers and fractions, both positive and negative, together with the number zero are called *rational numbers*. Every rational number may be represented in the form of a ratio, $\frac{p}{q}$, of two integers p and q; for example,

$$\frac{5}{7}, \quad 1.25=\frac{5}{4}.$$

In particular, the integer p may be regarded as a ratio of the integers $\frac{p}{1}$; for example,

$$6=\frac{6}{1}, \quad 0=\frac{0}{1}.$$

Rational numbers may be represented in the form of periodic terminating or nonterminating fractions. Numbers represented by nonterminating, but nonperiodic, decimal fractions are called *irrational numbers*; such are the numbers $\sqrt{2}$, $\sqrt{3}$, $5-\sqrt{2}$, etc.

The collection of all rational and irrational numbers makes up the set of *real numbers*. The real numbers are *ordered in magnitude*; that is to say, for each pair of real numbers x and y there is one, and only one, of the following relations:

$$x<y, \quad x=y, \quad x>y.$$

Real numbers may be depicted as points on a number scale. A *number scale* is an infinite straight line on which are chosen: 1) a certain point O called the origin, 2) a positive direction indicated by an arrow, and 3) a suitable unit of length. We shall usually make the number scale horizontal and take the positive direction to be from left to right.

If the number x_1 is positive, it is depicted as a point M_1 at a distance $OM_1=x_1$ to the right of the origin O; if the number x_2 is negative, it is represented by a point M_2 to the left of O at a

distance $OM_2 = -x_2$ (Fig. 1). The point O represents the number zero. It is obvious that every real number is represented by a definite point on the number scale. Two different real numbers are represented by different points on the number scale.

The following assertion is also true: each point on the number scale represents only one real number (rational or irrational).

To summarise, all real numbers and all points on the number scale are in one-to-one correspondence: to each number there corresponds only one point, and conversely, to each point there corresponds only one number. This frequently enables us to regard "the number x" and "the point x" as, in a certain sense, equivalent expressions. We shall make wide use of this circumstance in our course.

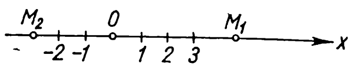


Fig. 1.

We state without proof the following important property of the set of real numbers: *both rational and irrational numbers may be found between any two arbitrary real numbers.* In geometrical terms, this proposition reads thus: *both rational and irrational points may be found between any two arbitrary points on the number scale.*

In conclusion we give the following theorem, which, in a certain sense, represents a bridge between theory and practice.

Theorem. *Every irrational number α may be expressed, to any degree of precision, with the aid of rational numbers.*

Indeed, let the irrational number $\alpha > 0$ and let it be required to evaluate α with an accuracy of $\frac{1}{n}$ $\left(\text{for example, } \frac{1}{10}, \frac{1}{100}, \text{ and so forth}\right)$.

No matter what α is, it lies between two integral numbers N and $N+1$. We divide the segment between N and $N+1$ into n parts; then α will lie somewhere between the rational numbers $N+\frac{m}{n}$ and $N+\frac{m+1}{n}$. Since their difference is equal to $\frac{1}{n}$, each of them expresses α to the given degree of accuracy, the former being smaller and the latter greater.

Example. The irrational number $\sqrt{2}$ is expressed by rational numbers:
1.4 and 1.5 to one decimal place,
1.41 and 1.42 to two decimal places,
1.414 and 1.415 to three decimal places, etc.

SEC. 2. THE ABSOLUTE VALUE OF A REAL NUMBER

Let us introduce a concept which we shall need later on: the absolute value of a real number.

Definition. *The absolute value (or modulus)* of a real number x (written $|x|$) is a nonnegative real number that satisfies the conditions

$$|x| = x \quad \text{if } x \geqslant 0;$$
$$|x| = -x \quad \text{if } x < 0.$$

Examples. $|2| = 2$; $|-5| = 5$; $|0| = 0$.

From the definition it follows that the relationship $x \leqslant |x|$ holds for any x.

Let us examine some of the properties of absolute values.

1. *The absolute value of an algebraic sum of several real numbers is no greater than the sum of the absolute values of the terms* $|x+y| \leqslant |x| + |y|$.

Proof. Let $x+y \geqslant 0$, then

$$|x+y| = x+y \leqslant |x| + |y| \text{ (since } x \leqslant |x| \text{ and } y \leqslant |y|\text{)}.$$

Let $x+y < 0$, then

$$|x+y| = -(x+y) = (-x) + (-y) \leqslant |x| + |y|.$$

This completes the proof.

The foregoing proof is readily extended to any number of terms.

Examples.

$$|-2+3| < |-2| + |3| = 2+3 = 5 \text{ or } 1 < 5;$$
$$|-3-5| = |-3| + |-5| = 3+5 = 8 \text{ or } 8 = 8.$$

2. *The absolute value of a difference is no less than the difference of the absolute values of the minuend and subtrahend*:

$$|x-y| \geqslant |x| - |y|.$$

Proof. Let $x-y=z$, then $x=y+z$ and from what has been proved

$$|x| = |y+z| \leqslant |y| + |z| = |y| + |x-y|,$$

whence

$$|x| - |y| \leqslant |x-y|,$$

thus completing the proof.

3. *The absolute value of a product is equal to the product of the absolute values of the factors*:

$$|xyz| = |x|\ |y|\ |z|.$$

4. *The absolute value of a quotient is equal to the quotient of the absolute values of the dividend and the divisor*:

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

The latter two properties follow directly from the definition of absolute value.

SEC. 3. VARIABLES AND CONSTANTS

The numerical values of such physical quantities as time, length, area, volume, mass, velocity, pressure, temperature, etc., are determined by measurement. Mathematics deals with quantities divested of any specific content. From now on, when speaking of quantities, we shall have in view their numerical values. In various phenomena, the numerical values of certain quantities vary, while the numerical values of others remain fixed. For instance, in uniform motion of a point, time and distance change, while the velocity remains constant.

A *variable* is a quantity that takes on various numerical values. A *constant* is a quantity whose numerical values remain fixed. We shall use the letters x, y, z, u, ..., etc., to designate variables, and the letters a, b, c, ..., etc., to designate constants.

Note. In mathematics, a constant is frequently regarded as a special case of variable whose numerical values are the same.

It should be noted that when considering specific physical phenomena it may happen that one and the same quantity in one phenomenon is a constant while in another it is a variable. For example, the velocity of uniform motion is a constant, while the velocity of uniformly accelerated motion is a variable. Quantities that have the same value under all circumstances are called *absolute constants*. For example, the ratio of the circumference of a circle to its diameter is an absolute constant: $\pi = 3.14159$.

As we shall see throughout this course, the concept of a variable quantity is the basic concept of differential and integral calculus. In "Dialectics of Nature", Friedrich Engels wrote: "The turning point in mathematics was Descartes' variable magnitude. With that came *motion* and hence *dialectics* in mathematics, and *at once*, too, *of necessity* the differential and integral calculus."

SEC. 4. THE RANGE OF A VARIABLE

A variable takes on a series of numerical values. The collection of these values may differ depending on the character of the problem. For example, the temperature of water heated under ordinary conditions will vary from room temperature (15-18°C) to the boiling point, 100°C. The variable quantity $x = \cos\alpha$ can take on all values from -1 to $+1$.

The values of a variable are geometrically depicted as points on a number scale. For instance, the values of the variable $x = \cos\alpha$ for all possible values of α are depicted as the set of points of an interval on the number scale, from -1 to 1, including the points -1 and 1 (Fig. 2).

Definition. The set of all numerical values of a variable quantity is called the *range* of the variable.

We shall now define the following ranges of a variable that will be frequently used later on.

An *open interval* is the collection of all numbers x lying between and excluding the given numbers a and b $(a < b)$; it is denoted (a, b) or by means of the inequalities $a < x < b$.

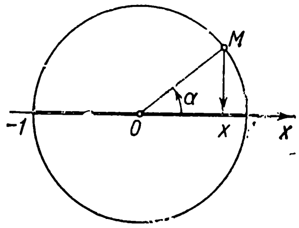


Fig. 2

A *closed interval* is the set of all numbers x lying between and including the two given numbers a and b; it is denoted $[a, b]$ or, by means of inequalities, $a \leqslant x \leqslant b$.

If one of the numbers a or b (say, a) belongs to the interval, while the other does not, we have a *partly closed interval*, which may be given by the inequalities

$$a \leqslant x < b$$

and is denoted $[a, b)$. If the number b belongs to the set and a does not, we have the partly closed interval $(a, b]$, which may be given by the inequalities

$$a < x \leqslant b.$$

If the variable x assumes all possible values greater than a, such an interval is denoted (a, ∞) and is represented by the conditional inequalities

$$a < x < \infty.$$

In the same way we regard the infinite intervals and partly closed infinite intervals represented by the conditional inequalities $a \leqslant x < \infty$; $-\infty < x < c$; $-\infty < x \leqslant c$; $-\infty < x < \infty$.

Example. The range of the variable $x = \cos\alpha$ for all possible values of α is the interval $[-1, 1]$ and is defined by the inequalities $-1 \leqslant x \leqslant 1$.

The foregoing definitions may be formulated for a "point" in place of a "number".

An *interval* is the set of all points x lying between the given points a and b (*the end points*) and is called closed or open accordingly as it does or does not include its end points.

The *neighbourhood* of a given point x_0 is an arbitrary interval (a, b) containing this point within it; that is, the interval (a, b) whose end points satisfy the condition $a < x_0 < b$. One often

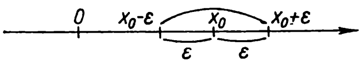


Fig. 3.

considers the neighbourhood (a, b) of the point x_0 for which x_0 is the midpoint. Then x_0 is called the *centre of the neighbourhood* and the quantity $\frac{b-a}{2}$, the *radius of the neighbourhood*. Fig. 3 shows the neighbourhood $(x_0-\varepsilon, x_0+\varepsilon)$ of the point x_0 with radius ε.

SEC. 5. ORDERED VARIABLES. INCREASING AND DECREASING VARIABLES. BOUNDED VARIABLES

We shall say that the variable x is an *ordered variable quantity* if its range is known and if about each of any two of its values it may be said which value is the preceding one and which is the following one. Here, the notions "preceding" and "following" are not connected with time, but serve as a way to "order" the values of the variable, i. e., to establish the order of the respective values of the variable.

Definition 1. A variable is called *increasing* if each subsequent value of it is greater than the preceding value. A variable is called *decreasing* if each subsequent value is less than the preceding value.

Increasing variable quantities and decreasing variable quantities are called *monotonically varying* variables or simply *monotonic quantities*.

Example. When the number of sides of a regular polygon inscribed in a circle is doubled, the area s of the polygon is an increasing variable. The area of a regular polygon circumscribed about a circle, when the number of sides is doubled, is a decreasing variable. It may be noted that not every variable quantity is necessarily increasing or decreasing. Thus, if α is an increasing variable over the interval $[0, 2\pi]$, the variable $x=\sin\alpha$ is not a monotonic quantity. It first increases from 0 to 1, then decreases from 1 to -1, and then increases from -1 to 0.

Definition 2. The variable x is called *bounded* if there exists a constant $M>0$ such that all subsequent values of the variable, after a certain one, satisfy the condition

$$-M \leqslant x \leqslant M, \text{ that is, } |x| \leqslant M.$$

In other words, a variable is called bounded if it is possible to indicate an interval $[-M, M]$ such that all subsequent values of the variable, after a certain one, will belong to this interval. However, one should not think that the variable will necessarily assume all values of the interval $[-M, M]$. For example, the variable that assumes all possible rational values on the interval $[-2, 2]$ is bounded, and nevertheless it does not assume all values on $[-2, 2]$, namely, the irrational values.

SEC. 6. FUNCTION

In the study of natural phenomena and the solution of technical and mathematical problems, one finds it necessary to consider the variation of one quantity as dependent on the variation of another. For instance, in studies of motion, the path traversed is regarded as a variable which varies with time. Here, the path traversed is a **function** of the time.

Let us consider another example. We know that the area of a circle, in terms of the radius, is $Q=\pi R^2$. If the radius R takes on a variety of numerical values, the area Q will also assume various numerical values. Thus, the variation of one variable brings about a variation in the other. Here, the area of a circle Q is a function of the radius R. Let us formulate a definition of the concept "function".

Definition 1. If to each value of the variable x (within a certain range) there corresponds one definite value of another variable y, then y is a *function* of x or, in functional notation, $y=f(x)$, $y=\varphi(x)$, and so forth.

The variable x is called the *independent variable* or *argument*. The relation between the variables x and y is called a *functional relation*. The letter f in the functional notation $y=f(x)$ indicates that some kind of operations must be performed on the value of x in order to obtain the value of y. In place of the notation $y=f(x)$, $u=\varphi(x)$, etc., one occasionally finds $y=y(x)$, $u=u(x)$, etc., the letters y, u designating both the dependent variable and the symbol of the totality of operations to be performed on x.

The notation $y=C$, where C is a constant, denotes a function whose value for any value of x is the same and is equal to C.

Definition 2. The set of values of x for which the values of the function y are determined by virtue of the rule $f(x)$ is called the *domain of definition of the function*.

Example 1. The function $y=\sin x$ is defined for all values of x. Therefore, its domain of definition is the infinite interval $-\infty<x<\infty$.

Note 1. If we have a functional relation of two variable quantities x and $y=f(x)$ and if x and $y=f(x)$ are regarded as ordered variables, then of the two values of the function $y^*=f(x^*)$ and $y^{**}=f(x^{**})$ corresponding to two values of the argument x^* and x^{**}, the subsequent value of the function will be that one which corresponds to the subsequent value of the argument. The following definition is, therefore, natural.

Definition 3. If the function $y=f(x)$ is such that to a greater value of the argument x there corresponds a greater value of the

function, then the function $y=f(x)$ is called *increasing*. A *decreasing* function is similarly defined.

Example 2. The function $Q=\pi R^2$ for $0 < R < \infty$ is an increasing function because to a greater value of R there corresponds a greater value of Q.

Note 2. The definition of function is sometimes broadened so that to each value of x, within a certain range, there corresponds not one but several values of y or even an infinitude of values of y. In this case we have a *multiple-valued* function in contrast to the one defined above, which is called a *single-valued* function. Henceforward, when speaking of a function, we shall have in view only **single-valued** functions. If it becomes necessary to deal with multiple-valued functions we shall specify this fact.

SEC. 7. WAYS OF REPRESENTING FUNCTIONS

I. Tabular representation of a function

Here, the values of the argument $x_1, x_2, \ldots, x_n$ and the corresponding values of the function $y_1, y_2, \ldots, y_n$ are written out in a definite order.

x	x_1	x_2	— — — — —	x_n
y	y_1	y_2	— — — — —	y_n

Examples are tables of trigonometric functions, tables of logarithms, and so on.

An experimental study of phenomena can result in tables that express a functional relation between the measured quantities. For example, temperature measurements of the air at a meteorological station on a definite day yield a table like the following.

The temperature T (in degrees) is dependent on the time t (in hours).

t	1	2	3	4	5	6	7	8	9
T	0	−1	−2	−2	−0.5	1	3	3.5	4

This table defines T as a function of t.

II. Graphical representation of a function

If in a rectangular coordinate system on a plane we have a set of points $M(x, y)$, and no two points lie on a straight line parallel to the y-axis, this set of points defines a certain single-valued function $y = f(x)$; the abscissas of the points are the values of the argument, the corresponding ordinates are the values of the function (Fig. 4).

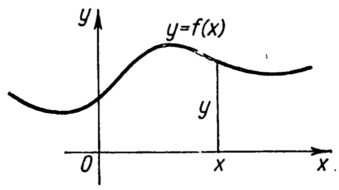


Fig. 4.

The collection of points in the xy-plane whose abscissas are the values of the independent variable and whose ordinates are the corresponding values of the function is called a *graph* of the given function.

III. Analytical representation of a function

Let us first explain what "analytical expression" means. By *analytical expression* we will understand a series of symbols denoting a totality of known mathematical operations that are performed in a definite sequence on numbers and letters which designate constant or variable quantities.

By totality of known mathematical operations we mean not only the mathematical operations familiar from the course of secondary school (addition, subtraction, extraction of roots, etc.) but also those which will be defined as we proceed in this course.

The following are examples of analytical expressions:

$$x^4 - 2; \quad \frac{\log x - \sin x}{5x^2 + 1}; \quad 2^x - \sqrt{5 + 3x},$$

etc.

If the functional relation $y = f(x)$ is such that f denotes an analytical expression, we say that the function y of x is *represented analytically*.

Examples of functions represented analytically are: 1) $y = x^4 - 2$; 2) $y = \frac{x+1}{x-1}$; 3) $y = \sqrt{1 - x^2}$; 4) $y = \sin x$; 5) $Q = \pi R^2$, and so forth.

Here, the functions are represented analytically by means of a single formula (a formula is understood to be the equality of two analytical expressions). In such cases one may speak of the *natural* domain of definition of the function.

The set of values of x for which the analytical expression on the right-hand side has a fully definite value is the *natural domain*

of definition of a function represented analytically. Thus, the natural domain of definition of the function $y=x^4-2$ is the infinite interval $-\infty<x<\infty$, because the function is defined for all values of x. The function $y=\frac{x+1}{x-1}$ is defined for all values of x, with the exception of $x=1$, because for this value of x the denominator vanishes. For the function $y=\sqrt{1-x^2}$, the natural domain of definition is the closed interval $-1\leqslant x\leqslant 1$, and so on.

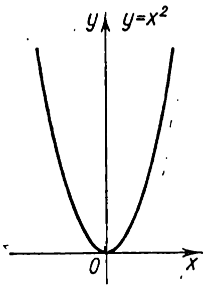


Fig. 5.

Note. It is sometimes necessary to consider only a part of the natural domain of a function, and not the whole domain. For instance, the dependence of the area Q of a circle upon the radius R is defined by the function $Q=\pi R^2$. The domain of this function, when considering a given geometrical problem, is the infinite interval $0<R<+\infty$. But the natural domain of this function is the infinite interval $-\infty<R<+\infty$.

If the function $y=f(x)$ is represented analytically, it may be shown graphically on a coordinate xy-plane. Thus, the graph of the function $y=x^2$ is a parabola as shown in Fig. 5.

SEC. 8. BASIC ELEMENTARY FUNCTIONS. ELEMENTARY FUNCTIONS

The basic elementary functions are the following analytically represented functions.

I. *Power function:* $y=x^\alpha$, where α is a real number.*)

II. *Exponential function:* $y=a^x$, where a is a positive number not equal to unity.

III. *Logarithmic function:* $y=\log_a x$, where the base of logarithms a is a positive number not equal to unity.

IV. *Trigonometric functions:* $y=\sin x$, $y=\cos x$, $y=\tan x$, $y=\cot x$, $y=\sec x$, $y=\csc x$.

V. *Inverse trigonometric functions:*

$$y=\arcsin x,\quad y=\arccos x,\quad y=\arctan x,$$

$$y=\operatorname{arc}\cot x,\quad y=\operatorname{arc}\sec x,\quad y=\operatorname{arc}\csc x.$$

Let us consider the domains of definition and the graphs of the basic elementary functions.

*) If α is irrational, this function is evaluated by taking logarithms and antilogarithms: $\log y=\alpha\log x$. It is assumed that $x>0$.

Power function $y = x^{\alpha}$.

1. α is a positive integer. The function is defined in the infinite interval $-\infty < x < +\infty$. In this case, the graphs of the function for certain values of α have the form shown in Figs. 6 and 7.

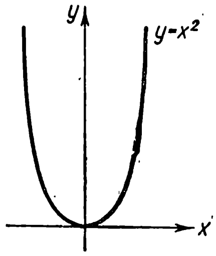


Fig. 6.

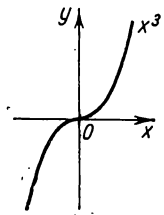


Fig. 7.

2. α is a negative integer. In this case, the function is defined for all values of x with the exception of $x = 0$. The graphs of the functions for certain values of α have the form shown in Figs. 8 and 9.

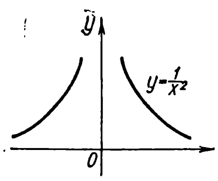


Fig. 8.

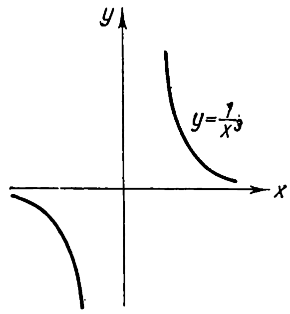


Fig. 9.

Figs. 10, 11, and 12 show graphs of a power function with fractional rational values of α.

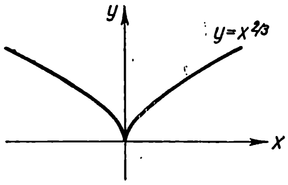


Fig. 10.

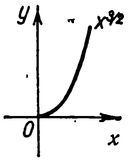


Fig. 11.

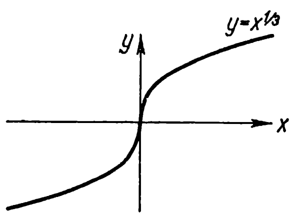


Fig. 12.

Exponential function, $y=a^x$, $a>0$ and $a \neq 1$. This function is defined for all values of x. Its graph is shown in Figs. 13 and 14.

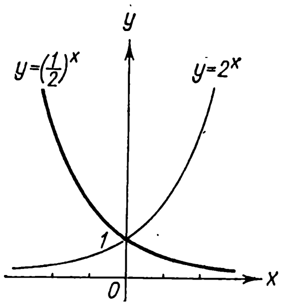


Fig. 13.

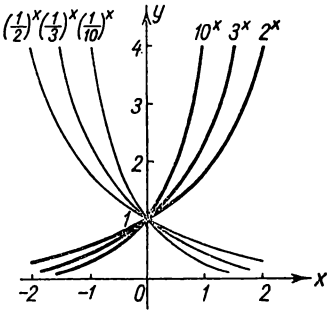


Fig. 14.

Logarithmic function, $y=\log_a x$, $a>0$ and $a \neq 1$. This function is defined for $x>0$. Its graph is shown in Fig. 15.

Trigonometric functions. In the formulas $y=\sin x$, etc., the independent variable x is expressed in radians. All the enumerated trigonometric functions are periodic. Let us give a general definition of a periodic function.

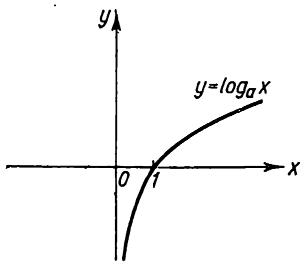


Fig. 15.

Definition 1. The function $y=f(x)$ is called *periodic* if there exists a constant C, which, when added to (or subtracted from) the argument x, does not change the value of the function: $f(x+C)=f(x)$. The least such number is called the *period* of the function; it will henceforward be designated as $2l$.

From the definition it follows directly that $y=\sin x$ is a periodic function with a period 2π: $\sin x = \sin(x+2\pi)$. The period of $\cos x$ is likewise 2π. The functions $y=\tan x$ and $y=\cot x$ have a period equal to π.

The functions $y=\sin x$, $y=\cos x$ are defined for all values of x; the functions $y=\tan x$ and $y=\sec x$ are defined everywhere except the points $x=(2k+1)\frac{\pi}{2}$ $(k=0, 1, 2, \ldots)$; the functions $y=\cot x$ and $y=\csc x$ are defined for all values of x except the points $x=k\pi$ $(k=0, 1, 2, \ldots)$. Graphs of trigonometric functions are shown in Figs. 16, 17, 18, and 19.

The inverse trigonometric functions will be discussed in more detail later on.

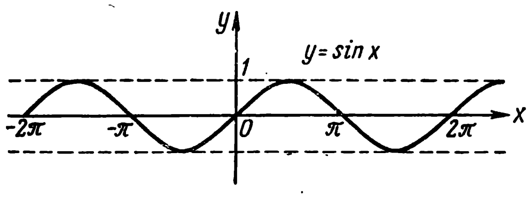


Fig. 16.

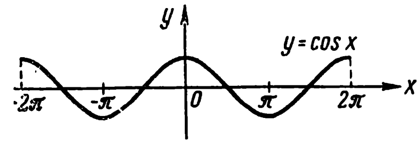


Fig. 17.

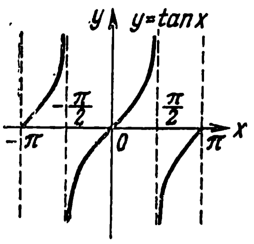


Fig. 18.

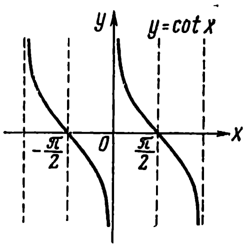


Fig. 19.

Let us now introduce the concept of a function of a function. If y is a function of u, and u (in turn) is dependent on the variable x, then y is also dependent on x. Let

$$y = F(u)$$
$$u = \varphi(x).$$

We get y as a function of x

$$y = F[\varphi(x)].$$

This function is called a *function of a function* or a *composite function.*

Example 1. Let $y = \sin u$, $u = x^2$. The function $y = \sin(x^2)$ is a composite function of x.

Note. The domain of definition of the function $y = F[\varphi(x)]$ is either the entire domain of the function, $u = \varphi(x)$, or that part

of it in which those values of u are defined that do not go beyond the domain of the function $F(u)$.

Example 2. The domain of definition of the function $y=\sqrt{1-x^2}$ ($y=\sqrt{u}$, $u=1-x^2$) is the closed interval $[-1, 1]$, because when $|x|>1$ $u<0$ and, consequently, the function $\sqrt{u}$ is not defined (although the function $u=1-x^2$ is defined for all values of x). The graph of this function is the upper half of a circle with centre at the origin of the coordinate system and with radius unity.

The operation "function of a function" may be performed any number of times. For instance, the function $y=\ln[\sin(x^2+1)]$ is obtained as a result of the following operations (defining the following functions):

$$v=x^2+1,\quad u=\sin v,\quad y=\ln u.$$

Let us now define an elementary function.

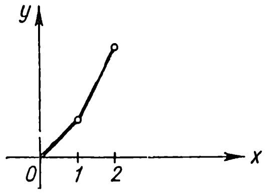


Fig. 20.

Definition 2. An *elementary function* is a function which may be represented by a single formula of the type $y=f(x)$, where the expression on the right-hand side is made up of basic elementary functions and constants by means of a finite number of operations of addition, subtraction, multiplication, division and taking the function of a function.

From the definition it follows that elementary functions are functions represented analytically.

Examples of elementary functions:

$$y=\sqrt{1+4\sin^2 x};\quad y=\frac{\log x+4\sqrt[3]{x}+2\tan x}{10^x-x+10};$$

and the like.

Examples of non-elementary functions:

1. The function $y=1\cdot2\cdot3\cdot\ldots\cdot n$ [$y=f(n)$] is not elementary because the number of operations that must be performed to obtain y increases with n, that is to say, it is not bounded.

2. The function given in Fig. 20 is not elementary either because it is represented by means of two formulas:

$$f(x)=x,\quad \text{if } 0\leqslant x\leqslant 1,$$
$$f(x)=2x-1,\quad \text{if } 1\leqslant x\leqslant 2.$$

SEC. 9. ALGEBRAIC FUNCTIONS

Algebraic functions include elementary functions of the following kind:

I. The rational integral function, or **polynomial**

$$y = a_0x^n + a_1x^{n-1} + \ldots + a_n,$$

where $a_0, a_1, \ldots, a_n$ are constants called coefficients, and n is a nonnegative integer called the degree of the polynomial. It is obvious that this function is defined for all values of x, that is, it is defined in an infinite interval.

Examples: 1. $y = ax + b$ is a *linear function*. When $b = 0$, the linear function $y = ax$ expresses y as being directly proportional to x. For $a = 0$, $y = b$, the function is a constant.

2. $y = ax^2 + bx + c$ is a *quadratic function*. The graph of a quadratic function is a parabola (Fig. 21). These functions are considered in detail in analytic geometry.

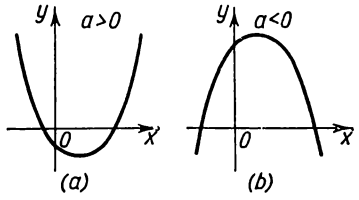


Fig. 21.

II. Fractional rational function. This function is defined as the ratio of two polynomials:

$$y = \frac{a_0x^n + a_1x^{n-1} + \ldots + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_m}.$$

For example, the following is a fractional rational function:

$$y = \frac{a}{x},$$

it expresses inverse variation. Its graph is shown in Fig. 22. It is obvious that a fractional rational function is defined for all values of x with the exception of those for which the denominator becomes zero.

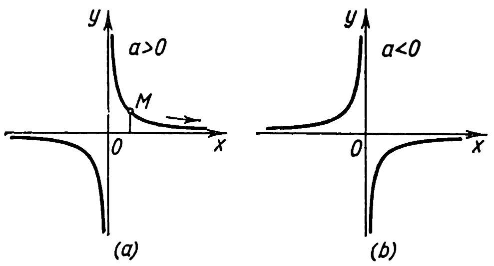


Fig. 22.

III. Irrational function. If in the formula $y = f(x)$, operations of addition, subtraction, multiplication, division and raising to a power with rational non-integral exponents are performed on the right-hand side, the function $y = f(x)$ is called *irrational*. Examples of irrational functions are: $y = \frac{2x^2 + \sqrt{x}}{\sqrt{1 + 5x^2}}$; $y = \sqrt{x}$; etc.

Note 1. The above-mentioned three types of algebraic functions do not exhaust all algebraic functions. An *algebraic function* is any function $y=f(x)$ which satisfies an equation of the form

$$P_0(x)y^n+P_1(x)y^{n-1}+\ldots+P_n(x)=0, \qquad (1)$$

where $P_0(x)$, $P_1(x)$, $\ldots$, $P_n(x)$ are certain polynomials in x.

It may be proved that each of the enumerated three types of function satisfies a certain equation of type (1), but not every function that satisfies an equation like (1) is a function of one of the three types given above.

Note 2. A function which is not algebraic is called *transcendental*. Examples of transcendental functions are:

$$y=\cos x;\ y=10^x x$$

and the like.

SEC. 10. POLAR COORDINATE SYSTEM

The position of a point in a plane may be determined by means of a so-called *polar coordinate system.*

We choose a point O in a plane and call it the *pole;* the half-line issuing from this point is called the *polar axis.* The position of the point M in the plane may be specified by two numbers: the number ϱ, which expresses the distance of M from the pole, and the number φ, which is the angle formed by the line segment OM and the polar axis. The positive direction of the angle φ is reckoned counterclockwise. The numbers ϱ and φ are called the *polar coordinates* of the point M (Fig. 23).

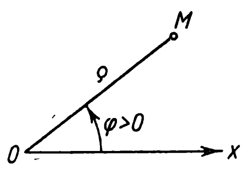


Fig. 23.

We will always consider the radius vector ϱ nonnegative. If the polar angle φ is taken within the limits $0\leqslant\varphi<2\pi$, then to each point of the plane (with the exception of the pole) there corresponds a definite number pair ϱ and φ. For the pole, $\varrho=0$ and φ is arbitrary.

Let us now see how the polar and rectangular Cartesian coordinates are related. Let the origin of the rectangular coordinate system coincide with the pole, and the positive direction of the x-axis, with the polar axis. We establish a relationship between the rectangular and polar coordinates of one and the same point. From Fig. 24 it follows directly that

$$x=\varrho\cos\varphi,\ y=\varrho\sin\varphi$$

and, conversely, that

$$\varrho=\sqrt{x^2+y^2},\ \tan\varphi=\frac{y}{x}.$$

Note. To find φ, it is necessary to take into account the quadrant in which the point is located and then take the correspond-

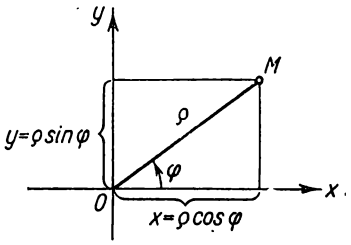


Fig. 24.

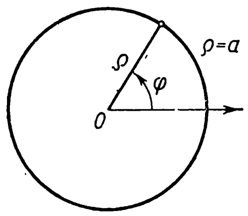


Fig. 25.

ing value of φ. The equation $\varrho = F(\varphi)$ in polar coordinates defines a certain line.

Example 1. Equation $\varrho = a$, where $a = \text{const}$, defines in polar coordinates a circle with centre in the pole and with radius a. The equation of this circle in a rectangular coordinate system situated as shown in Fig. 24 is

$$\sqrt{x^2+y^2} = a \text{ or } x^2+y^2 = a^2$$

(Fig. 25).

Example 2. $\varrho = a\varphi$, where $a = \text{const}$.

Let us tabulate the values of ϱ for certain values of φ

φ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{3}{2}\pi$	2π	3π	4π
ϱ	0	$\approx 0.78a$	$\approx 1.57a$	$\approx 2.36a$	$\approx 3.14a$	$\approx 4.71a$	$\approx 6.28a$	$\approx 9.42a$	$\approx 12.56a$

The corresponding curve is shown in Fig. 26. It is called the *spiral of Archimedes*.

Example 3.

$$\varrho = 2a \cos \varphi.$$

This is the equation of a circle of radius a, the centre of which is at the point $\varrho_0 = a$, $\varphi = 0$ (Fig. 27). Let us write the equation of this circle in rect-

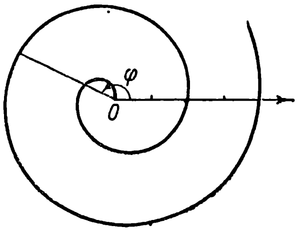


Fig. 26.

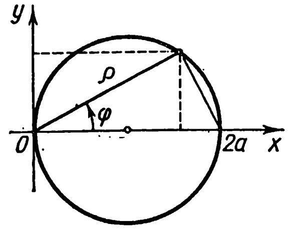


Fig. 27.

angular coordinates. Substituting $\varrho=\sqrt{x^2+y^2}$, $\cos\varphi=\frac{x}{\sqrt{x^2+y^2}}$ into the given equation, we get

$$\sqrt{x^2+y^2}=2a\frac{x}{\sqrt{x^2+y^2}}$$

or

$$x^2+y^2-2ax=0.$$

Exercises on Chapter I

1. Given the function $f(x)=x^2+6x-4$. Verify the equalities $f(1)=3$, $f(3)=23$.

2. $f(x)=x^2+1$. Evaluate: a) $f(4)$. *Ans.* 17. b) $f(\sqrt{2})$. *Ans.* 3. c) $f(a+1)$. *Ans.* a^2+2a+2. d) $f(a)+1$. *Ans.* a^2+2. e) $f(a^2)$. *Ans.* a^4+1. f) $[f(a)]^2$. *Ans.* a^4+2a^2+1. g) $f(2a)$. *Ans.* $4a^2+1$.

3. $\varphi(x)=\frac{x-1}{3x+5}$. Write the expressions $\varphi\left(\frac{1}{x}\right)$ and $\frac{1}{\varphi(x)}$. *Ans.* $\varphi\left(\frac{1}{x}\right)=\frac{1-x}{3+5x}$; $\frac{1}{\varphi(x)}=\frac{3(x)+5}{1-x}$.

4. $\psi(x)=\sqrt{x^2+4}$. Write the expressions $\psi(2x)$ and $\psi(0)$. *Ans.* $\psi(2x)=2\sqrt{x^2+1}$; $\psi(0)=2$.

5. $f(\theta)=\tan\theta$. Verify the equality $f(2\theta)=\frac{2f(\theta)}{1-[f(\theta)]^2}$.

6. $\varphi(x)=\log\frac{1-x}{1+x}$. Verify the equality $\varphi(a)+\varphi(b)=\varphi\left(\frac{a+b}{1+ab}\right)$.

7. $f(x)=\log x$; $\varphi(x)=x^3$. Write the expressions: a) $f[\varphi(2)]$. *Ans.* $3\log 2$. b) $f[\varphi(a)]$. *Ans.* $3\log a$. c) $\varphi[f(a)]$. *Ans.* $[\log a]^3$.

8. Find the natural domain of definition of the function $y=2x^2+1$ *Ans.* $-\infty<x<+\infty$.

9. Find the natural domains of definition of the functions: a) $\sqrt{1-x^2}$. *Ans.* $-1\leqslant x\leqslant+1$. b) $\sqrt{3+x}+\sqrt[4]{7-x}$. *Ans.* $-3\leqslant x\leqslant 7$. c) $\sqrt[3]{x+a}-\sqrt[5]{x-b}$. *Ans.* $-\infty<x<+\infty$. d) $\frac{a+x}{a-x}$. *Ans.* $x\neq a$. e) $\arcsin^2 x$. *Ans.* $-1\leqslant x\leqslant 1$. f) $y=\log x$. *Ans.* $x>0$. g) $y=a^x$ $(a>0)$. *Ans.* $-\infty<x<+\infty$.

Construct the graphs of the functions:

10. $y=-3x+5$. **11.** $y=\frac{1}{2}x^2+1$. **12.** $y=3-2x^2$. **13.** $y=x^2+2x-1$. **14.** $y=\frac{1}{x-1}$. **15.** $y=\sin 2x$. **16.** $y=\cos 3x$. **17.** $y=x^2-4x+6$. **18.** $y=\frac{1}{1-x^2}$. **19.** $y=\sin\left(x+\frac{\pi}{4}\right)$. **20.** $y=\cos\left(x-\frac{\pi}{3}\right)$. **21.** $y=\tan\frac{1}{2}x$. **22.** $y=\cot\frac{1}{4}x$. **23.** $y=3^x$. **24.** $y=2^{-x^2}$. **25.** $y=\log_2\frac{1}{x}$. **26.** $y=x^3+1$. **27.** $y=4-x^3$. **28.** $y=\frac{1}{x^2}$. **29.** $y=x^4$. **30.** $y=x^5$. **31.** $y=x^{\frac{1}{2}}$. **32.** $y=x^{-\frac{1}{2}}$. **33.** $y=x^{\frac{1}{3}}$. **34.** $y=|x|$. **35.** $y=\log_2|x|$. **36.** $y=\log_2(1-x)$. **37.** $y=3\sin\left(2x+\frac{\pi}{3}\right)$. **38.** $y=$

$=4\cos\left(x+\frac{\pi}{2}\right)$. **39.** The function $f(x)$ is defined on the interval $[-1, 1]$ as follows:

$$f(x)=1+x \quad \text{for } -1\leqslant x<0;$$
$$f(x)=1-2x \quad \text{for } 0\leqslant x<1.$$

40. The function $f(x)$ is defined on the interval $[0, 2]$ as follows:

$$f(x)=x^3 \quad \text{for } 0\leqslant x\leqslant 1;$$
$$f(x)=x \quad \text{for } 1\leqslant x\leqslant 2.$$

Plot the curves given by the polar equations: **41.** $\varrho=\frac{a}{\varphi}$ (hyperbolic spiral). **42.** $\varrho=a^{\varphi}$ (logarithmic spiral). **43.** $\varrho=a\sqrt{\cos 2\varphi}$ (lemniscate). **44.** $\varrho=a(1-\cos\varphi)$ (cardioid). **45.** $\varrho=a\sin 3\varphi$.

CHAPTER II

LIMIT. CONTINUITY OF A FUNCTION

SEC. 1. THE LIMIT OF A VARIABLE. AN INFINITELY LARGE VARIABLE

In this section we shall consider ordered variables that vary in a special way defined as follows: "the variable approaches a limit". Throughout the remainder of the course, the concept of limit of a variable will play a fundamental role, for it is intimately bound up with the basic concepts of mathematical analysis, such as derivative, integral, etc.

Definition 1. A constant number a is said to be the *limit* of a variable x, if for every preassigned arbitrarily small positive number ε it is possible to indicate a value of the variable x such that all subsequent values of the variable will satisfy the inequality

$$|x-a|<\varepsilon.$$

If the number a is the limit of the variable x, one says that x approaches the limit a; in symbols we have

$$x \longrightarrow a \text{ or } \lim x = a.$$

In geometric terms, limit may be defined as follows.

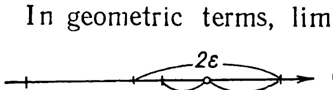


Fig. 28.

The constant number a is the *limit* of the variable x if for any preassigned arbitrarily small neighbourhood with centre in the point a and with radius ε there is a value of x such that all points corresponding to subsequent values of the variable will be within this neighbourhood (Fig. 28). Let us consider several cases of variables approaching limits.

Example 1. The variable x takes on successive values:

$$x_1=2;\ x_2=1\frac{1}{2};\ \ x_3=1\frac{1}{3};\ \ldots;\ x_n=1\frac{1}{n};\ \ldots$$

We shall prove that this variable has unity as its limit. We have

$$|x_n-1|=\left|\left(1+\frac{1}{n}\right)-1\right|=\frac{1}{n}.$$

For any ε, all subsequent values of the variable begin with n, where $\frac{1}{n}<\varepsilon$, or $n>\frac{1}{\varepsilon}$ will satisfy the inequality $|x_n-1|<\varepsilon$ and the proof is complete.

It will be noted here that the variable quantity decreases as it approaches the limit.

Example 2. The variable x takes on successive values: $x_1 = 1 - \frac{1}{2}$; $x^2 = 1 + \frac{1}{2^2}$; $x_3 = 1 - \frac{1}{2^3}$; $x_4 = 1 - \frac{1}{2^4}$; ...; $x_n = 1 + (-1)^n \frac{1}{2^n}$; ...
This variable has a limit of unity. Indeed,

$$|x_n - 1| = \left|\left(1 + (-1)^n \frac{1}{2^n}\right) - 1\right| = \frac{1}{2^n}.$$

For any ε, beginning with n, which satisfies the relation

$$\frac{1}{2^n} < \varepsilon,$$

from which it follows that

$$2^n > \frac{1}{\varepsilon},$$

$$n \log 2 > \log \frac{1}{\varepsilon}$$

or

$$n > \frac{\log \frac{1}{\varepsilon}}{\log 2},$$

all subsequent values of x will satisfy the relation

$$|x_n - 1| < \varepsilon.$$

It will be noted here that the values of the variable are greater than or less than the limit, and the variable approaches its limit by "oscillating about it".

Note 1. As was pointed out in Sec. 3 (see Ch. 1), a constant quantity c is frequently regarded as a variable whose values all coincide: $x = c$.

Obviously, the limit of a constant is equal to the constant itself, since we always have the inequality $|x - c| = |c - c| = 0 < \varepsilon$ for any ε.

Note 2. From the definition of a limit it follows that a variable cannot have two limits. Indeed, if $\lim x = a$ and $\lim x = b$ $(a < b)$, then x must satisfy, at one and the same time, two inequalities:

$$|x - a| < \varepsilon \text{ and } |x - b| < \varepsilon$$

for an arbitrarily small ε; but this is impossible if $\varepsilon < \frac{b-a}{2}$ (Fig. 29).

Note 3. One should not think that every variable has a limit. Let the variable x take on the following successive values:

$$x_1=\frac{1}{2};\ x_2=1-\frac{1}{4};\ x_3=\frac{1}{8};\ x_4=1-\frac{1}{16};\ \ldots;$$

$$x_{2k}=1-\frac{1}{2^{2k}};\ x_{2k+1}=\frac{1}{2^{2k+1}}$$

(Fig. 30). For k sufficiently large, the value x_{2k} and all subsequent values with even labels will differ from unity by as small a

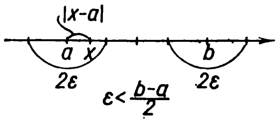


Fig. 29.

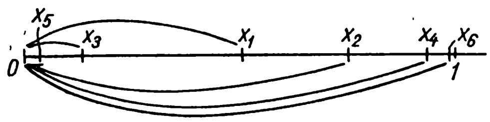


Fig. 30.

number as we please, while the next value x_{2k+1} and all subsequent values of x with odd labels will differ from zero by as small a number as we please. Consequently, the variable x does not approach a limit.

In the definition of a limit it is stated that if the variable approaches the limit a, then a is a constant. But the word "approaches" is used also to describe another type of variation of a variable, as will be seen from the following definition.

Definition 2. A variable x approaches infinity if for every preassigned positive number M it is possible to indicate a value of x such that, beginning with this value, all subsequent values of the variable will satisfy the inequality $|x|>M$.

If the variable x approaches infinity, it is called an *infinitely large* variable and we write $x\to\infty$.

Example 3. The variable x takes on the values

$$x_1=-1;\ x_2=-2;\ x_3=-3;\ \ldots;\ x_n=(-1)^n n\ \ldots$$

This is an infinitely large variable quantity, since for an arbitrary $M>0$ all values of the variable, beginning with a certain one, are, in absolute magnitude, greater than M.

The variable x "approaches plus infinity", $x\to+\infty$, if for an arbitrary $M>0$ all subsequent values of the variable, beginning with a certain one, satisfy the inequality $M<x$.

An example of a variable quantity approaching plus infinity is the variable x that takes on the values $x_1=1$, $x_2=2$, $\ldots$, $x_n=n$, $\ldots$

A variable approaches minus infinity, $x \to -\infty$, if for an arbitrary $M>0$, all subsequent values of the variable, beginning with a certain one, satisfy the inequality $x<-M$.

For example, a variable x that assumes the values $x_1=-1$, $x_2=-2$, ..., $x_n=-n$, ..., approaches minus infinity.

SEC. 2. THE LIMIT OF A FUNCTION

In this section we shall consider certain cases of the variation of a function when the argument x approaches a certain limit a or infinity.

Definition 1. Let the function $y=f(x)$ be defined in a certain neighbourhood of the point a or at certain points of this neighbourhood. *The function $y=f(x)$ approaches the limit b $(y \to b)$ as x approaches a $(x \to a)$*, if for every positive number ε, no matter how small, it is possible to indicate a positive number δ such that for all x, different from a and satisfying the inequality*)

$$|x-a|<\delta,$$

we have the inequality

$$|f(x)-b|<\varepsilon.$$

If b is the *limit of the function* $f(x)$ as $x \to a$, we write

$$\lim_{x \to a} f(x) = b$$

or $f(x) \to b$ as $x \to a$.

If $f(x) \to b$ as $x \to a$, this is illustrated on the graph of the function $y=f(x)$ as follows (Fig. 31).

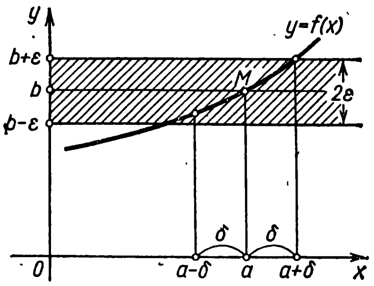


Fig. 31.

Since from the inequality $|x-a|<\delta$ there follows the inequality $|f(x)-b|<\varepsilon$, this means that for all points x

*) Here we mean the values of x that satisfy the inequality $|x-a|<\delta$ and belong to the domain of definition of the function. We shall encounter similar circumstances in the future. For instance, when considering the behaviour of a function as $x \to \infty$, it may happen that the function is defined only for positive integral values of x. And so in this case $x \to \infty$, assuming only positive integral values. We shall not specify this when it comes up later on.

that are not more distant from the point a than δ, the points M of the graph of the function $y=f(x)$ lie within a band of width 2ε bounded by the lines $y=b-\varepsilon$ and $y=b+\varepsilon$.

Note 1. We may define the limit of the function $f(x)$ as $x \longrightarrow a$ as follows.

Let a variable x assume values such (that is, ordered in such fashion) that if

$$|x^{*}-a|>|x^{**}-a|,$$

then x^{**} is the subsequent value and x^{*} is the preceding value; but if

$$|\bar{x}^{*}-a|=|\bar{x}^{**}-a| \text{ and } \bar{x}^{*}<\bar{x}^{**},$$

then $\bar{x}^{**}$ is the subsequent value and $\bar{x}^{*}$ is the preceding value.

In other words, of two points on a number scale, the subsequent one is that which is closer to the point a; at equal distances, the subsequent one is that which is to the right of the point a.

Let a variable quantity x ordered in this fashion approach the limit $a\,[x \longrightarrow a$ or $\lim x=a]$.

Let us further consider the variable $y=f(x)$. We shall here and henceforward consider that of the two values of a function, the subsequent one is that which corresponds to the subsequent value of the argument.

If, as $x \longrightarrow a$, a variable y thus defined approaches a certain limit b, we shall write

$$\lim_{x \to a} f(x)=b$$

and we shall say that the function $y=f(x)$ approaches the limit b as $x \longrightarrow a$.

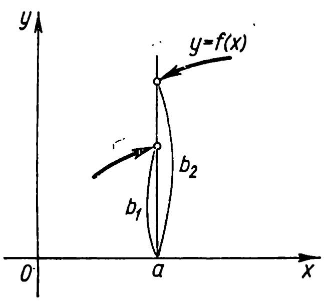


Fig. 32.

It is easy to prove that both definitions of the limit of a function are equivalent.

Note 2. If $f(x)$ approaches the limit b_1 as x approaches a certain number a, so that x takes on only values less than a, we write $\lim\limits_{x \to a-0} f(x)=b_1$ and call b_1 the limit of the function $f(x)$ on the left of the point a. If x takes on only values greater than a, we write $\lim\limits_{x \to a+0} f(x)=b_2$ and call b_2 the limit of the function on the right of the point a (Fig. 32).

It can be proved that if the limit on the right and the limit on the left exist and are equal, that is, $b_1=b_2=b$, then b will be

the limit in the sense of the foregoing definition of a limit at the point a. And conversely, if there exists a limit b of a function at the point a, then there exist limits of the function at the point a both on the right and on the left and they are equal.

Example 1. Let us prove that $\lim\limits_{x \to 2}(3x+1)=7$. Indeed, let an arbitrary $\varepsilon > 0$ be given; for the inequality $|(3x+1)-7| < \varepsilon$ to be fulfilled it is necessary to have the following inequalities fulfilled:

$$|3x-6| < \varepsilon, \quad |x-2| < \frac{\varepsilon}{3}, \quad -\frac{\varepsilon}{3} < x-2 < \frac{\varepsilon}{3}.$$

Thus, given any ε, for all values of x satisfying the inequality $|x-2| < \frac{\varepsilon}{3} = \delta$, the value of the function $3x+1$ will differ from 7 by less than ε. And this means that 7 is the limit of the function as $x \to 2$.

Note 3. For a function to have a limit as $x \to a$, it is not necessary that the function be defined at the point $x=a$. When finding the limit we consider the values of the function in the neighbourhood of the point a that are different from a; this is clearly illustrated in the following case.

Example 2. We shall prove that $\lim\limits_{x \to 2}\frac{x^2-4}{x-2}=4$. Here, the function $\frac{x^2-4}{x-2}$ is not defined for $x=2$.

It is necessary to prove that for an arbitrary ε, there will be a δ such that the following inequality will be fulfilled:

$$\left|\frac{x^2-4}{x-2}-4\right| < \varepsilon \qquad (1)$$

if $|x-2| < \delta$. But when $x \neq 2$ inequality (1) is equivalent to the inequality

$$\left|\frac{(x-2)(x+2)}{x-2}-4\right| = |(x+2)-4| < \varepsilon$$

$$|x-2| < \varepsilon. \qquad (2)$$

Thus, for an arbitrary ε, inequality (1) will be fulfilled if inequality (2) is fulfilled (here, $\delta=\varepsilon$), which means that the given function has the number 4 as its limit as $x \to 2$.

Let us now consider certain cases of variation of a function as $x \to \infty$.

Definition 2. *The function $f(x)$ approaches the limit b as $x \to \infty$* if for each arbitrarily small positive number ε it is possible to indicate a positive number N such that for all values of x that satisfy the inequality $|x| > N$ the inequality $|f(x)-b| < \varepsilon$ will be fulfilled.

Example 3. To prove that

$$\lim_{x\to\infty}\left(\frac{x+1}{x}\right)=1$$

or

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)=1.$$

It is necessary to prove that, for an arbitrary ε, the following inequality will be fulfilled

$$\left|\left(1+\frac{1}{x}\right)-1\right|<\varepsilon, \tag{3}$$

provided $|x|>N$, where N is determined by the choice of ε. Inequality (3) is equivalent to the following inequality: $\left|\frac{1}{x}\right|<\varepsilon$, which will be fulfilled if

$$|x|>\frac{1}{\varepsilon}=N.$$

And this means that $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)=\lim\limits_{x\to\infty}\frac{x+1}{x}=1$ (Fig. 33).

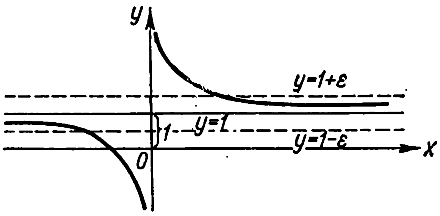


Fig. 33.

Knowing the meanings of the symbols $x\to\infty$ and $x\to-\infty$ the meanings of the following expressions are obvious:
"$f(x)$ approaches b as $x\to+\infty$" and
"$f(x)$ approaches b as $x\to-\infty$", or, in symbols,

$$\lim_{x\to+\infty} f(x)=b,$$

$$\lim_{x\to-\infty} f(x)=b.$$

SEC. 3. A FUNCTION THAT APPROACHES INFINITY. BOUNDED FUNCTIONS

We have considered cases when the function $f(x)$ approaches a certain limit b as $x\to a$ or as $x\to\infty$.

Let us now take the case when the function $y=f(x)$ approaches infinity when the argument varies in some way.

Definition 1. The function $f(x)$ approaches infinity as $x \to a$, i.e., it is an *infinitely large* quantity as $x \to a$, if for each positive number M, no matter how large, it is possible to find a $\delta > 0$ such that for all values of x different from a and satisfying the condition $|x-a| < \delta$, we have the inequality $|f(x)| > M$.

If $f(x)$ approaches infinity as $x \to a$, we write

$$\lim_{x \to a} f(x) = \infty$$

or $f(x) \to \infty$ as $x \to a$.

If $f(x)$ approaches infinity as $x \to a$ and, in the process, assumes only positive or only negative values, the appropriate notation is $\lim\limits_{x \to a} f(x) = +\infty$ or $\lim\limits_{x \to a} f(x) = -\infty$.

Example 1. We shall prove that $\lim\limits_{x \to 1} \frac{1}{(1-x)^2} = +\infty$. Indeed, for any $M > 0$ we will have

$$\frac{1}{(1-x)^2} > M,$$

provided

$$(1-x)^2 < \frac{1}{M}, \quad |1-x| < \frac{1}{\sqrt{M}} = \delta.$$

The function $\frac{1}{(1-x)^2}$ assumes only positive values (Fig. 34).

Example 2. We shall prove that $\lim\limits_{x \to 0} \left(-\frac{1}{x}\right) = \infty$. Indeed, for any $M > 0$ we will have

$$\left|-\frac{1}{x}\right| > M,$$

provided

$$|x| = |x-0| < \frac{1}{M} = \delta.$$

Here $\left(-\frac{1}{x}\right) > 0$ for $x < 0$ and $\left(-\frac{1}{x}\right) < 0$ for $x > 0$ (Fig. 35).

If the function $f(x)$ approaches infinity as $x \to \infty$, we write

$$\lim_{x \to \infty} f(x) = \infty,$$

and we may have the particular cases:

$$\lim_{x \to +\infty} f(x) = \infty, \qquad \lim_{x \to -\infty} f(x) = \infty, \qquad \lim_{x \to +\infty} f(x) = -\infty.$$

For example,

$$\lim_{x \to \infty} x^2 = +\infty, \qquad \lim_{x \to -\infty} x^3 = -\infty$$

Note 1. The function $y=f(x)$ as $x \to a$ or as $x \to \infty$ may not approach a finite limit or infinity.

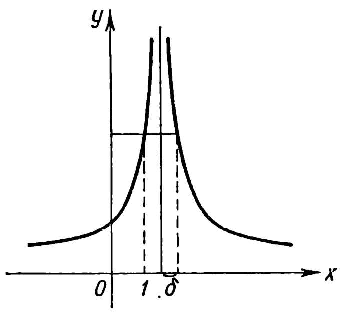


Fig. 34.

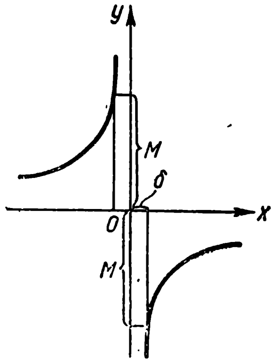


Fig. 35.

Example 3. The function $y=\sin x$ defined on the infinite interval $-\infty < x < +\infty$, as $x \to +\infty$, does not approach either a finite limit or infinity (Fig. 36).

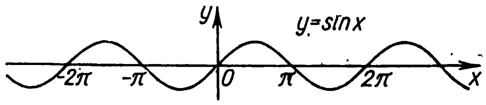


Fig. 36.

Example 4. The function $y=\sin\frac{1}{x}$ defined for all values of x, except $x=0$, does not approach either a finite limit or infinity as $x \to 0$. The graph of this function is shown in Fig. 37.

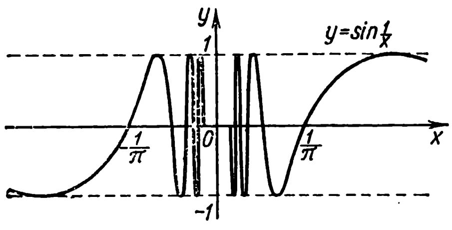


Fig. 37.

Definition 2. The function $y=f(x)$ is called *bounded* in a given range of the argument x if there exists a positive number M such that for all values of x in the range under consideration the

inequality $|f(x)| \leqslant M$ will be fulfilled. If there is no such number M, the function $f(x)$ is called *unbounded* in the given range.

Example 5. The function $y = \sin x$, defined in the infinite interval $-\infty < x < +\infty$, is bounded, since for all values of x

$$|\sin x| \leqslant 1 = M.$$

Definition 3. The function $f(x)$ is called *bounded as* $x \to a$ if there exists a neighbourhood with centre at the point a, in which the given function is bounded.

Definition 4. The function $y = f(x)$ is called *bounded as* $x \to \infty$ if there exists a number $N > 0$ such that for all values of x satisfying the inequality $|x| > N$, the function $f(x)$ is bounded.

The boundedness of a function approaching a limit is decided by the following theorem.

Theorem 1. *If* $\lim\limits_{x \to a} f(x) = b$, *where* b *is a finite number, the function* $f(x)$ *is bounded as* $x \to a$.

Proof. From the equality $\lim\limits_{x \to a} f(x) = b$ it follows that for any $\varepsilon > 0$ there will be a δ such that in the neighbourhood $a - \delta < x < a + \delta$ the inequality

$$|f(x) - b| < \varepsilon$$

or

$$|f(x)| < |b| + \varepsilon$$

will be fulfilled, which means that the function $f(x)$ is bounded as $x \to a$.

Note 2. From the definition of a bounded function $f(x)$ it follows that if

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad \lim_{x \to \infty} f(x) = \infty,$$

that is, if $f(x)$ is an infinitely large function, it is unbounded. The converse is not true: an unbounded function may not be infinitely large.

For example, the function $y = x \sin x$ as $x \to \infty$ is unbounded because, for any $M > 0$, values of x can be found such that $|x \sin x| > M$. But the function $y = x \sin x$ is not infinitely large because it becomes zero when $x = 0, \pi, 2\pi, \ldots$ The graph of the unction $y = x \sin x$ is shown in Fig. 38.

Theorem 2. *If* $\lim\limits_{x \to a} f(x) = b \neq 0$, *then the function* $y = \frac{1}{f(x)}$ *is a bounded function as* $x \to a$.

Proof. From the statement of the theorem it follows that for an arbitrary $\varepsilon > 0$ in a certain neighbourhood of the point $x = a$ we

will have $|f(x)-b|<\varepsilon$, or $||f(x)|-|b||<\varepsilon$, or $-\varepsilon<|f(x)|-|b|<\varepsilon$, or $|b|-\varepsilon<|f(x)|<|b|+\varepsilon$.

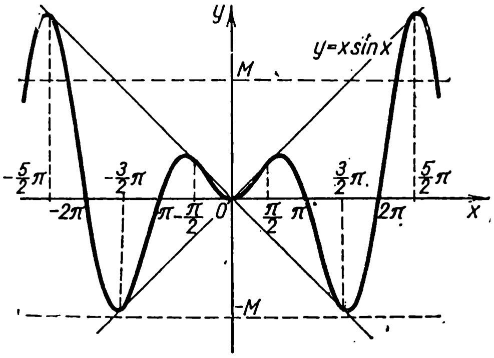


Fig. 38.

From the latter inequality it follows that

$$\frac{1}{|b|-\varepsilon}>\frac{1}{|f(x)|}>\frac{1}{|b|+\varepsilon}.$$

For example, taking $\varepsilon=\frac{1}{10}|b|$, we get

$$\frac{10}{9|b|}>\frac{1}{|f(x)|}>\frac{10}{11|b|},$$

which means that the function $\frac{1}{f(x)}$ is bounded.

SEC. 4. INFINITESIMALS AND THEIR BASIC PROPERTIES

In this section we shall consider functions approaching zero as the argument varies in a certain manner.

Definition. The function $\alpha=\alpha(x)$ is called *infinitesimal* as $x \to a$ or as $x \to \infty$ if $\lim\limits_{x \to a} \alpha(x)=0$ or $\lim\limits_{x \to \infty} \alpha(x)=0$.

From the definition of a limit it follows that if, for example, $\lim\limits_{x \to a} \alpha(x)=0$, this means that for any preassigned arbitrarily small positive ε there will be a $\delta>0$ such that for all x satisfying the condition $|x-a|<\delta$, the condition $|\alpha(x)|<\varepsilon$ will be satisfied.

Example 1. The function $\alpha=(x-1)^2$ is an infinitesimal as $x \to 1$ because $\lim\limits_{x \to 1} \alpha=\lim\limits_{x \to 1}(x-1)^2=0$ (Fig. 39).

Example 2. The function $\alpha=\frac{1}{x}$ is an infinitesimal as $x\to\infty$ (Fig. 40) (see Example 3, Sec. 2).

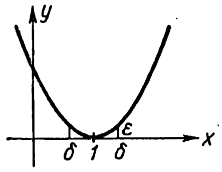


Fig. 39.

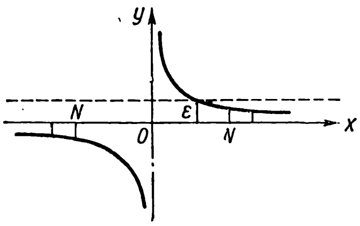


Fig. 40.

Let us establish a relationship that will be important later on.

Theorem 1. *If the function $y=f(x)$ is in the form of a sum of a constant b and an infinitesimal α:*

$$y=b+\alpha, \tag{1}$$

then

$$\lim y=b \quad (\text{as } x\to a \text{ or } x\to\infty).$$

Conversely, *if $\lim y=b$, we may write $y=b+\alpha$, where α is an infinitesimal.*

Proof. From equality (1) it follows that $|y-b|=|\alpha|$. But for an arbitrary ε, all values of α, from a certain value onwards, satisfy the relationship $|\alpha|<\varepsilon$; consequently, the inequality $|y-b|<\varepsilon$ will be fulfilled for all values of y from a certain value onwards. And this means that $\lim y=b$.

Conversely: if $\lim y=b$, then given an arbitrary ε, for all values of y, from a certain value onwards, we will have $|y-b|<\varepsilon$. But if we denote $y-b=\alpha$, then it follows that for all values of α, from a certain one onwards, we will have $|\alpha|<\varepsilon$; and this means that α is an infinitesimal.

Example 3. Let a function be given (Fig. 41)

$$y=1+\frac{1}{x},$$

then

$$\lim_{x\to\infty} y=1,$$

and, conversely, if

$$\lim_{x\to\infty} y=1$$

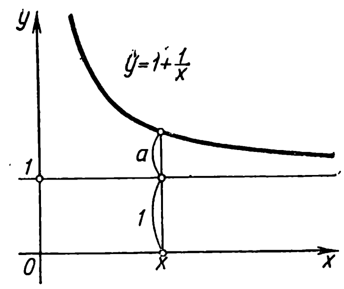


Fig. 41.

the variable y may be represented in the form of a sum of the limit 1 and the infinitesimal $\alpha=\frac{1}{x}$; that is (Fig. 41),

$$y=1+\alpha.$$

Theorem 2. *If* $\alpha=\alpha(x)$ *approaches zero as* $x\to a$ *(or as* $x\to\infty$*) and does not become zero, then* $y=\frac{1}{\alpha}$ *approaches infinity.*

Proof. For any $M>0$, no matter how large, the inequality $\frac{1}{|\alpha|}>M$ will be fulfilled provided the inequality $|\alpha|<\frac{1}{M}$ is fulfilled. The latter inequality will be fulfilled for all values of α, from a certain one onwards, since $\alpha(x)\to 0$.

Theorem 3. *The algebraic sum of two, three and, in general, a definite number of infinitesimals is an infinitesimal function.*

Proof. We shall prove the theorem for two terms, since the proof is similar for any number of terms.

Let $u(x)=\alpha(x)+\beta(x)$, where $\lim\limits_{x\to a}\alpha(x)=0$, $\lim\limits_{x\to a}\beta(x)=0$. We shall prove that for any $\varepsilon>0$, no matter how small, there will be a $\delta>0$ such that when the inequality $|x-a|<\delta$ is satisfied, the inequality $|u|<\varepsilon$ will be fulfilled. Since $\alpha(x)$ is an infinitesimal, a δ will be found such that in a neighbourhood with centre at the point a and radius δ_1, we will have

$$|\alpha(x)|<\frac{\varepsilon}{2}.$$

Since $\beta(x)$ is an infinitesimal, we will have $|\beta(x)|<\frac{\varepsilon}{2}$ in the neighbourhood of the point a with radius δ_2.

Let us take δ equal to the smaller of the two quantities δ_1 and δ_2, then the inequalities $|\alpha|<\frac{\varepsilon}{2}$ and $|\beta|<\frac{\varepsilon}{2}$ will be fulfilled in the neighbourhood of the point a with radius δ. Hence, in this neighbourhood we will have

$$|u|=|\alpha(x)+\beta(x)|\leqslant|\alpha(x)|+|\beta(x)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

and so $|u|<\varepsilon$, as required.

The proof is similar for the case when

$$\lim_{x\to\infty}\alpha(x)=0,\quad \lim_{x\to\infty}\beta(x)=0.$$

Note. Later on we shall have to consider sums of infinitesimals such that the number of terms increases with a decrease in each term. In this case, the theorem may not hold. To take an example, consider $u=\underbrace{\frac{1}{x}+\frac{1}{x}+\ldots+\frac{1}{x}}_{x \text{ terms}}$ where x takes on only positive integral values ($x=1, 2, 3, \ldots, n, \ldots$). It is obvious that as $x \to \infty$ each term is an infinitesimal, but the sum $u=1$ is not an infinitesimal.

Theorem 4. *The product of the function of an infinitesimal $\alpha=\alpha(x)$ by a function bounded by $z=z(\alpha)$, as $x \to a$ (or $x \to \infty$) is an infinitesimal quantity (function).*

Proof. Let us prove the theorem for the case $x \to a$. For a certain $M>0$ there will be a neighbourhood of the point $x=a$ in which the inequality $|z|<M$ will be satisfied. For any $\varepsilon>0$ there will be a neighbourhood in which the inequality $|\alpha|<\frac{\varepsilon}{M}$ will be fulfilled. The following inequality will be fulfilled in the least of these two neighbourhoods:

$$|\alpha z|<\frac{\varepsilon}{M}M=\varepsilon$$

which means that αz is an infinitesimal. The proof is similar for the case $x \to \infty$. Two corollaries follow from this theorem.

Corollary 1. If $\lim \alpha=0$, $\lim \beta=0$, then $\lim \alpha\beta=0$ because $\beta(x)$ is a bounded quantity. This holds for any finite number of factors.

Corollary 2. If $\lim \alpha=0$ and $c=\text{const}$, then $\lim c\alpha=0$.

Theorem 5. *The quotient $\frac{\alpha(x)}{z(x)}$ obtained by dividing the infinitesimal $\alpha(x)$ by a function whose limit differs from zero is an infinitesimal.*

Proof. Let $\lim \alpha(x)=0$, $\lim z(x)=b \neq 0$. By Theorem 2, Sec. 3, it follows that $\frac{1}{z(\alpha)}$ is a bounded quantity. For this reason, the fractions $\frac{\alpha(x)}{z(x)}=\alpha(x)\frac{1}{z(x)}$ are a product of an infinitesimal by a bounded quantity, that is, an infinitesimal.

SEC. 5. BASIC THEOREMS ON LIMITS

In this section, as in the preceding one, we shall consider sets of functions that depend on the same argument x, where $x \to a$ or $x \to \infty$.

We shall carry out the proof for one of these cases, since the other is proved analogously. Sometimes we will not even write $x \to a$ or $x \to \infty$, but will take them for granted.

Theorem 1. *The limit of an algebraic sum of two, three and, in general, any definite number of variables is equal to the algebraic sum of the limits of these variables:*

$$\lim (u_1 + u_2 + \ldots + u_k) = \lim u_1 + \lim u_2 + \ldots + \lim u_k.$$

Proof. We shall carry out the proof for two terms, since it is the same for any number of terms. Let $\lim u_1 = a_1$, $\lim u_2 = a_2$. Then on the basis of Theorem 1, Sec. 4, we can write

$$u_1 = a_1 + \alpha_1, \quad u_2 = a_2 + \alpha_2$$

where α_1 and α_2 are infinitesimals. Consequently,

$$u_1 + u_2 = (a_1 + a_2) + (\alpha_1 + \alpha_2).$$

Since $(a_1 + a_2)$ is a constant and $(\alpha_1 + \alpha_2)$ is an infinitesimal, again by Theorem 1, Sec. 4, we conclude that

$$\lim (u_1 + u_2) = a_1 + a_2 = \lim u_1 + \lim u_2.$$

Example 1.

$$\lim_{x \to \infty} \frac{x^2 + 2x}{x^2} = \lim_{x \to \infty} \left(1 + \frac{2}{x}\right) = \lim_{x \to \infty} 1 + \lim_{x \to \infty} \frac{2}{x} = 1 + \lim_{x \to \infty} \frac{2}{x} = 1 + 0 = 1.$$

Theorem 2. *The limit of a product of two, three and, in general, any definite number of variables is equal to the product of the limits of these variables:*

$$\lim u_1 \cdot u_2 \ \ldots \ u_k = \lim u_1 \cdot \lim u_2 \ \ldots \ \lim u_k.$$

Proof. To save space we shall carry out the proof for two factors. Let $\lim u_1 = a_1$, $\lim u_2 = a_2$. Therefore,

$$u_1 = a_1 + \alpha_1, \quad u_2 = a_2 + \alpha_2,$$

$$u_1 u_2 = (a_1 + \alpha_1)(a_2 + \alpha_2) = a_1 a_2 + a_1 \alpha_2 + a_2 \alpha_1 + \alpha_1 \alpha_2.$$

The product $a_1 a_2$ is a constant. By the theorems of Sec. 4, the quantity $a_1 \alpha_2 + a_2 \alpha_1 + \alpha_1 \alpha_2$ is an infinitesimal. Hence, $\lim u_1 u_2 = a_1 a_2 = \lim u_1 \cdot \lim u_2$.

Corollary. A constant factor may be taken outside the limit sign. Indeed, if $\lim u_1 = a_1$, c is a constant and, consequently, $\lim c = c$, then $\lim (cu_1) = \lim c \cdot \lim u_1 = c \cdot \lim u_1$, as required.

Example 2.

$$\lim_{x \to 2} 5x^3 = 5 \lim_{x \to 2} x^3 = 5 \cdot 8 = 40.$$

Theorem 3. *The limit of a quotient of two variables is equal to the quotient of the limits of these variables if the limit of the denominator is not zero:*

$$\lim \frac{u}{v} = \frac{\lim u}{\lim v} \text{ if } \lim v \neq 0.$$

Proof. Let $\lim u = a$, $\lim v = b \neq 0$. Then $u = a + \alpha$, $v = b + \beta$, where α and β are infinitesimals.

We write the identities

$$\frac{u}{v} = \frac{a+\alpha}{b+\beta} = \frac{a}{b} + \left(\frac{a+\alpha}{b+\beta} - \frac{a}{b}\right) = \frac{a}{b} + \frac{\alpha b - \beta a}{b(b+\beta)}.$$

or

$$\frac{u}{v} = \frac{a}{b} + \frac{\alpha b - \beta a}{b(b+\beta)}.$$

The fraction $\frac{a}{b}$ is a constant number, while the fraction $\frac{\alpha b - \beta a}{b(b+\beta)}$ is an infinitesimal variable by virtue of Theorems 4 and 5 (Sec. 4), since $\alpha b - \beta a$ is an infinitesimal, while the denominator $b(b+\beta)$ has the limit $b^2 \neq 0$. Thus, $\lim \frac{u}{v} = \frac{a}{b} = \frac{\lim u}{\lim v}$.

Example 3.

$$\lim_{x \to 1} \frac{3x+5}{4x-2} = \frac{\lim\limits_{x \to 1} (3x+5)}{\lim\limits_{x \to 1} (4x-2)} = \frac{3 \lim\limits_{x \to 1} x + 5}{4 \lim\limits_{x \to 1} x - 2} = \frac{3 \cdot 1 + 5}{4 \cdot 1 - 2} = \frac{8}{2} = 4.$$

Here, we made use of the already proved theorem for the limit of a fraction because the limit of the denominator differs from zero as $x \to 1$. If the limit of the denominator is zero, the theorem for the limit of a fraction is not applicable, and special considerations have to be invoked.

Example 4. Find $\lim\limits_{x \to 2} \frac{x^2-4}{x-2}$.

Here the denominator and numerator approach zero as $x \to 2$, and, consequently, Theorem 3 is inapplicable. Perform the following identical transformation:

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2.$$

This transformation holds for all values of x different from 2. And so, having in view the definition of a limit, we can write

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2} (x+2) = 4.$$

Example 5. Find $\lim\limits_{x \to 1} \frac{1}{x-1}$. As $x \to 1$ the denominator approaches zero but the numerator does not (it approaches unity). Thus, the limit of the reciprocal quantity is zero:

$$\lim_{x \to 1} \frac{x-1}{x} = \frac{\lim\limits_{x \to 1} (x-1)}{\lim\limits_{x \to 1} x} = \frac{0}{1} = 0.$$

Whence, by Theorem 2 of the preceding section, we have

$$\lim_{x \to 1} \frac{x}{x-1} = \infty .$$

Theorem 4. *If the inequalities* $u \leqslant z \leqslant v$ *are fulfilled between the corresponding values of three functions* $u = u(x)$, $z = z(x)$, *and* $v = v(x)$, *where* $u(x)$ *and* $v(x)$, *as* $x \to a$ (*or as* $x \to \infty$), *approach one and the same limit* b, *then* $z = z(x)$ *as* $x \to a$ (*or as* $x \to \infty$) *approaches the same limit.*

Proof. For definiteness we shall consider variations of the functions as $x \to a$. From the inequalities $u \leqslant z \leqslant v$ follow the inequalities

$$u - b \leqslant z - b \leqslant v - b;$$

it is given that

$$\lim_{x \to a} u = b, \quad \lim_{x \to a} v = b.$$

Consequently, for any $\varepsilon > 0$ there will be a certain neighbourhood with centre at the point a, in which the inequality $|u - b| < \varepsilon$ will be fulfilled; likewise, there will be a certain neighbourhood with centre at the point a in which the inequality $|v - b| < \varepsilon$ will be fulfilled. The following inequalities will be fulfilled in the smaller of these neighbourhoods:

$$-\varepsilon < u - b < \varepsilon \quad \text{and} \quad -\varepsilon < v - b < \varepsilon,$$

and thus the inequalities

$$-\varepsilon < z - b < \varepsilon$$

will be fulfilled; that is,

$$\lim_{x \to a} z = b.$$

Theorem 5. *If as* $x \to a$ (*or as* $x \to \infty$) *the function* y *takes on nonnegative values* $y \geqslant 0$ *and, at the same time, approaches the limit* b, *then* b *is a nonnegative number* $b \geqslant 0$.

Proof. Assume that $b < 0$, then $|y - b| \geqslant b$; that is, the difference modulus $|y - b|$ is greater than the positive number $|b|$ and, hence, does not approach zero as $x \to a$. But then y does not approach b as $x \to a$; this contradicts the statement of the theorem. Thus, the assumption that $b < 0$ leads to a contradiction. Consequently, $b \geqslant 0$.

In similar fashion we can prove that if $y \leqslant 0$, then $\lim y \leqslant 0$.

Theorem 6. *If the inequality* $v \geqslant u$ *is fulfilled between corresponding values of two functions* $u = u(x)$ *and* $v = v(x)$ *which approach limits as* $x \to a$ (*or as* $x \to \infty$), *then* $\lim v \geqslant \lim u$.

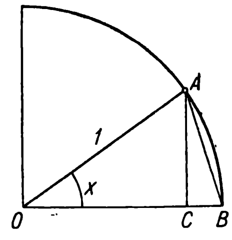


Fig. 42.

Proof. It is given that $v-u\geqslant 0$. Hence, by Theorem 5, $\lim(v-u)\geqslant 0$ or $\lim v-\lim u\geqslant 0$, and so $\lim v\geqslant\lim u$.

Example 6. Prove that $\lim\limits_{x\to 0}\sin x=0$.

From Fig. 42 it follows that if $OA=1$, $x>0$, then $AC=\sin x$, $\overset{\frown}{AB}=x$, $\sin x<x$. Obviously, when $x<0$ we will have $|\sin x|<|x|$. By Theorems 5 and 6, it follows, from these inequalities, that $\lim\limits_{x\to 0}\sin x=0$.

Example 7 Prove that $\lim\limits_{x\to 0}\sin\frac{x}{2}=0$. Indeed, $\left|\sin\frac{x}{2}\right|<|\sin x|$. Consequently, $\lim\limits_{x\to 0}\sin\frac{x}{2}=0$.

Example 8. Prove that $\lim\limits_{x\to 0}\cos x=1$; note that

$$\cos x=1-2\sin^2\frac{x}{2},$$

therefore,

$$\lim_{x\to 0}\cos x=\lim_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)=1-2\lim_{x\to 0}\sin^2\frac{x}{2}=1-0=1.$$

In some investigations concerning the limits of variables, one has to solve two independent problems:

1) to prove that the limit of the variable exists and to establish the boundaries within which the limit under consideration exists;

2) to calculate the limit to the necessary degree of accuracy.

The first problem is sometimes solved by means of the following theorem which will be important later on.

Theorem 7. *If a variable v is an increasing variable, that is, each subsequent value is greater than the preceding value, and if it is bounded, that is, $v<M$, then this variable has the limit $\lim v=a$, where $a\leqslant M$.*

A similar assertion may be made with respect to a decreasing bounded variable quantity.

We do not give the proof of this theorem here since it is based on the theory of real numbers, which we shall not consider in this text.

In the following two sections we shall derive the limits of two functions that find wide application in mathematics.

SEC. 6. THE LIMIT OF THE FUNCTION $\frac{\sin x}{x}$ AS $x \to 0$

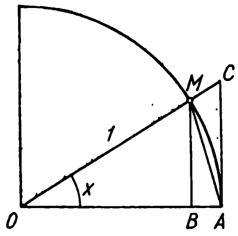


Fig. 43.

This function is not defined for $x=0$ since the numerator and denominator of the fraction become zero. Let us find the limit of this function as $x \to 0$. Let us consider a circle of radius 1 (Fig. 43); denote the central angle MOB by x; $0 < x < \frac{\pi}{2}$. From Fig. 43 it follows in straightforward fashion that

area $\triangle MOA <$ area of sector $MOA <$ area $\triangle COA$. (1)

The area $\triangle MOA = \frac{1}{2} OA \cdot MB = \frac{1}{2} \cdot 1 \cdot \sin x = \frac{1}{2} \sin x$.

The area of sector $MOA = \frac{1}{2} OA \cdot \widehat{AM} = \frac{1}{2} \cdot 1 \cdot x = \frac{1}{2} x$.

The area of $\triangle COA = \frac{1}{2} OA \cdot AC = \frac{1}{2} \cdot 1 \cdot \tan x = \frac{1}{2} \tan x$.

After cancelling $\frac{1}{2}$, inequality (1) is rewritten

$$\sin x < x < \tan x.$$

Divide all terms by $\sin x$:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

or

$$1 > \frac{\sin x}{x} > \cos x.$$

We derived this inequality on the assumption that $x > 0$; noting that $\frac{\sin(-x)}{(-x)} = \frac{\sin x}{x}$ and $\cos(-x) = \cos x$, we conclude that it holds for $x < 0$ as well But $\lim_{x \to 0} \cos x = 1$, $\lim_{x \to 0} 1 = 1$.

Hence, the variable $\frac{\sin x}{x}$ lies between two quantities that have the same limit (unity). Thus by Theorem 4 of the preceding section,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

The graph of the function $y = \frac{\sin x}{x}$ is shown in Fig. 44.

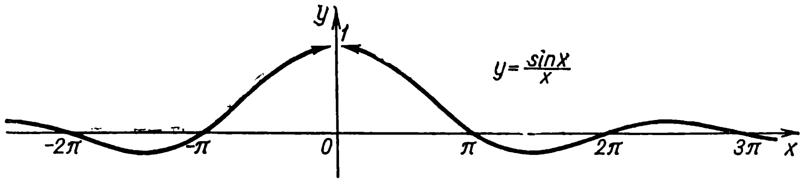


Fig. 44.

Examples.

1) $\lim\limits_{x\to 0}\frac{\tan x}{x}=\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\lim\limits_{x\to 0}\frac{1}{\cos x}=1\cdot\frac{1}{1}=1.$

2) $\lim\limits_{x\to 0}\frac{\sin kx}{x}=\lim\limits_{x\to 0}k\frac{\sin kx}{kx}=k\lim\limits_{\substack{x\to 0\\(kx\to 0)}}\frac{\sin(kx)}{(kx)}=k\cdot 1=k\quad(k=\text{const}).$

3) $\lim\limits_{x\to 0}\frac{1-\cos x}{x}=\lim\limits_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x}=\lim\limits_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\sin\frac{x}{2}=1\cdot 0=0.$

4) $$\lim_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}=\lim_{x\to 0}\frac{\alpha}{\beta}\cdot\frac{\frac{\sin\alpha x}{\alpha x}}{\frac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}\frac{\lim\limits_{x\to 0}\frac{\sin\alpha x}{\alpha x}}{\lim\limits_{x\to 0}\frac{\sin\beta x}{\beta x}}=$$
$$=\frac{\alpha}{\beta}\cdot\frac{1}{1}=\frac{\alpha}{\beta}\quad(\alpha=\text{const},\quad\beta=\text{const}).$$

SEC. 7. THE NUMBER *E*

Let us consider the variable

$$\left(1+\frac{1}{n}\right)^n,$$

where n is an increasing variable that takes on the values 1, 2, 3, ...

Theorem 1. *The variable* $\left(1+\frac{1}{n}\right)^n$, *as* $n\to\infty$, *has a limit between the numbers* 2 *and* 3.

Proof. By Newton's binomial formula we have

$$\left(1+\frac{1}{n}\right)^n=1+\frac{n}{1}\frac{1}{n}+\frac{n(n-1)}{1\cdot 2}\cdot\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{1}{n}\right)^3+\ldots$$
$$\ldots+\frac{n(n-1)(n-2)\ldots[n-(n-1)]}{1\cdot 2\cdot\ldots\cdot n}\left(\frac{1}{n}\right)^n. \qquad (1)$$

Carrying out the obvious algebraic manipulations in (1), we get

$$\left(1+\frac{1}{n}\right)^n=1+1+\frac{1}{1\cdot 2}\left(1-\frac{1}{n}\right)+\frac{1}{1\cdot 2\cdot 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots$$
$$\ldots+\frac{1}{1\cdot 2\cdot\ldots\cdot n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right). \qquad (2)$$

From the latter equality it follows that the variable $\left(1+\frac{1}{n}\right)^n$ is an increasing variable as n increases.

Indeed, when passing from the value n to the value $n+1$, each term in the latter sum increases,

$$\frac{1}{1\cdot 2}\left(1-\frac{1}{n}\right)<\frac{1}{1\cdot 2}\left(1-\frac{1}{n+1}\right) \quad \text{and so forth,}$$

and another term is added. (All terms of the expansion are positive.)

We shall show that the variable $\left(1+\frac{1}{n}\right)^n$ is bounded. Noting that $\left(1-\frac{1}{n}\right)<1$; $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)<1$, etc., we obtain from expression (2) the inequality

$$\left(1+\frac{1}{n}\right)^n<1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\ldots+\frac{1}{1\cdot 2\cdot 3\cdot\ldots\cdot n}.$$

Further noting that

$$\frac{1}{1\cdot 2\cdot 3}<\frac{1}{2^2}; \quad \frac{1}{1\cdot 2\cdot 3\cdot 4}<\frac{1}{2^3}; \quad \frac{1}{1\cdot 2\cdot\ldots\cdot n}<\frac{1}{2^{n-1}},$$

we can write the inequality

$$\left(1+\frac{1}{n}\right)^n<1+\underbrace{1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}}.$$

The grouped terms on the right-hand side of this inequality form a geometric progression with the common ratio $q=\frac{1}{2}$ and the first term $a=1$, and so

$$\left(1+\frac{1}{n}\right)^n< 1+\left[1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}\right]=$$
$$=1+\frac{a-aq^n}{1-q}=1+\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}=1+\left[2-\left(\frac{1}{2}\right)^{n-1}\right]<3.$$

Consequently, for all n we get

$$\left(1+\frac{1}{n}\right)^n < 3.$$

From equality (2) it follows that

$$\left(1+\frac{1}{n}\right)^n \geqslant 2.$$

Thus, we get the inequality

$$2 \leqslant \left(1+\frac{1}{n}\right)^n < 3. \qquad (3)$$

This proves that the variable $\left(1+\frac{1}{n}\right)^n$ is bounded.

Thus, the variable $\left(1+\frac{1}{n}\right)^n$ is an increasing and bounded variable; therefore, by Theorem 7, Sec. 5, it has a limit. This limit is denoted by the letter e.

Definition. The limit of the variable $\left(1+\frac{1}{n}\right)^n$ as $n \to \infty$ is the *number e:*

$$e = \lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n. \text{*)}$$

By Theorem 6, Sec. 5, it follows from inequality (3) that the number e satisfies the inequality $2 \leqslant e \leqslant 3$. The theorem is thus proved.

The number e is an irrational number. Later on, a method will be shown that permits calculating e to any degree of accuracy. Its value to ten significant decimal places is

$$e = 2.7182818284\ldots$$

Theorem 2. *The function* $\left(1+\frac{1}{x}\right)^x$ *approaches the limit e as x approaches infinity,* $\lim\limits_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e.$

Proof. It has been shown that $\left(1+\frac{1}{n}\right)^n \to e$ as $n \to \infty$, if n takes on positive integral values. Now let x approach infinity while taking on fractional and negative values.

---

*) It may be shown that $\left(1+\frac{1}{n}\right)^n \to e$ as $n \to +\infty$ even if n is not an increasing variable quantity.

1) Let $x \to +\infty$. Each of its values lies between two positive integral numbers,

$$n \leqslant x < n+1.$$

The following inequalities will be fulfilled:

$$\frac{1}{n} \geqslant \frac{1}{x} > \frac{1}{n+1},$$

$$1+\frac{1}{n} \geqslant 1+\frac{1}{x} > 1+\frac{1}{n+1},$$

$$\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{x}\right)^{x} > \left(1+\frac{1}{n+1}\right)^{n}.$$

If $x \to \infty$, it is obvious that $n \to \infty$. Let us find the limits of the variables between which the variable $\left(1+\frac{1}{x}\right)^{x}$ lies:

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) =$$
$$= \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\lim_{n\to+\infty}\left(1+\frac{1}{n}\right) = e\cdot 1 = e,$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^{n} = \lim_{n\to+\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}} =$$
$$= \frac{\lim\limits_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim\limits_{n\to+\infty}\left(1+\frac{1}{n+1}\right)} = \frac{e}{1} = e.$$

Hence, by Theorem 4, Sec. 5,

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x} = e. \tag{4}$$

2) Let $x \to -\infty$. We introduce a new variable $t = -(x+1)$ or $x = -(t+1)$. When $t \to +\infty$ then $x \to -\infty$. We can write

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{t\to+\infty}\left(1-\frac{1}{t+1}\right)^{-t-1} = \lim_{t\to+\infty}\left(\frac{t}{t+1}\right)^{-t-1} =$$
$$= \lim_{t\to+\infty}\left(\frac{t+1}{t}\right)^{t+1} = \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t+1} =$$
$$= \lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}\left(1+\frac{1}{t}\right) = e\cdot 1 = e.$$

The theorem is proved. The graph of the function $y=\left(1+\frac{1}{x}\right)^x$ is shown in Fig. 45.

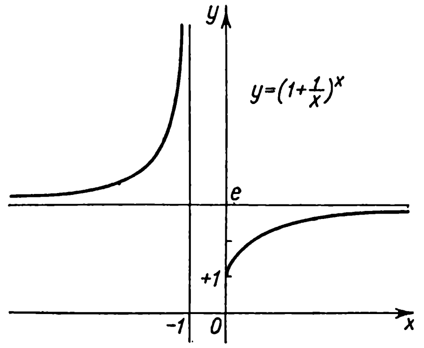


Fig. 45.

If in equality (4) we put $\frac{1}{x}=\alpha$, then as $x\to\infty$ we have $\alpha\to 0$ (but $\alpha\neq 0$) and we get

$$\lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e.$$

Examples:

1) $$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+5}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{5}=$$
$$=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{5}=e\cdot 1=e.$$

2) $$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{x}=$$
$$=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\cdot\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\cdot\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e\cdot e\cdot e=e^3.$$

3) $$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{2y}=e^2.$$

4) $$\lim_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{x+3}=\lim_{x\to\infty}\left(\frac{x-1+4}{x-1}\right)^{x+3}=\lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{x+3}=$$
$$=\lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{(x-1)+4}=\lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y+4}=$$
$$=\lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y}\cdot\lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{4}=e^4\cdot 1=e^4.$$

SEC. 8. NATURAL LOGARITHMS

In Sec. 8 of Chapter I we defined the logarithmic function $y=\log_a x$. The number a is called the base of the logarithms. If $a=10$, then y is the decimal (common) logarithm of the number x and is denoted $y=\log x$. In school courses of mathematics we have tables of common logarithms, which are called Briggs' logarithms after the English mathematician Briggs (1556-1630).

Logarithms to the base $e=2.71828\ldots$ are called *natural* or *Napierian logarithms* after one of the first inventors of logarithmic

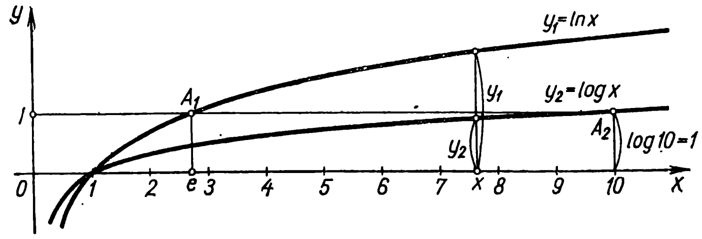


Fig. 46.

tables, the mathematician Napier (1550-1617).*) Therefore, if $e^y=x$, then y is called the natural logarithm of the number x. In writing we have $y=\ln x$ (after the initial letters of *logarithmus naturalis*) in place of $y=\log_e x$. Graphs of the function $y=\ln x$ and $y=\log x$ are plotted in Fig. 46.

Let us now establish a relationship between decimal and natural logarithms of one and the same number x.

Let $y=\log x$ or $x=10^y$. We take logarithms of the left and right sides of the latter equality to the base e and get $\ln x=y\ln 10$. We determine $y=\frac{1}{\ln 10}\ln x$, or, substituting the value of y, we have $\log x=\frac{1}{\ln 10}\ln x$.

Thus, if we know the natural logarithm of a number x, the common (decimal) logarithm of this number is found by multiplying by the factor $M=\frac{1}{\ln 10}\approx 0.434294$, which factor is independent of x. The number M is the *modulus* of common logarithms with respect to natural logarithms:

$$\log x = M\ln x.$$

*) The first logarithmic tables were constructed by the Swiss mathematician Bürgi (1552-1632) to a base close to the number e.

If in this identity we put $x=e$, we obtain an expression of the number M in terms of common logarithms:

$$\log e = M \, (\ln e = 1).$$

Natural logarithms are expressed in terms of common logarithms as follows:

$$\ln x = \frac{1}{M} \log x$$

where

$$\frac{1}{M} = 2.302585.$$

SEC. 9. CONTINUITY OF FUNCTIONS

Let the function $y=f(x)$ be defined for some value x_0 and in some neighbourhood with centre at x_0. Let $y_0=f(x_0)$.

If x receives some positive or negative (it is immaterial which) increment Δx and assumes the value $x=x_0+\Delta x$, then the function y too will receive an increment Δy. The new increased value of the function will be $y_0+\Delta y=f(x_0+\Delta x)$ (Fig. 47). The increment of the function Δy will be expressed by the formula

$$\Delta y = f(x_0+\Delta x) - f(x_0).$$

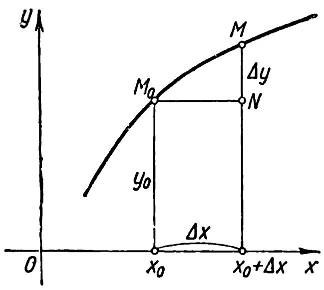


Fig. 47.

Definition 1. The function $y=f(x)$ is called *continuous for the value* $x=x_0$ (or *at the point* x_0) if it is defined in some neighbourhood of the point x_0 (obviously, at the point x_0 as well) and if

$$\lim_{\Delta x \to 0} \Delta y = 0 \tag{1}$$

or, which is the same thing,

$$\lim_{\Delta x \to 0} [f(x_0+\Delta x) - f(x_0)] = 0. \tag{2}$$

In descriptive geometrical terms, the continuity of a function at a given point signifies that the difference of the ordinates of the graph of the function $y=f(x)$ at the points $x_0+\Delta x$ and x_0 will, in absolute magnitude, be arbitrarily small, provided $|\Delta x|$ is sufficiently small.

Example 1. We shall prove that the function $y=x^2$ is continuous at an arbitrary point x_0. Indeed,

$$y_0=x_0^2,\ y_0+\Delta y=(x_0+\Delta x)^2,\ \Delta y=(x_0+\Delta x)^2-x_0^2=2x_0\,\Delta x+\Delta x^2,$$

$$\lim_{\Delta x\to 0}\Delta y=\lim_{\Delta x\to 0}(2x_0\,\Delta x+\Delta x^2)=2x\lim_{\Delta x\to 0}\Delta x+\lim_{\Delta x\to 0}\Delta x\cdot\lim_{\Delta x\to 0}\Delta x=0$$

for any way that Δx may approach zero (Figs. 48, *a* and 48, *b*).

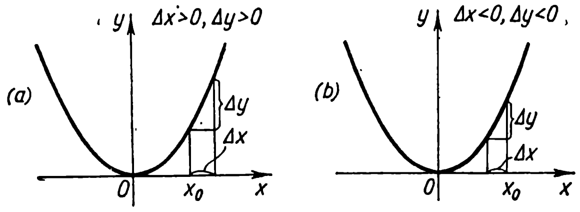


Fig. 48.

Example 2. We shall prove that the function $y=\sin x$ is continuous at an arbitrary point x_0. Indeed,

$$y_0=\sin x_0,\quad y_0+\Delta y=\sin(x_0+\Delta x),$$

$$\Delta y=\sin(x_0+\Delta x)-\sin x_0=2\sin\frac{\Delta x}{2}\cdot\cos\left(x_0+\frac{\Delta x}{2}\right).$$

It was shown that $\lim\limits_{\Delta x\to 0}\sin\frac{\Delta x}{2}=0$ (Example 7, Sec. 5). The function $\cos\left(x+\frac{\Delta x}{2}\right)$ is bounded. Therefore, $\lim\limits_{\Delta x\to 0}\Delta y=0$.

In similar fashion, it is possible to prove the following theorem by considering each basic elementary function and each elementary function.

Theorem. *Every elementary function is continuous at each point at which it is defined.*

The condition of continuity (2) may be written thus:

$$\lim_{\Delta x\to 0}f(x_0+\Delta x)=f(x_0)$$

or

$$\lim_{x\to x_0}f(x)=f(x_0),$$

but

$$x_0=\lim_{x\to x_0}x.$$

Consequently,

$$\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x). \tag{3}$$

In other words, in order to find the limit of a continuous function as $x \to x_0$ it is sufficient to substitute into the expression of the function the value of the argument, x_0, in place of the argument x.

Example 3. The function $y = x^2$ is continuous at every point x_0 and therefore

$$\lim_{x \to x_0} x^2 = x_0^2,$$

$$\lim_{x \to 3} x^2 = 3^2 = 9.$$

Example 4. The function $y = \sin x$ is continuous at every point and therefore

$$\lim_{x \to \frac{\pi}{4}} \sin x = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Example 5. The function $y = e^x$ is continuous at every point and therefore $\lim_{x \to a} e^x = e^a$.

Example 6. $\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \ln[(1+x)^{\frac{1}{x}}]$. Since $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$ and the function $\ln z$ is continuous for $z > 0$, and, consequently, for $z = e$,

$$\lim_{x \to 0} \ln[(1+x)^{\frac{1}{x}}] = \ln[\lim_{x \to 0} (1+x)^{\frac{1}{x}}] = \ln e = 1.$$

Definition 2. If the function $y = f(x)$ is continuous at each point of a certain interval (a, b), where $a < b$, then it is said that the function *is continuous in this interval*.

If the function is also defined for $x = a$ and $\lim_{x \to a+0} f(x) = f(a)$, it is said that $f(x)$ at the point $x = a$ *is continuous on the right*. If $\lim_{x \to b-0} f(x) = f(b)$, it is said that the function $f(x)$ *is continuous on the left* of the point $x = b$.

If the function $f(x)$ is continuous at each point of the interval (a, b) and is continuous at the end points of the interval, on the right and left, respectively, it is said that the function $f(x)$ *is continuous over the closed interval* $[a, b]$.

Example 7. The function $y = x^2$ is continuous in any closed interval $[a, b]$. This follows from Example 1.

If at some point $x=x_0$, at least one of the conditions of continuity is not fulfilled for the function $y=f(x)$, that is, if for $x=x_0$ the function is not defined or there does not exist a limit $\lim\limits_{x\to x_0} f(x)$ or $\lim\limits_{x\to x_0} f(x)\neq f(x_0)$ in the arbitrary approach of $x\longrightarrow x_0$, although the expressions on the right and left exist, then at $x=x_0$ the function $y=f(x)$ is *discontinuous*. In this case, the point $x=x_0$ is called the *point of discontinuity* of the function.

Example 8. The function $y=\frac{1}{x}$ is discontinuous at $x=0$. Indeed, the function is not defined at $x=0$.

$$\lim_{x\to 0+0}\frac{1}{x}=+\infty;\quad \lim_{x\to 0-0}\frac{1}{x}=-\infty$$

(see Fig. 35). It is easy to show that this function is continuous for any value $x\neq 0$.

Example 9. The function $y=2^{\frac{1}{x}}$ is discontinuous at $x=0$. Indeed, $\lim\limits_{x\to 0+0} 2^{\frac{1}{x}}=\infty$, $\lim\limits_{x\to 0-0} 2^{\frac{1}{x}}=0$. The function is not defined at $x=0$ (Fig. 49).

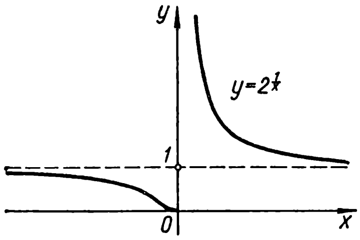


Fig. 49.

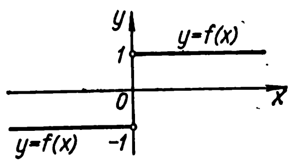


Fig. 50.

Example 10. Consider the function $f(x)=\frac{x}{|x|}$. At $x<0$, $\frac{x}{|x|}=-1$; at $x>0$, $\frac{x}{|x|}=1$. Hence,

$$\lim_{x\to 0-0} f(x)=\lim_{x\to 0-0}\frac{x}{|x|}=-1;$$

$$\lim_{x\to 0+0} f(x)=\lim_{x\to 0+0}\frac{1}{|x|}=1;$$

the function is not defined at $x=0$. We have thus established the fact that the function $f(x)=\frac{x}{|x|}$ is discontinuous at $x=0$ (Fig. 50).

Example 11. The earlier examined function $y=\sin\frac{1}{x}$ is discontinuous at $x=0$.

Definition 3. If the function $f(x)$ is such that there exist finite limits $\lim\limits_{x\to x_0+0} f(x)=f(x_0+0)$ and $\lim\limits_{x\to x_0-0} f(x)=f(x_0-0)$, but either $\lim\limits_{x\to x_0+0} f(x)\neq \lim\limits_{x\to x_0-0} f(x)$ or the value of the function $f(x)$ at $x=x_0$ is not defined, then $x=x_0$ is called a *point of discontinuity of the first kind*. (For example, for the function considered in Example 10, the point $x=0$ is a point of discontinuity of the first kind).

SEC. 10. CERTAIN PROPERTIES OF CONTINUOUS FUNCTIONS

In this section we shall consider a number of properties of functions that are continuous on an interval. These properties will be stated in the form of theorems given without proof.

Theorem 1. *If a function $y=f(x)$ is continuous on some interval $[a, b]$ $(a\leqslant x\leqslant b)$, there will be, on this interval at least one point $x=x_1$ such that the value of the function at this point will satisfy the relation*

$$f(x_1)\geqslant f(x),$$

where x is any other point of the interval, and there will be at least one point x_2 such that the value of the function at this point will satisfy the relation

$$f(x_2)\leqslant f(x).$$

We shall call the value of the function $f(x_1)$ the *greatest value* of the function $y=f(x)$ on the interval $[a, b]$, and the value of the function $f(x_2)$ the *smallest (least) value* of the function on the interval $[a, b]$.

This theorem is briefly stated as follows:

A function continuous on the interval $a\leqslant x\leqslant b$ attains on this interval (at least once) a greatest value M and a smallest value m.

The meaning of this theorem is clearly illustrated in Fig. 51.

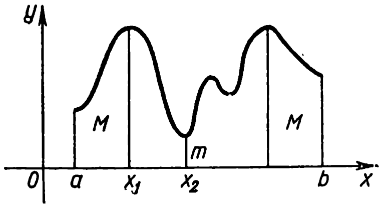


Fig. 51.

Note. The assertion that there exists a greatest value of the function may prove incorrect if one considers the values of the function in the interval $a<x<b$. For instance, if we consider the function $y=x$ in the interval $0<x<1$, there will be no

greatest and no least (smallest) values among them. Indeed, there is no least value or greatest value of x in the interval. (There is no extreme left point, since no matter what point x^* we take there will be a point left of it, for instance, the point $\frac{x^*}{2}$; likewise, there is no extreme right point; consequently, there is no least and no greatest value of the function $y = x$.)

Theorem 2. *Let the function $y = f(x)$ be continuous on the interval $[a, b]$ and at the end point of this interval let it take on values of different sign; then between the points a and b there will be at least one point $x = c$, at which the function becomes zero:*

$$f(c) = 0, \quad a < c < b.$$

This theorem has a simple geometrical meaning. The graph of a continuous function $y = f(x)$ joining the points $M_1[a, f(a)]$ and $M_2[b, f(b)]$, where $f(a) < 0$ and $f(b) > 0$ or $f(a) > 0$ and $f(b) < 0$, cuts the x-axis at least at one point (Fig. 52).

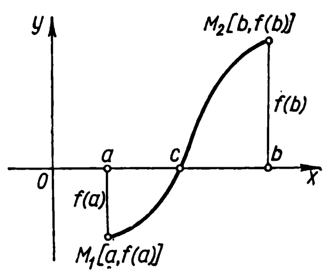


Fig. 52.

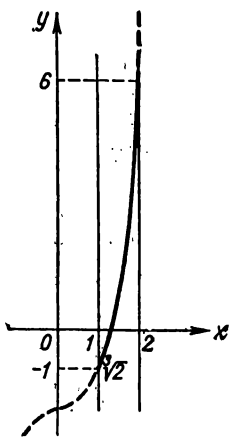


Fig. 53.

Example. Given the function $y = x^3 - 2$. $y_{x=1} = -1$, $y_{x=2} = 6$. It is continuous in the interval $[1, 2]$. Hence, in this interval there is a point where $y = x^3 - 2$ becomes zero. Indeed, $y = 0$ when $x = \sqrt[3]{2}$ (Fig. 53).

Theorem 3. *Let the function $y = f(x)$ be defined and continuous in the interval $[a, b]$. If at the end points of this interval the function takes on unequal values $f(a) = A$, $f(b) = B$, then no matter what the number μ between numbers A and B, there will be a point $x = c$ between a and b such that $f(c) = \mu$.*

The meaning of this theorem is clearly illustrated in Fig. 54. In the given case, any straight line $y = \mu$ cuts the graph of the function $y = f(x)$.

Note. It will be noted that Theorem 2 is a particular case of this theorem, for if A and B have different signs, then for μ one can take 0, and then $\mu = 0$ will lie between the numbers A and B.

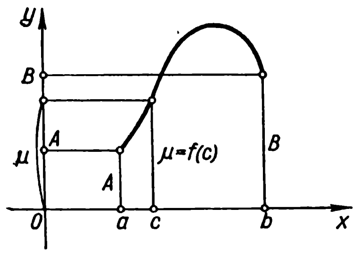


Fig. 54.

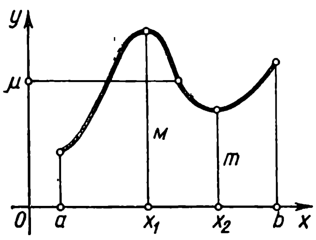


Fig. 55.

Corollary of Theorem 3. *If a function $y = f(x)$ is continuous in some interval and takes on a greatest value and a least value, then in this interval it takes on, at least once, any value lying between the greatest and least values.*

Indeed, let $f(x_1) = M$, $f(x_2) = m$. Consider the interval $[x_1, x_2]$. By Theorem 3, in this interval the function $y = f(x)$ takes on any value μ lying between M and m. But the interval $[x_1, x_2]$ lies inside the interval under consideration in which the function $f(x)$ is defined (Fig. 55).

SEC. 11. COMPARING INFINITESIMALS

Let several infinitesimal quantities

$$\alpha, \beta, \gamma, \ldots$$

be at the same time functions of one and the same argument x and let them approach zero as x approaches some limit a or infinity. We shall describe the approach of these variables to zero when we consider their ratios. *)

We shall, in future, make use of the following definitions.

Definition 1. If the ratio $\frac{\beta}{\alpha}$ has a finite nonzero limit, that is, if $\lim \frac{\beta}{\alpha} = A \neq 0$, and therefore, $\lim \frac{\alpha}{\beta} = \frac{1}{A} \neq 0$, the infinitesimals β and α are called *infinitesimals of the same order*.

*) We assume that the infinitesimal in the denominator does not vanish in some neighbourhood of the point a.

Example 1. Let $\alpha=x$, $\beta=\sin 2x$, where $x\to 0$. The infinitesimals α and β are of the same order because

$$\lim_{x\to 0}\frac{\beta}{\alpha}=\lim_{x\to 0}\frac{\sin 2x}{x}=2.$$

Example 2. When $x\to 0$, the infinitesimals x, $\sin 3x$, $\tan 2x$, $7\ln(1+x)$ are infinitesimals of the same order. The proof is similar to that given in Example 1.

Definition 2. If the ratio of two infinitesimals $\frac{\beta}{\alpha}$ approaches zero, that is, if $\lim\frac{\beta}{\alpha}=0$ $\left(\text{and } \lim\frac{\alpha}{\beta}=\infty\right)$, then the infinitesimal β is called *an infinitesimal of higher order than* α, and the infinitesimal α is called, *an infinitesimal of lower order than* β.

Example 3. Let $\alpha=x$, $\beta=x^n$, $n>1$, $x\to 0$. The infinitesimal β is an infinitesimal of higher order than the infinitesimal α, since

$$\lim_{x\to 0}\frac{x^n}{x}=\lim_{x\to 0}x^{n-1}=0.$$

Here, the infinitesimal α is an infinitesimal of lower order than β.

Definition 3. An infinitesimal β is called *an infinitesimal of the kth order relative to an infinitesimal* α, if β and α^k are infinitesimals of the same order, that is, if $\lim\frac{\beta}{\alpha^k}=A\neq 0$.

Example 4. If $\alpha=x$, $\beta=x^3$, then as $x\to 0$ the infinitesimal β is an infinitesimal of the third order relative to the infinitesimal α since

$$\lim_{x\to 0}\frac{\beta}{\alpha^3}=\lim_{x\to 0}\frac{x^3}{(x)^3}=1.$$

Definition 4. If the ratio of two infinitesimals $\frac{\beta}{\alpha}$ approaches unity, that is, if $\lim\frac{\beta}{\alpha}=1$, the infinitesimals β and α are called *equivalent infinitesimals* and we write $\alpha\sim\beta$.

Example 5. Let $\alpha=x$ and $\beta=\sin x$, where $x\to 0$. The infinitesimals α and β are equivalent, since

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Example 6. Let $\alpha=x$, $\beta=\ln(1+x)$, where $x\to 0$. The infinitesimals α and β are equivalent, since

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$$

(see Example 6, Sec. 9).

Theorem 1. *If α and β are equivalent infinitesimals, their difference $\alpha-\beta$ is an infinitesimal of higher order than α and than β.*

Proof. Indeed,

$$\lim \frac{\alpha-\beta}{\alpha}=\lim\left(1-\frac{\beta}{\alpha}\right)=1-\lim\frac{\beta}{\alpha}=1-1=0.$$

Theorem 2. *If the difference of two infinitesimals $\alpha-\beta$ is an infinitesimal of higher order than α and than β, then α and β are equivalent infinitesimals.*

Proof. Let $\lim\frac{\alpha-\beta}{\alpha}=0$, then $\lim\left(1-\frac{\beta}{\alpha}\right)=0$, or $1-\lim\frac{\beta}{\alpha}=0$, or $1=\lim\frac{\beta}{\alpha}$, i. e., $\alpha\approx\beta$. If $\lim\frac{\alpha-\beta}{\beta}=0$, then $\lim\left(\frac{\alpha}{\beta}-1\right)=0$, $\lim\frac{\alpha}{\beta}=1$, that is, $\alpha\approx\beta$.

Example 7. Let $\alpha=x$, $\beta=x+x^3$, where $x\longrightarrow 0$.

The infinitesimals α and β are equivalent, since their difference $\beta-\alpha=x^3$ is an infinitesimal of higher order than α and than β. Indeed,

$$\lim_{x\to 0}\frac{\beta-\alpha}{\alpha}=\lim_{x\to 0}\frac{x^3}{x}=\lim_{x\to 0}x^2=0,$$

$$\lim\frac{\alpha-\beta}{\beta}=\lim_{x\to 0}\frac{x^3}{x+x^3}=\lim_{x\to 0}\frac{x^2}{1+x^2}=0.$$

Example 8. As $x\longrightarrow\infty$ the infinitesimals $\alpha=\frac{x+1}{x^2}$ and $\beta=\frac{1}{x}$ are equivalent infinitesimals, since their difference $\alpha-\beta=\frac{x+1}{x^2}-\frac{1}{x}=\frac{1}{x^2}$ is an infinitesimal of higher order than α and than β. The limit of the ratio of α and β is unity:

$$\lim\frac{\beta}{\alpha}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{x+1}{x^2}}=\lim_{x\to\infty}\frac{x+1}{x}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)=1.$$

Note. If the ratio of two infinitesimals $\frac{\beta}{\alpha}$ has no limit and does not approach infinity, then β and α are not comparable in the above sense.

Example 9. Let $\alpha=x$, $\beta=x\sin\frac{1}{x}$, where $x\longrightarrow 0$. The infinitesimals α and β cannot be compared because their ratio $\frac{\beta}{\alpha}=\sin\frac{1}{x}$ as $x\longrightarrow 0$ does not approach either a finite limit or infinity (see Example 4, Sec. 3).

Exercises on Chapter II

Find the indicated limits:

1. $\lim\limits_{x\to 1}\frac{x^2+2x+5}{x^2+1}$. *Ans.* 4. 2. $\lim\limits_{x\to\frac{\pi}{2}}[2\sin x-\cos x+\cot x]$. *Ans.* 2.

3. $\lim\limits_{x\to 2}\frac{x-2}{\sqrt{2+x}}$. *Ans.* 0. 4. $\lim\limits_{x\to\infty}\left(2-\frac{1}{x}+\frac{4}{x^2}\right)$. *Ans.* 2. 5. $\lim\limits_{x\to\infty}\frac{4x^3-2x^2+1}{3x^3-5}$. *Ans.* $\frac{4}{3}$. 6. $\lim\limits_{x\to\infty}\frac{x+1}{x}$. *Ans.* 1. 7. $\lim\limits_{n\to\infty}\frac{1+2+\ldots+n}{n^2}$. *Ans.* $\frac{1}{2}$.

8. $\lim\limits_{n\to\infty}\frac{1^2+2^2+3^2+\ldots+n^2}{n^3}$. *Ans.* $\frac{1}{3}$.

Hint. Write the formula $(k+1)^3-k^3=3k^2+3k+1$ for $k=0, 1, 2, \ldots, n$.

$$1^3=1;$$
$$2^3-1^3=3\cdot 1^2+3\cdot 1+1;$$
$$3^3-2^3=3\cdot 2^2+3\cdot 2+1;$$
$$\cdots\cdots\cdots$$
$$(n+1)^3-n^3=3n^2+3n+1.$$

Adding the left and right sides, we get

$$(n+1)^3=3(1^2+2^2+\ldots+n^2)+3(1+2+\ldots+n)+(n+1),$$
$$(n+1)^3=3(1^2+2^2+\ldots+n^2)-3\frac{n(n+1)}{2}+(n+1),$$

whence

$$1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

9. $\lim\limits_{x\to\infty}\frac{x^2+x-1}{2x+5}$. *Ans.* ∞. 10. $\lim\limits_{x\to\infty}\frac{3x^2-2x-1}{x^3+4}$. *Ans.* 0.

11. $\lim\limits_{x\to 0}\frac{4x^3-2x^2+x}{3x^2+2x}$. *Ans.* $\frac{1}{2}$. 12. $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$. *Ans.* 4. 13. $\lim\limits_{x\to 1}\frac{x^3-1}{x-1}$. *Ans.* 3.

14. $\lim\limits_{x\to 2}\frac{x^2-5x+6}{x^2-12x+20}$. *Ans.* $\frac{1}{8}$. 15. $\lim\limits_{x\to 2}\frac{x^2+3x-10}{3x^2-5x-2}$. *Ans.* 1.

16. $\lim\limits_{y\to -2}\frac{y^3+3y^2+2y}{y^2-y-6}$. *Ans.* $-\frac{2}{5}$. 17. $\lim\limits_{u\to -2}\frac{u^3+4u^2+4u}{(u+2)(u-3)}$. *Ans.* 0.

18. $\lim\limits_{h\to 0}\frac{(x+h)^3-x^3}{h}$. *Ans.* $3x^2$. 19. $\lim\limits_{x\to 1}\left[\frac{1}{1-x}-\frac{3}{1-x^3}\right]$. *Ans.* -1.

20. $\lim\limits_{x\to 1}\frac{x^n-1}{x-1}$. *Ans.* n (n is a positive integer). 21. $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x}$. *Ans.* $\frac{1}{2}$.

22. $\lim\limits_{x\to 4}\dfrac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}}$. *Ans.* $\dfrac{2\sqrt{2}}{3}$. 23. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+p^2}-p}{\sqrt{x^2+q^2}-q}$. *Ans.* $\dfrac{q}{p}$.

24. $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1}$. *Ans.* $\dfrac{2}{3}$. 25. $\lim\limits_{x\to a}\dfrac{\sqrt[m]{x}-\sqrt[m]{a}}{x-a}$. *Ans.* $\dfrac{\sqrt[m]{a}}{ma}$.

26. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x+x^2}-1}{x}$. *Ans.* $\dfrac{1}{2}$. 27. $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2-3}}{\sqrt[3]{x^3+1}}$. *Ans.* 1.

28. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+1}}{x+1}$. *Ans.* 1 as $x\to+\infty$, -1 as $x\to-\infty$. 29. $\lim\limits_{x\to\infty}(\sqrt{x^2+1}-\sqrt{x^2-1})$. *Ans.* 0. 30. $\lim\limits_{x\to\infty}x(\sqrt{x^2+1}-x)$. *Ans.* $\dfrac{1}{2}$ as $x\to+\infty$, $-\infty$ as $x\to-\infty$. 31. $\lim\limits_{x\to 0}\dfrac{\sin x}{\tan x}$. *Ans* 1. 32. $\lim\limits_{x\to 0}\dfrac{\sin 4x}{x}$. *Ans.* 4. 33. $\lim\limits_{x\to 0}\dfrac{\sin^2\frac{x}{3}}{x^2}$. *Ans.* $\dfrac{1}{9}$. 34. $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1-\cos x}}$. *Ans.* $\dfrac{2}{\sqrt{2}}$. 35. $\lim\limits_{x\to 0}x\cot x$. *Ans.* 1.

36. $\lim\limits_{v\to\frac{\pi}{3}}\dfrac{1-2\cos v}{\sin\left(v-\frac{\pi}{3}\right)}$. *Ans.* $\sqrt{3}$. 37. $\lim\limits_{z\to 1}(1-z)\tan\dfrac{\pi z}{2}$. *Ans.* $\dfrac{2}{\pi}$.

38. $\lim\limits_{x\to 0}\dfrac{2\arcsin x}{3x}$. *Ans.* $\dfrac{2}{3}$. 39. $\lim\limits_{x\to 0}\dfrac{\sin(a+x)-\sin(a-x)}{x}$. *Ans.* $2\cos a$.

40. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$. *Ans.* $\dfrac{1}{2}$. 41. $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^x$. *Ans.* e^2.

42. $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^x$. *Ans.* $\dfrac{1}{e}$. 43. $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^x$. *Ans.* $\dfrac{1}{e}$.

44. $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n+5}$. *Ans.* e. 45. $\lim\limits_{n\to\infty}\{n[\ln(n+1)-\ln n]\}$. *Ans.* 1.

46. $\lim\limits_{x\to\frac{\pi}{2}}(1+\cos x)^{3\sec x}$. *Ans.* e^3. 47. $\lim\limits_{x\to 0}\dfrac{\ln(1+\alpha x)}{x}$. *Ans.* α.

48. $\lim\limits_{x\to\infty}\left(\dfrac{2x+3}{2x+1}\right)^{x+1}$. *Ans.* e. 49. $\lim\limits_{x\to 0}(1+3\tan^2 x)^{\cot^2 x}$. *Ans.* e^3.

50. $\lim\limits_{m\to\infty}\left(\cos\dfrac{x}{m}\right)^m$. *Ans.* 1. 51. $\lim\limits_{\alpha\to\infty}\dfrac{\ln(1+e^\alpha)}{\alpha}$. *Ans.* 1 as $\alpha\to+\infty$, 0 as $\alpha\to-\infty$. 52. $\lim\limits_{x\to 0}\dfrac{\sin\alpha x}{\sin\beta x}$. *Ans.* $\dfrac{\alpha}{\beta}$. 53. $\lim\limits_{x\to\infty}\dfrac{a^x-1}{x}$ $(a>1)$. *Ans.* $+\infty$

as $x \to +\infty$, 0 as $x \to -\infty$. 54. $\lim\limits_{n \to \infty} n\left[a^{\frac{1}{n}} - 1\right]$. *Ans.* $\ln a$. 55. $\lim\limits_{x \to 0} \frac{e^{\alpha x} - e^{\beta x}}{x}$. *Ans.* $\alpha - \beta$. 56. $\lim\limits_{x \to 0} \frac{e^{\alpha x} - e^{\beta x}}{\sin \alpha x - \sin \beta x}$ *Ans.* 1.

Determine the points of discontinuity of the functions:

57. $y = \frac{x-1}{x(x+1)(x^2-4)}$. *Ans.* Discontinuities of second kind for $x = -2; -1; 0; 2$. 58. $y = \tan \frac{1}{x}$. *Ans.* Discontinuities of second kind for $x = 0$ and $x = \pm \frac{2}{\pi}; \pm \frac{2}{3\pi}; \ldots; \pm \frac{2}{(2n+1)\pi}; \ldots$

59. Find the points of discontinuity of the functions $y = 1 + 2^{\frac{1}{x}}$ and construct the graph of this function. *Ans.* Discontinuity of second kind at $x = 0$ ($y \to +\infty$ as $x \to 0+0$, $y \to 1$ as $x \to -0-0$).

60. From among the following infinitesimals (as $x \to 0$); x^2, $\sqrt{x(1-x)}$, $\sin 3x$, $2x \cos x \sqrt[3]{\tan^2 x}$, xe^{2x}, select infinitesimals of the same order as x, and also of higher and lower order than x. *Ans.* Infinitesimals of the same order are $\sin 3x$ and xe^{2x}; infinitesimals of higher order, x^2 and $2x \cos x \sqrt[3]{\tan^2 x}$, infinitesimals of lower order, $\sqrt{x(1-x)}$.

61. Choose from among the same infinitesimals (as $x \to 0$) such that are equivalent to the infinitesimal x: $2 \sin x$, $\frac{1}{2} \tan 2x$, $x - 3x^2$, $\sqrt{2x^2 + x^3}$, $\ln(1+x)$, $x^3 + 3x^4$. *Ans.* $\frac{1}{2} \tan 2x$, $x - 3x^2$, $\ln(1+x)$.

62. Check to see that as $x \to 1$, the infinitesimals $1 - x$ and $1 - \sqrt[3]{x}$ are of the same order of smallness. Are they equivalent? *Ans.* $\lim\limits_{x \to 1} \frac{1-x}{1-\sqrt[3]{x}} = 3$; hence, these infinitesimals are of the same order, but they are not equivalent.

CHAPTER III

DERIVATIVE AND DIFFERENTIAL

SEC. 1. VELOCITY MOTION

Let us consider the rectilinear motion of some solid, say a stone, thrown vertically upwards, or the motion of a piston in the cylinder of an engine, etc. Idealising the situation and disregarding dimensions and shapes, we shall always represent such a body in the form of a moving point M. The distance s of the moving point reckoned from some initial position M_0 will depend on the time t; in other words, s will be a function of time t:

$$s = f(t). \tag{1}$$

At some instant of time*) t, let the moving point M be at a distance s from the initial position M_0, and at some later instant $t + \Delta t$ let the point be at M_1, a distance $s + \Delta s$ from the initial position (Fig. 56). Thus, during the interval of time Δt the distance s changed by the quantity Δs. In such cases, one says that during the time Δt the quantity s received an increment Δs.

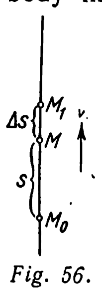


Fig. 56.

Let us consider the ratio $\frac{\Delta s}{\Delta t}$; it gives us the average velocity of motion of the point during the time Δt:

$$v_{av} = \frac{\Delta s}{\Delta t}. \tag{2}$$

The average velocity cannot in all cases give an exact picture of the rate of translation of the point M at time t. If, for example, the body moved very fast at the beginning of the interval Δt and very slow at the end, the average velocity obviously cannot reflect these peculiarities in the motion of the point and give us a correct idea of the true velocity of motion at time t. In order to express more precisely this true velocity in terms of the average velocity, one has to take a small interval of time Δt. The most complete description of the rate of motion of the point at time t is given by the limit which the average velocity approaches as $\Delta t \to 0$.

*) Here and henceforward we shall denote the specific value of a variable and the variable itself by the same letter.

This limit is called the **rate of motion at a given instant:**

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}. \quad (3)$$

Thus, the *rate (velocity) of motion at a given instant* is the limit of the ratio of increment of path Δs to increment of time Δt, as the time increment approaches zero.

Let us write equality (3) in full. Since

$$\Delta s = f(t+\Delta t) - f(t),$$

$$v = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}. \quad (3')$$

This is the velocity of variable motion. It is thus obvious that the notion of velocity of variable motion is intimately related to the concept of a limit. It is only with the aid of the limit concept that we can determine the velocity of variable motion.

From formula (3′) it follows that v is independent of the increment in time Δt, but depends on the value of t and the type of function $f(t)$.

Example. Find the velocity of uniformly accelerated motion at an arbitrary time t and at $t=2$ sec if the relation of the path traversed to the time is expressed by the formula

$$s = \frac{1}{2} g t^2.$$

Solution. At time t we have $s = \frac{1}{2} g t^2$; at time $t+\Delta t$ we get

$$s + \Delta s = \frac{1}{2} g (t+\Delta t)^2 = \frac{1}{2} g (t^2 + 2t\,\Delta t + \Delta t^2).$$

We find Δs:

$$\Delta s = \frac{1}{2} g (t^2 + 2t\,\Delta t + \Delta t^2) - \frac{1}{2} g t^2 = g t\,\Delta t + \frac{1}{2} g\,\Delta t^2.$$

We form the ratio $\frac{\Delta s}{\Delta t}$:

$$\frac{\Delta s}{\Delta t} = \frac{g t\,\Delta t + \frac{1}{2} g \Delta t^2}{\Delta t} = g t + \frac{1}{2} g\,\Delta t;$$

by definition we have

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \left(g t + \frac{1}{2} g \Delta t \right) = g t.$$

Thus, the velocity at an arbitrary time t is $v = gt$.
At $t=2$ we have $(\cdot)_{t=2} = g \cdot 2 = 9.8 \cdot 2 = 19.6$ m/sec.

SEC. 2. DEFINITION OF DERIVATIVE

Let there be a function

$$y=f(x) \tag{1}$$

defined in a certain interval. The function $y=f(x)$ has a definite value for each value of the argument x in this interval.

Let the argument x receive a certain increment Δx (it is immaterial whether it be positive or negative). Then the function y will receive a certain increment Δy. Thus, with the value of the argument x we will have $y=f(x)$, with the value of the argument $x+\Delta x$ we will have $y+\Delta y=f(x+\Delta x)$.

Let us find the increment of the function Δy:

$$\Delta y=f(x+\Delta x)-f(x). \tag{2}$$

Forming the ratio of the increment of the function to the increment of the argument, we get

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}. \tag{3}$$

We then find the limit of this ratio as $\Delta x \to 0$. If this limit exists, it is called the **derivative** of the given function $f(x)$ and is denoted $f'(x)$. Thus, by definition,

$$f'(x)=\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

or

$$f'(x)=\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}. \tag{4}$$

Consequently, the *derivative* of a given function $y=f(x)$ with respect to the argument x is the limit of the ratio of the increment of the function Δy to the increment of the argument Δx, when the latter approaches zero in arbitrary fashion.

It will be noted that in the general case, the derivative $f'(x)$ has a definite value for each value of x, which means that the derivative is also a **function** of x.

The designation $f'(x)$ is not the only one used for a derivative. Alternative symbols are

$$y', \ y'_x, \ \frac{dy}{dx}.$$

The specific value of the derivative for $x=a$ is denoted $f'(a)$ or $y'|_{x=a}$.

The operation of finding the derivative of a function $f(x)$ is called *differentiation* of the function.

Example 1. Given the function $y=x^2$; find its derivative y':
1) at an arbitrary point x,
2) at $x=3$.

Solution. 1) For the value of the argument x, we have $y=x^2$. When the value of the argument is $x+\Delta x$, we have $y+\Delta y=(x+\Delta x)^2$.

Find the increment of the function:

$$\Delta y=(x+\Delta x)^2-x^2=2x\Delta x+(\Delta x)^2.$$

Forming the ratio $\frac{\Delta y}{\Delta x}$, we have

$$\frac{\Delta y}{\Delta x}=\frac{2x\Delta x+(\Delta x)^2}{\Delta x}=2x+\Delta x.$$

Passing to the limit, we get the derivative of the given function:

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x.$$

Hence, the derivative of the function $y=x^2$ at an arbitrary point is $y'=2x$.

2) When $x=3$ we have

$$y'\big|_{x=3}=2\cdot 3=6.$$

Example 2. $y=\frac{1}{x}$; find y'.

Solution. Reasoning as before, we get

$$y=\frac{1}{x};\ y+\Delta y=\frac{1}{x+\Delta x};$$

$$\Delta y=\frac{1}{x+\Delta x}-\frac{1}{x}=\frac{x-x-\Delta x}{x(x+\Delta x)}=-\frac{\Delta x}{x(x+\Delta x)};$$

$$\frac{\Delta y}{\Delta x}=-\frac{1}{x(x+\Delta x)};$$

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\lim_{\Delta x\to 0}\left[-\frac{1}{x(x+\Delta x)}\right]=-\frac{1}{x^2}.$$

Note. In the preceding section it was established that if the dependence upon time t of the distance s of a moving point is expressed by the formula

$$s=f(t),$$

the velocity v at time t is expressed by the formula

$$v=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}=\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}.$$

Hence

$$v=s'_t=f'(t),$$

or, the velocity is equal to the derivative*) of the distance with respect to the time.

*) When we say "the derivative with respect to x" or "the derivative with respect to t" we mean that in computing the derivative we consider the variable x (or the time t, etc.) the argument (independent variable).

SEC. 3. GEOMETRIC MEANING OF THE DERIVATIVE

We approached the notion of a derivative by regarding the velocity of a moving body (point), that is to say, by proceeding from **mechanical** concepts. We shall now give a no less important **geometric** interpretation of the derivative. To do this we must first define a **line tangent** to a curve at a given point.

We take a curve with a fixed point M_0 on it. Taking a point M_1 on the curve we draw the secant M_0M_1 (Fig .57). If the point M_1 approaches the point M_0 without limit, the secant M_0M_1 will occupy various positions M_0M_1', M_0M_1'', and so on.

If, in the limitless approach of the point M_1 (along the curve) to the point M_0 **from either side**, the secant tends to occupy the position of a definite straight line M_0T, this line is called the **tangent** to the curve at the point M_0 (the concept "tends to occupy" will be explained later on).

Let us consider the function $f(x)$ and the corresponding curve

$$y=f(x)$$

in a rectangular coordinate system (Fig. 58). At a certain value of x the function has the value $y=f(x)$. Corresponding to these values of x and y on the curve we have the point $M_0(x, y)$. Let us increase

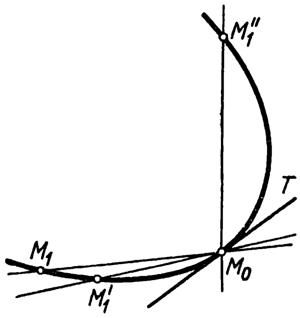


Fig. 57.

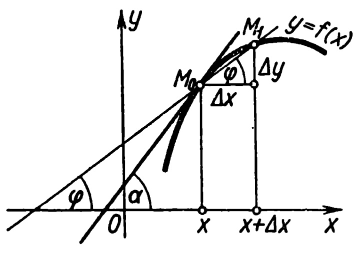


Fig. 58.

the argument x by Δx. Corresponding to the new value of the argument, $x+\Delta x$, we have an increased value of the function, $y+\Delta y=f(x+\Delta x)$. Another corresponding point on the curve will be $M_1(x+\Delta x, y+\Delta y)$. Draw the secant M_0M_1 and denote by φ the angle formed by the secant and the positive direction of the x-axis. Form the ratio $\frac{\Delta y}{\Delta x}$. From Fig. 58 it follows immediately that

$$\frac{\Delta y}{\Delta x}=\tan\varphi. \tag{1}$$

Now if Δx approaches zero, the point M_1 will move along the curve always approaching M_0. The secant M_0M_1 will turn about M_0 and the angle φ will change in Δx. If as $\Delta x \to 0$ the angle φ approaches a certain limit α, the straight line passing through M_0 and forming an angle α with the positive direction of the abscissa axis will be the sought-for line tangent. It is easy to find its slope:

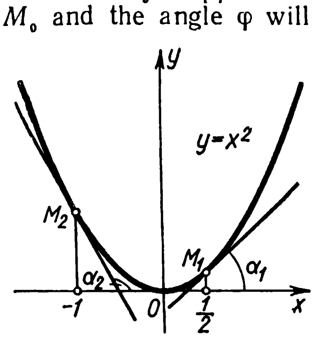


Fig. 59.

$$\tan \alpha = \lim_{\Delta x \to 0} \tan \varphi = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x).$$

Hence,

$$f'(x) = \tan \alpha, \tag{2}$$

which means that *the values of the derivative $f'(x)$, for a given value of the argument x, is equal to the tangent of the angle formed with the positive direction of the x-axis by the line tangent to the graph of the function $f(x)$ at the corresponding point $M_0(x, y)$.*

Example. Find the tangents of the angles of inclination of the line tangent to the curve $y = x^2$ at the points $M_1\left(\frac{1}{2}, \frac{1}{4}\right)$; $M_2(-1, 1)$ (Fig. 59).

Solution. On the basis of Example 1, Sec. 2, we have $y' = 2x$; hence,

$$\tan \alpha_1 = y'\Big|_{x=\frac{1}{2}} = 1; \quad \tan \alpha_2 = y'\Big|_{x=-1} = -2.$$

SEC. 4. DIFFERENTIABILITY OF FUNCTIONS

Definition. If the function

$$y = f(x) \tag{1}$$

has a derivative at the point $x = x_0$, that is, if there exists

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}, \tag{2}$$

we say that for the given value $x = x_0$ the function is *differentiable* or (which is the same thing) has a derivative.

If a function is differentiable at every point of some interval $[a, b]$ or (a, b), we say that it is *differentiable over the interval.*

Theorem. *If a function $y = f(x)$ is differentiable at some point $x = x_0$, it is continuous at this point.*

Indeed, if

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0),$$

then

$$\frac{\Delta y}{\Delta x} = f'(x_0) + \gamma,$$

where γ is a quantity that approaches zero as $\Delta x \to 0$. But then

$$\Delta y = f'(x_0)\,\Delta x + \gamma \Delta x;$$

whence it follows that $\Delta y \to 0$ as $\Delta x \to 0$; and this means that the function $f(x)$ is continuous at the point x_0 (see Sec. 9, Ch. II).

In other words, **a function cannot have a derivative at points of discontinuity.** The converse is not true; from the fact that at some point $x = x_0$ the function $y = f(x)$ is continuous, it does not yet follow that it is differentiable at this point: the function $f(x)$ may not have a derivative at the point x_0. To convince ourselves of this, let us examine several cases.

Example 1. A function $f(x)$ is defined in an interval $[0, 2]$ as follows (see Fig. 60):

$$f(x) = x \quad \text{when } 0 \leqslant x \leqslant 1,$$
$$f(x) = 2x - 1 \quad \text{when } 1 < x \leqslant 2.$$

At $x = 1$ this function has no derivative, although it is continuous at this point.

Indeed, when $\Delta x > 0$ we have

$$\lim_{\Delta x \to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[2(1+\Delta x) - 1] - [2 \cdot 1 - 1]}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = 2.$$

when $\Delta x < 0$ we get

$$\lim_{\Delta x \to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[1+\Delta x] - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

Thus, this limit depends on the sign of Δx, and this means that the function has no derivative *) at the point $x = 1$. Geometrically, this is in accord with the fact that at the point $x = 1$ the given "curve" does not have a definite line tangent.

Now the continuity of the function at the point $x = 1$ follows from the fact that

$$\Delta y = \Delta x \quad \text{when } \Delta x < 0,$$
$$\Delta y = 2\Delta x \quad \text{when } \Delta x > 0,$$

and, therefore, in both cases $\Delta y \to 0$ as $\Delta x \to 0$.

---

*) The definition of a derivative requires that the ratio $\frac{\Delta y}{\Delta x}$ should (as $\Delta x \to 0$) approach one and the same limit regardless of *the way* in which Δx approaches zero.

Example 2. A function $y=\sqrt[3]{x}$, the graph of which is shown in Fig. 61, is defined and continuous for all values of the independent variable.

Let us try to find out whether this function has a derivative at $x=0$; to do this, we find the values of the function at $x=0$ and at $x=0+\Delta x$: at $x=0$ we have $y=0$, at $x=0+\Delta x$ we have $y+\Delta y=\sqrt[3]{(\Delta x)}$.

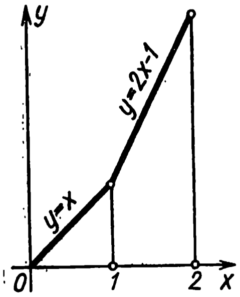


Fig. 60.

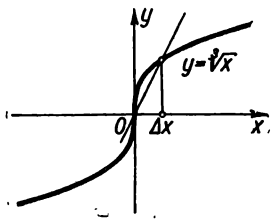


Fig. 61.

Therefore,

$$\Delta y=\sqrt[3]{(\Delta x)}.$$

Find the limit of the ratio of the increment of the function to the increment of the argument:

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt[3]{(\Delta x)}}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\sqrt[3]{\Delta x^2}}=+\infty.$$

Thus, the ratio of the increment of the function to the increment of the argument at the point $x=0$ approaches infinity as $\Delta x\to 0$ (hence there is no limit). Consequently, this function is not differentiable at the point $x=0$. The line tangent to the curve at this point forms, with the x-axis, an angle $\frac{\pi}{2}$, which means that it coincides with the y-axis.

SEC. 5. FINDING THE DERIVATIVES OF ELEMENTARY FUNCTIONS. THE DERIVATIVE OF THE FUNCTION $y=x^n$, WHERE n IS POSITIVE AND INTEGRAL

To find the derivative of a given function $y=f(x)$, it is necessary to carry out the following operations (on the basis of the general definition of a derivative):

1) increase the argument x by Δx, calculate the increased value of the function:

$$y+\Delta y=f(x+\Delta x);$$

2) find the corresponding increment of the function:

$$\Delta y = f(x+\Delta x) - f(x);$$

3) form the ratio of the increment of the function to the increment of the argument:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x};$$

4) find the limit of this ratio as $\Delta x \to 0$:

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Here and in the following sections, we shall apply this general method for evaluating the derivatives of certain elementary functions.

Theorem. *The derivative of the function $y = x^n$, where n is a positive integer, is equal to nx^{n-1}; that is,*

$$\textit{if } y = x^n, \textit{ then } y' = nx^{n-1}. \tag{I}$$

Proof. We have the function

$$y = x^n.$$

1) If x receives an increment Δx, then

$$y + \Delta y = (x+\Delta x)^n.$$

2) Applying Newton's binomial formula, we find

$$\Delta y = (x+\Delta x)^n - x^n = x^n + \frac{n}{1} x^{n-1}\Delta x + \\ + \frac{n(n-1)}{1\cdot 2} x^{n-2}(\Delta x)^2 + \ldots + (\Delta x)^n - x^n$$

or

$$\Delta y = nx^{n-1}\Delta x + \frac{n(n-1)}{1\cdot 2} x^{n-2}(\Delta x)^2 + \ldots + (\Delta x)^n.$$

3) We find the ratio

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{1\cdot 2} x^{n-2}\Delta x + \ldots + (\Delta x)^{n-1}.$$

4) Then we find the limit of this ratio

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \\ = \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)}{1\cdot 2} x^{n-1}\Delta x + \ldots + (\Delta x)^{n-1} \right] = nx^{n-1},$$

consequently, $y' = nx^{n-1}$, and thus we have proved the theorem.

Example 1. $y=x^5$, $y'=5x^{5-1}=5x^4$.

Example 2. $y=x$, $y'=1x^{1-1}$, $y'=1$. The latter result has a simple geometric interpretation: the line tangent to the straight line $y=x$ for any value of x coincides with this line and, consequently, forms with the positive direction of the x-axis an angle, the tangent of which is 1.

Note that formula (1) also holds true when n is fractional or negative. (This will be proved in Sec. 12).

Example 3. $y=\sqrt{x}$.
Let us represent the function in the form of a power:

$$y=x^{\frac{1}{2}};$$

then by formula (1), taking into consideration what we have just said, we get

$$y'=\frac{1}{2}x^{\frac{1}{2}-1}$$

or

$$y'=\frac{1}{2\sqrt{x}}.$$

Example 4. $y=\frac{1}{x\sqrt{x}}$.
Represent y in the form of a power function:

$$y=x^{-\frac{3}{2}}.$$

Then

$$y'=-\frac{3}{2}x^{-\frac{3}{2}-1}=-\frac{3}{2}x^{-\frac{5}{2}}=-\frac{3}{2x^2\sqrt{x}}.$$

SEC. 6. DERIVATIVES OF THE FUNCTIONS $y=\sin x$ $y=\cos x$

Theorem 1. *The derivative of* $\sin x$ *is* $\cos x$, or

$$\textit{if } y=\sin x, \textit{ then } y'=\cos x. \qquad \text{(II)}$$

Proof. Increase the argument x by the increment Δx; then

1) $y+\Delta y=\sin(x+\Delta x)$;

2) $\Delta y=\sin(x+\Delta x)-\sin x=2\sin\frac{x+\Delta x-x}{2}\cos\frac{x+\Delta x+x}{2}=$
$$=2\sin\frac{\Delta x}{2}\cdot\cos\left(x+\frac{\Delta x}{2}\right);$$

3) $\frac{\Delta y}{\Delta x}=\frac{2\sin\frac{\Delta x}{2}\cos\left(x+\frac{\Delta x}{2}\right)}{\Delta x}=\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cos\left(x+\frac{\Delta x}{2}\right);$

4) $$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right),$$

but since

$$\lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1,$$

we get

$$y' = \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) = \cos x.$$

This latter equality is obtained on the grounds that $\cos x$ is a continuous function.

Theorem 2. *The derivative of* $\cos x$ *is* $-\sin x$, or

$$\text{if } y = \cos x, \text{ then } y' = -\sin x. \qquad \text{(III)}$$

Proof. Increase the argument x by the increment Δx, then

$$y + \Delta y = \cos(x + \Delta x);$$

$$\Delta y = \cos(x + \Delta x) - \cos x = -2\sin\frac{x + \Delta x - x}{2} \sin\frac{x + \Delta x + x}{2} =$$
$$= -2 \sin\frac{\Delta x}{2} \sin\left(x + \frac{\Delta x}{2}\right);$$

$$\frac{\Delta y}{\Delta x} = -\frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \sin\left(x + \frac{\Delta x}{2}\right);$$

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \sin\left(x + \frac{\Delta x}{2}\right) = -\lim_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right);$$

taking into account the fact that $\sin x$ is a continuous function, we finally get

$$y' = -\sin x.$$

SEC. 7. DERIVATIVES OF: A CONSTANT, THE PRODUCT OF A CONSTANT BY A FUNCTION. A SUM, A PRODUCT, AND A QUOTIENT

Theorem 1. *The derivative of a constant is equal to zero; that is,*

$$\textit{if } y=C, \textit{ where } C=\text{const}, \textit{ then } y'=0. \qquad \text{(IV)}$$

Proof. $y=C$ is a function of x such that the values of it are equal to C for all x.

Hence, for any value of x

$$y=f(x)=C.$$

We increase the argument x by an increment $\Delta x\,(\Delta x \neq 0)$. Since the function y retains the value C for all values of the argument, we have

$$y+\Delta y=f(x+\Delta x)=C.$$

Therefore, the increment of the function is

$$\Delta y=f(x+\Delta x)-f(x)=0,$$

the ratio of the increment of the function to the increment of the argument

$$\frac{\Delta y}{\Delta x}=0,$$

and, consequently,

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=0,$$

that is,

$$y'=0.$$

The latter result has a simple geometric interpretation. The graph of the function $y=C$ is a straight line parallel to the x-axis. Obviously, the line tangent to the graph at any one of its points coincides with this straight line and, therefore, forms with the x-axis an angle whose tangent y' is zero.

Theorem 2. *A constant factor may be taken outside the derivative sign, i.e.,*

$$\textit{if } y=Cu(x) \ (C=\text{const}), \textit{ then } y'=Cu'(x). \qquad \text{(V)}$$

Proof. Reasoning as in the proof of the preceding theorem, we have

$$y=Cu(x);$$

$$y+\Delta y=Cu(x+\Delta x);$$

$$\Delta y=Cu(x+\Delta x)-Cu(x)=C\,[u(x+\Delta x)-u(x)],$$

$$\frac{\Delta y}{\Delta x}=C\frac{u(x+\Delta x)-u(x)}{\Delta x},$$

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=C\lim_{\Delta x\to 0}\frac{u(x+\Delta x)-u(x)}{\Delta x}, \text{ i. e. } y'=Cu'(x).$$

Example 1. $y=3\frac{1}{\sqrt{x}}$.

$$y'=3\left(\frac{1}{\sqrt{x}}\right)'=3\left(x^{-\frac{1}{2}}\right)'=3\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}=-\frac{3}{2}x^{-\frac{3}{2}},$$

or

$$y'=-\frac{3}{2x\sqrt{x}}.$$

Theorem 3. *The derivative of the sum of a finite number of differentiable functions is equal to the corresponding sum of the derivatives of these functions.* *)

For the case of three terms, for example, we have

$$y=u(x)+v(x)+w(x);\quad y'=u'(x)+v'(x)+w'(x). \qquad \text{(VI)}$$

Proof. For the values of the argument x

$$y=u+v+w$$

(for the sake of brevity we drop the argument x in denoting the function).

For the value of the argument $x+\Delta x$ we have

$$y+\Delta y=(u+\Delta u)+(v+\Delta v)+(w+\Delta w),$$

where Δy, Δu, Δv, and Δw are increments of the functions y, u, v and w, which correspond to the increment Δx in the argument x. Hence,

$$\Delta y=\Delta u+\Delta v+\Delta w,\quad \frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}+\frac{\Delta w}{\Delta x},$$

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}+\lim_{\Delta x\to 0}\frac{\Delta w}{\Delta x}$$

or

$$y'=u'(x)+v'(x)+w'(x).$$

Example 2. $y=3x^4-\frac{1}{\sqrt[3]{x}}$,

*) The expression $y=u(x)-v(x)$ is equivalent to $y=u(x)+(-1)v(x)$ and $y'=[u(x)+(-1)v(x)]'=u'(x)+[-v(x)]'=u'(x)-v'(x)$.

$$y' = 3(x^4)' - \left(x^{-\frac{1}{3}}\right)' = 3 \cdot 4x^3 - \left(-\frac{1}{3}\right) x^{-\frac{1}{3}-1},$$

and so

$$y' = 12x^3 + \frac{1}{3} \frac{1}{x\sqrt[3]{x}}.$$

Theorem 4. *The derivative of a product of two differentiable functions is equal to the product of the derivative of the first function by the second function plus the product of the first function by the derivative of the second function; that is,*

$$\text{if } y = uv, \text{ then } y' = u'v + uv'. \qquad \text{(VII)}$$

Proof. Reasoning as in the proof of the preceding theorem, we get

$$y = uv,$$

$$y + \Delta y = (u + \Delta u)(v + \Delta v),$$

$$\Delta y = (u + \Delta u)(v + \Delta v) - uv = \Delta u v + u \Delta v + \Delta u \, \Delta v,$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} v + u \frac{\Delta v}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x},$$

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} v + \lim_{\Delta x \to 0} u \frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \Delta u \frac{\Delta v}{\Delta x} =$$

$$= \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right) v + u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \Delta u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

(since u and v are independent of Δx).

Let us consider the last term on the right-hand side:

$$\lim_{\Delta x \to 0} \Delta u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}.$$

Since $u(x)$ is a differentiable function, it is continuous. Consequently, $\lim\limits_{\Delta x \to 0} \Delta u = 0$. Also,

$$\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = v' \neq \infty.$$

Thus, the term under consideration is zero and we finally get

$$y' = u'v + uv'.$$

The theorem just proved readily gives us the rule for differentiating the product of any number of functions.

Thus, if we have a product of three functions

$$y = uvw,$$

then by representing the right-hand side as the product of u and (vw), we get $y' = u'(vw) + u(vw)' = u'vw + u(v'w + vw') = u'vw + uv'w + uvw'$.

In this way we can obtain a similar formula for the derivative of the product of any (finite) number of functions. Namely, if $y = u_1 u_2 \ldots u_n$, then

$$y' = u_1' u_2 \ldots u_{n-1} u_n + u_1 u_2' \ldots u_{n-1} u_n + \ldots + u_1 u_2 \ldots u_{n-1} u_n'.$$

Example 3. If $y = x^2 \sin x$, then

$$y' = (x^2)' \sin x + x^2 (\sin x)' = 2x \sin x + x^2 \cos x.$$

Example 4. If $y = \sqrt{x} \sin x \cos x$, then

$$y' = (\sqrt{x})' \sin x \cos x + \sqrt{x} (\sin x)' \cos x + \sqrt{x} \sin x (\cos x)' =$$
$$= \frac{1}{2\sqrt{x}} \sin x \cos x + \sqrt{x} \cos x \cos x + \sqrt{x} \sin x (-\sin x) =$$
$$= \frac{1}{2\sqrt{x}} \sin x \cos x + \sqrt{x} (\cos^2 x - \sin^2 x) = \frac{\sin 2x}{4\sqrt{x}} + \sqrt{x} \cos 2x.$$

Theorem 5. *The derivative of a fraction (that is, the quotient obtained by the division of two functions) is equal to a fraction whose denominator is the square of the denominator of the given fraction, and the numerator is the difference between the product of the denominator by the derivative of the numerator, and the product of the numerator by the derivative of the denominator; i. e.,*

$$\text{if } y = \frac{u}{v}, \text{ then } y' = \frac{u'v - uv'}{v^2}. \tag{VIII}$$

Proof. If Δy, Δu, and Δv are increments of the functions y, u, and v, corresponding to the increment Δx of the argument x, then

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v},$$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v \Delta u - u \Delta v}{v(v + \Delta v)},$$

$$\frac{\Delta y}{\Delta x} = \frac{\frac{v \Delta u - u \Delta v}{\Delta x}}{v(v + \Delta v)} = \frac{\frac{\Delta u}{\Delta x} v - u \frac{\Delta v}{\Delta x}}{v(v + \Delta v)},$$

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta u}{\Delta x} v - u \frac{\Delta v}{\Delta x}}{v(v + \Delta v)} = \frac{v \lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} - u \lim\limits_{\Delta x \to 0} \frac{\Delta v}{\Delta x}}{v \lim\limits_{\Delta x \to 0} (v + \Delta v)}.$$

Whence, noting that $\Delta v \to 0$ as $\Delta x \to 0$, *) we get

$$y' = \frac{u'v - uv'}{v^2}.$$

Example 5. If $y = \frac{x^3}{\cos x}$, then

$$y' = \frac{(x^3)' \cos x - x^3 (\cos x)'}{\cos^2 x} = \frac{3x^2 \cos x + x^3 \sin x}{\cos^2 x}.$$

Note. If we have a function of the form

$$y = \frac{u(x)}{c},$$

where the denominator c is a constant, then when differentiating this function we do not need to use formula (VIII); it is better to make use of formula (V):

$$y' = \left(\frac{1}{c} u\right)' = \frac{1}{c} u' = \frac{u'}{c}.$$

Of course, the same result is obtained if formula (VIII) is applied.

Example 6. If $y = \frac{\cos x}{7}$, then

$$y' = \frac{(\cos x)'}{7} = -\frac{\sin x}{7}.$$

SEC. 8. THE DERIVATIVE OF A LOGARITHMIC FUNCTION

Theorem. *The derivative of the function* $\log_a x$ *is* $\frac{1}{x} \log_a e$, *that is,*

$$\textit{if } y = \log_a x, \textit{ then } y' = \frac{1}{x} \log_a e. \qquad \text{(IX)}$$

Proof. If Δy is an increment of the function $y = \log_a x$ that corresponds to the increment Δx of the argument x, then

$$y + \Delta y = \log_a (x + \Delta x);$$

$$\Delta y = \log_a (x + \Delta x) - \log_a x = \log_a \frac{x + \Delta x}{x} = \log_a \left(1 + \frac{\Delta x}{x}\right);$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right).$$

*) $\lim\limits_{\Delta x \to 0} \Delta v = 0$ since $v(x)$ is a differentiable and, consequently, continuous function.

Multiply and divide by x the expression on the right-hand side of the latter equality:

$$\frac{\Delta y}{\Delta x}=\frac{1}{x}\frac{x}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)=\frac{1}{x}\log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}.$$

We denote the quantity $\frac{\Delta x}{x}$ in terms of α. Obviously, for the given x, $\alpha\to 0$ as $\Delta x\to 0$. Consequently,

$$\frac{\Delta y}{\Delta x}=\frac{1}{x}\log_a(1+\alpha)^{\frac{1}{\alpha}}.$$

But, as we know from Sec. 7, Ch. II,

$$\lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e.$$

But if the expression under the sign of the logarithm approaches the number e, then the logarithm of this expression approaches $\log_a e$ (in virtue of the continuity of the logarithmic function). We therefore finally get

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\alpha\to 0}\frac{1}{x}\log_a(1+\alpha)^{\frac{1}{\alpha}}=\frac{1}{x}\log_a e.$$

Noting that $\log_a e=\frac{1}{\ln a}$, we can rewrite the formula as follows:

$$y'=\frac{1}{x}\frac{1}{\ln a}.$$

The following is an important particular case of this formula: if $a=e$, then $\ln a=\ln e=1$; that is,

$$\textit{if } y=\ln x, \textit{ then } y'=\frac{1}{x}. \qquad \text{(X)}$$

SEC. 9. THE DERIVATIVE OF A COMPOSITE FUNCTION

Given a composite function $y=f(x)$, that is, such that it may be represented in the following form:

$$y=F(u),\quad u=\varphi(x).$$

or $y=F[\varphi(x)]$ (see Ch. I, Sec. 8). In the expression $y=F(u)$, u is called the *intermediate argument.*

Let us establish a rule for differentiating composite functions.

Theorem. *If a function $u=\varphi(x)$ has, at some point x, a derivative $u'_x=\varphi'(x)$, and the function $y=F(u)$ has, at the corresponding*

value of u, the derivative $y'_u = F'(u)$, *then the composite function* $y = F[\varphi(x)]$ *at the given point x also has a derivative, which is equal to*

$$y'_x = F'_u(u)\,\varphi'(x),$$

where in place of u we must substitute the expression $u = \varphi(x)$. *Briefly,*

$$y'_x = y'_u u'_x.$$

In other words, the derivative of a composite function is equal to the product of the derivative of the given function with respect to the intermediate argument u by the derivative of the intermediate argument with respect to x.

Proof. For a definite value of x we will have

$$u = \varphi(x), \quad y = F(u).$$

For the increased value of the argument $x + \Delta x$,

$$u + \Delta u = \psi(x + \Delta x), \quad y + \Delta y = F(u + \Delta u).$$

Thus, to the increment Δx there corresponds an increment Δu, to which corresponds an increment Δy, whereby $\Delta u \to 0$ and $\Delta y \to 0$ as $\Delta x \to 0$. It is given that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} = y'_u.$$

From this relation (taking advantage of the definition of a limit) we get (for $\Delta u \neq 0$)

$$\frac{\Delta y}{\Delta u} = y'_u + \alpha, \tag{1}$$

where $\alpha \to 0$ as $\Delta u \to 0$. We rewrite (1) as

$$\Delta y = y'_u \Delta u + \alpha \Delta u. \tag{2}$$

Equality (2) also holds true when $\Delta u = 0$ for an arbitrary α, since it turns into an identity, $0 = 0$. For $\Delta u = 0$ we shall assume $\alpha = 0$. Divide all terms of (2) by Δx:

$$\frac{\Delta y}{\Delta x} = y'_u \frac{\Delta u}{\Delta x} + \alpha \frac{\Delta u}{\Delta x}. \tag{3}$$

It is given that

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = u'_x, \quad \lim_{\Delta x \to 0} \alpha = 0.$$

Passing to the limit as $\Delta x \longrightarrow 0$ in (3), we get

$$y'_x = y'_u u'_x, \tag{4}$$

which is the required proof.

Example 1. Given a function $y = \sin(x^2)$. Find y'_x. Represent the given function as a function of a function as follows:

$$y = \sin u, \quad u = x^2.$$

We find

$$y'_u = \cos u, \qquad u'_x = 2x.$$

Hence, by formula (4),

$$y'_x = y'_u u'_x = \cos u \cdot 2x.$$

Substituting, in place of u, its expression, we finally get

$$y'_x = 2x \cos(x^2).$$

Example 2. Given the function $y = (\ln x)^3$. Find y'_x.
Represent this function as follows:

$$y = u^3, \quad u = \ln x.$$

We find

$$y'_u = 3u^2, \qquad u'_x = \frac{1}{x}.$$

Hence,

$$y'_x = 3u^2 \frac{1}{x} = 3(\ln x)^2 \frac{1}{x}.$$

If a function $y = f(x)$ is such that it may be represented in the form

$$y = F(u), \quad u = \varphi(v), \quad v = \psi(x),$$

the derivative y'_x is found by a successive application of the foregoing theorem.
Applying the proved rule, we have

$$y'_x = y'_u u'_x.$$

Applying the same theorem to find u'_x, we have

$$u'_x = u'_v v'_x.$$

Substituting the expression of u'_x into the preceding equality, we get

$$y'_x = y'_u u'_v v'_x \tag{5}$$

or

$$y'_x = F'_u(u)\, \varphi'_v(v)\, \psi'_x(x).$$

Example 3. Given the function $y = \sin[(\ln x)^3]$. Find y'_x. Represent the function as follows:

$$y = \sin u, \quad u = v^3, \quad v = \ln x.$$

We then find

$$y'_u = \cos u, \quad u'_v = 3v^2, \quad v'_x = \frac{1}{x}.$$

In this way, by formula (5), we get

$$y'_x = y'_u u'_v v'_x = 3(\cos u)\, v^2 \frac{1}{x},$$

or finally,

$$y'_x = \cos[(\ln x)^3] \cdot 3(\ln x)^2 \frac{1}{x}.$$

It is to be noted that the function considered is defined only for $x > 0$.

SEC. 10. DERIVATIVES OF THE FUNCTIONS $y = \tan x$, $y = \cot x$, $y = \ln|x|$

Theorem 1. *The derivative of the function* $\tan x$ *is* $\frac{1}{\cos^2 x}$,

$$\text{or if } y = \tan x, \text{ then } y' = \frac{1}{\cos^2 x}. \qquad \text{(XI)}$$

Proof. Since

$$y = \frac{\sin x}{\cos x},$$

by the rule of differentiation of a fraction [see formula (VIII), Sec. 7, Ch. III] we get

$$y' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

Theorem 2. *The derivative of the function* $\cot x$ *is*

$$-\frac{1}{\sin^2 x}, \text{ or if } y = \cot x, \text{ then } y' = -\frac{1}{\sin^2 x}. \qquad \text{(XII)}$$

Proof. Since $y = \frac{\cos x}{\sin x}$, we have

$$y' = \frac{(\cos x)' \sin x - \cos x (\sin x)'}{\sin^2 x} = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} =$$

$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

Example 1. If $y = \tan \sqrt{x}$, then

$$y' = \frac{1}{\cos^2 \sqrt{x}} (\sqrt{x})' = \frac{1}{2\sqrt{x}} \frac{1}{\cos^2 \sqrt{x}}.$$

Example 2. If $y = \ln \cot x$, then

$$y' = \frac{1}{\cot x} (\cot x)' = \frac{1}{\cot x}\left(-\frac{1}{\sin^2 x}\right) = -\frac{1}{\cos x \sin x} = -\frac{2}{\sin 2x}.$$

Theorem 3. *The derivative of the function* $\ln|x|$ (Fig. 62) *is* $\frac{1}{x}$,

$$\text{or } if\ y = \ln|x|,\ then\ y' = \frac{1}{x}. \qquad \text{(XIII)}$$

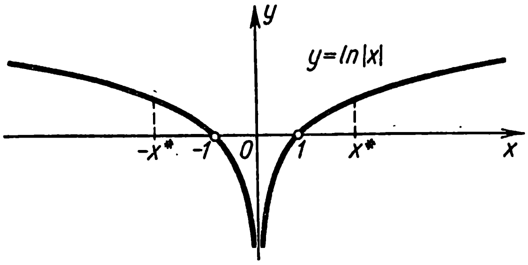


Fig. 62.

Proof. a) If $x > 0$, then $|x| = x$, $\ln|x| = \ln x$, and therefore

$$y' = \frac{1}{x}.$$

b) Let $x < 0$, then $|x| = -x$. But

$$\ln|x| = \ln(-x).$$

(It will be noted that if $x < 0$, then $-x > 0$.) Let us represent the function $y = \ln(-x)$ as a composite function by putting

$$y = \ln u; \quad u = -x.$$

Then

$$y'_x = y'_u u'_x = \frac{1}{u}(-1) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

And so for negative values of x we also have the equation

$$y'_x = \frac{1}{x}.$$

Hence, formula (XIII) has been proved for any value $x \neq 0$. (For $x = 0$ the function $\ln|x|$ is not defined.)

SEC. 11. AN IMPLICIT FUNCTION AND ITS DIFFERENTIATION

Let the values of two variables x and y be related by some equation, which we can symbolise as follows:

$$F(x, y) = 0. \qquad (1)$$

If the function $y=f(x)$, defined on some interval (a, b), is such that equation (1) becomes an identity in x when the expres-

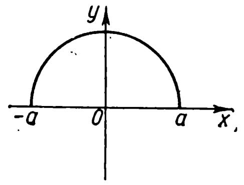


Fig. 63.

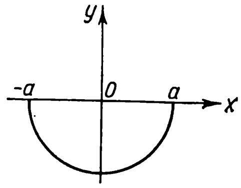


Fig. 64.

sion $f(x)$ is substituted into it in place of y, the function $y=f(x)$ is an *implicit function* defined by equation (1).

For example, the equation

$$x^2+y^2-a^2=0 \tag{2}$$

defines implicitly the following elementary functions (Figs. 63 and 64):

$$y=\sqrt{a^2-x^2}, \tag{3}$$

$$y=-\sqrt{a^2-x^2}. \tag{4}$$

Indeed, substitution into equation (2) yields the identity

$$x^2+(a^2-x^2)-a^2=0.$$

Expressions (3) and (4) were obtained by solving equation (2) for y. But not every implicitly defined function may be represented explicitly, that is, in the form $y=f(x)$, *) where $f(x)$ is an elementary function.

For instance, functions defined by the equations

$$y^6-y-x^2=0$$

or

$$y-x-\frac{1}{4}\sin y=0$$

are not expressible in terms of elementary functions; that is, these equations cannot be solved for y by means of elementary functions.

Note 1. Observe that the terms "explicit function" and "implicit function" do not characterise the nature of the function but merely the way it is defined. Every explicit function $y=f(x)$ may also be represented as an implicit function $y-f(x)=0$.

*) If a function is defined by an equation of the form $y=f(x)$, one says that the function is *defined explicitly* or is *explicit*.

We shall now give the rule for finding the derivative of an implicit function without transforming it into an explicit one, that is, without representing it in the form $y=f(x)$.

Assume the function is defined by the equation

$$x^2+y^2-a^2=0.$$

Here, if y is a function of x defined by this equality, then the equality is an identity.

Differentiating both sides of this identity with respect to x, and regarding y as a function of x, we get (via the rule of differentiating a composite function)

$$2x+2yy'=0,$$

whence

$$y'=-\frac{x}{y}.$$

Observe that if we were to differentiate the corresponding explicit function

$$y=\sqrt{a^2-x^2},$$

we would obtain

$$y'=-\frac{x}{\sqrt{a^2-x^2}}=-\frac{x}{y},$$

which is the same result.

Let us consider another case of an implicit function y of x:

$$y^6-y-x^2=0.$$

Differentiate with respect to x:

$$6y^5y'-y'-2x=0,$$

whence

$$y'=\frac{2x}{6y^5-1}.$$

Note 2. From the foregoing examples it follows that to find the value of the derivative of an implicit function for a given value of the argument x, one also has to know the value of the function y for a given value of x.

SEC. 12. DERIVATIVES OF A POWER FUNCTION FOR AN ARBITRARY REAL EXPONENT, OF AN EXPONENTIAL FUNCTION, AND A COMPOSITE EXPONENTIAL FUNCTION

Theorem 1. *The derivative of the function x^n, where n is any real number, is equal to nx^{n-1}; that is,*

$$\text{if } y=x^n, \text{ then } y'=nx^{n-1}. \qquad \text{(I}'\text{)}$$

Proof. Let $x>0$. Taking logarithms of this function, we get

$$\ln y = n \ln x.$$

Differentiate, with respect to x, both sides of the equality obtained, taking y to be a function of x:

$$\frac{y'}{y} = n\frac{1}{x}; \quad y' = yn\frac{1}{x}.$$

Substituting into this equation the value $y = x^n$, we finally get

$$y' = nx^{n-1}.$$

It is easy to show that this formula holds true also for $x<0$ provided x^n is meaningful. *)

Theorem 2. *The derivative of the function* a^x, *where* $a>0$, *is* $a^x \ln a$; *that is,*

$$\text{if } y = a^x, \text{ then } y' = a^x \ln a. \qquad \text{(XIV)}$$

Proof. Taking logarithms of the equality $y = a^x$, we get

$$\ln y = x \ln a.$$

Differentiate the equality obtained regarding y as a function of x:

$$\frac{1}{y}y' = \ln a; \quad y' = y \ln a$$

or

$$y' = a^x \ln a.$$

If the base is $a = e$, then $\ln e = 1$ and we have the formula

$$y = e^x, \quad y' = e^x. \qquad \text{(XIV')}$$

Example 1. Given the function

$$y = e^{x^2}.$$

Represent it as a composite function by introducing the intermediate argument u:

$$y = e^u, \quad u = x^2;$$

then

$$y'_u = e^u, \quad u'_x = 2x$$

and, therefore,

$$y'_x = e^u \cdot 2x = e^{x^2} \cdot 2x.$$

---

*) This formula was proved in Sec. 5, Ch. III, for the case when n is a *positive integer*. Formula (I) has now been proved for the general case (for any constant number n).

A composite exponential function is a function in which both the base and the exponent are functions of x, for instance, $(\sin x)^{x^2}$, $x^{\tan x}$, x^x, $(\ln x)^x$, and the like; generally, any function of the form

$$y = [u(x)]^{v(x)} \equiv u^v$$

is an exponential function (composite exponential function). *)

Theorem 3.

$$\textit{If } y = u^v, \textit{ then } y' = vu^{v-1}u' + u^v v' \ln u. \qquad \text{(XV)}$$

Proof. Taking logarithms of the function y, we have

$$\ln y = v \ln u.$$

Differentiating the resultant equation with respect to x, we get

$$\frac{1}{y}y' = v\frac{1}{u}u' + v' \ln u$$

whence

$$y' = y\left(v\frac{u'}{u} + v' \ln u\right).$$

Substituting into this equation the expression $y = u^v$, we obtain

$$y' = vu^{v-1}u' + u^v v' \ln u.$$

Thus, the derivative of an exponential function (composite exponential function) consists of two terms: the first term is obtained by assuming, when differentiating, that u is a function of x and v is a **constant** (that is to say, if we regard u^v as a **power** function); the second term is obtained on the assumption that v is a function of x, and $u = \text{const}$ (i. e., if we regard u^v as an **exponential** function).

Example 2. If $y = x^x$, then $y' = xx^{x-1}(x') + x^x(x') \ln x$ or $y' = x^x + x^x \ln x = x^x(1 + \ln x)$.

Example 3. If $y = (\sin x)^{x^2}$, then

$$y' = x^2(\sin x)^{x^2-1}(\sin x)' + (\sin x)^{x^2}(x^2)' \ln \sin x =$$
$$= x^2(\sin x)^{x^2-1}\cos x + (\sin x)^{x^2} 2x \ln \sin x.$$

The procedure applied in this section for finding derivatives (first finding the derivative **of the logarithm of the given function**) is widely used in differentiating functions. Very often the use of this method greatly simplifies calculations.

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*) In the Russian mathematical literature this function is also called an exponential-power function or a power-exponential function.

Example 4. To find the derivative of the function

$$y=\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3e^x}.$$

Solution. Taking logarithms we get

$$\ln y=2\ln(x+1)+\frac{1}{2}\ln(x-1)-3\ln(x+4)-x.$$

Differentiate both sides of this equality:

$$\frac{y'}{y}=\frac{2}{x+1}+\frac{1}{2(x-1)}-\frac{3}{x+4}-1.$$

Multiplying by y and substituting, in place of y, the expression $\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3e^x}$, we get

$$y'=\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3e^x}\left[\frac{2}{x+1}+\frac{1}{2(x-1)}-\frac{3}{x+4}-1\right].$$

Note. The expression $\frac{y'}{y}=(\ln y)'$, which is the derivative, with respect to x, of the natural logarithm of the given function $y=y(x)$, is called the *logarithmic derivative.*

SEC. 13. AN INVERSE FUNCTION AND ITS DIFFERENTIATION

Take an increasing or decreasing function (Fig. 65)

$$y=f(x) \tag{1}$$

defined in some interval (a, b) $(a<b)$ (see Sec. 6, Ch. I). Let $f(a)=c$, $f(b)=d$. For definiteness we shall henceforward consider an increasing function.

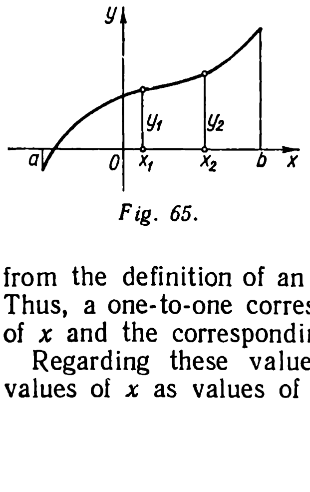


Fig. 65.

Let us consider two different values x_1 and x_2 in the interval (a, b). From the definition of an increasing function it follows that if $x_1<x_2$ and $y_1=f(x_1)$, $y_2=f(x_2)$, then $y_1<y_2$. Hence, to two different values x_1 and x_2 there correspond two different values of the function, y_1 and y_2. The converse is also true: if $y_1<y_2$, $y_1=f(x_1)$, and $y_2=f(x_2)$, then from the definition of an increasing function it follows that $x_1<x_2$. Thus, a one-to-one correspondence is established between the values of x and the corresponding values of y.

Regarding these values of y as values of the argument and the values of x as values of the function, we get x as a function of y:

$$x=\varphi(y). \tag{2}$$

This function is called the *inverse function* of $y=f(x)$. It is obvious too that the function $y=f(x)$ is the inverse of $x=\varphi(y)$. With similar reasoning it is possible to prove that a decreasing function also has an inverse.

Note 1. We state, without proof, that *if an increasing (or decreasing) function $y=f(x)$ is continuous on the interval $[a, b]$, where $f(a)=c$, $f(b)=d$, then the inverse function is defined and is continuous on the interval $[c, d]$.*

Example 1. Given the function $y=x^3$. This function is increasing on the infinite interval $-\infty<x<\infty$; it has an inverse function $x=\sqrt[3]{y}$ (Fig. 66).
It will be noted that the inverse function $x=\varphi(y)$ is found by solving the equation $y=f(x)$ for x.

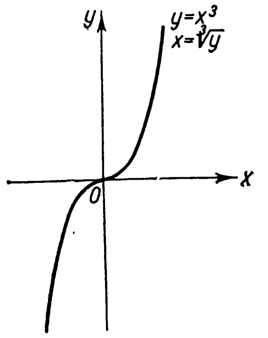


Fig. 66.

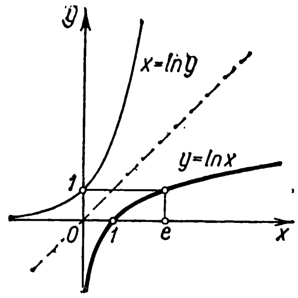


Fig. 67.

Example 2. Given the function $y=e^x$. This function is increasing on the infinite interval $-\infty<x<\infty$. It has an inverse $x=\ln y$. The domain of definition of the inverse function is $0<y<\infty$ (Fig 67).

Note 2. If the function $y=f(x)$ is neither increasing nor decreasing on a certain interval, it can have several inverse functions.*)

Example 3. The function $y=x^2$ is defined on an infinite interval $-\infty<x<+\infty$. It is neither increasing nor decreasing and does not have an inverse function. If we consider the interval $0\leqslant x<\infty$, then the function here is increasing and $x=\sqrt{y}$ is its inverse. But in the interval $-\infty<x<0$ the function is decreasing and its inverse is $x=-\sqrt{y}$ (Fig. 68).

Note 3. If the functions $y=f(x)$ and $x=\varphi(y)$ are reciprocal, their graphs are represented by a single curve. But if we again

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*) Let it be noted once again that when speaking of y as a function of x we have in mind that y is a single-valued function of x.

denote the argument of the inverse function by x, and the function by y and then construct them in a single coordinate system, we will get two different graphs.

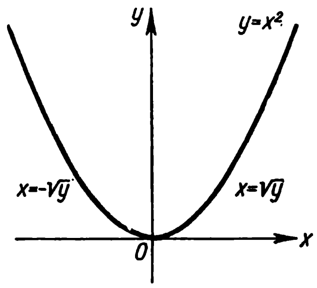


Fig. 68.

It will readily be seen that the graphs will be symmetric about the bisector of the first quadrantal angle.

Example 4. Fig. 67 gives the graphs of the function $y = e^x$ (or $x = \ln y$) and its inverse $y = \ln x$, which are considered in Example 2.

Let us now prove a theorem that permits finding the derivative of a function $y = f(x)$ if we know the derivative of the inverse function.

Theorem. *If for the function*

$$y = f(x) \tag{1}$$

there exists an inverse function

$$x = \varphi(y) \tag{2}$$

which at the point under consideration y has a nonzero derivative $\varphi'(y)$, then at the corresponding point x the function $y = f(x)$ has a derivative $f'(x)$ equal to $\frac{1}{\varphi'(y)}$; that is, the following formula is true

$$f'(x) = \frac{1}{\varphi'(y)}. \tag{XVI}$$

Thus, the derivative of one of two reciprocal functions is equal to unity divided by the derivative of the second function for corresponding values of x and y.*)

Proof. Differentiate, with respect to x, both sides of equality (2), taking y as a function of x **):

$$1 = \varphi'(y)\, y'_x,$$

*) When we write $f'(x)$ or y'_x we regard x as the independent variable when evaluating the derivative; but when we write $\varphi'(y)$ or x'_y we assume that y is the independent variable when evaluating the derivative. It should be noted that **after differentiating with respect to y, as indicated on the right side of formula (XVI), $f(x)$ must be substituted for y.**

**) Actually, here we find the derivative of a function of x defined implicitly by the equation

$$x - \varphi(y) = 0$$

whence

$$y'_x = \frac{1}{\varphi'(y)}.$$

Noting that $y'_x = f'(x)$ we get formula (XVI), which may also be written as

$$y'_x = \frac{1}{x'_y}.$$

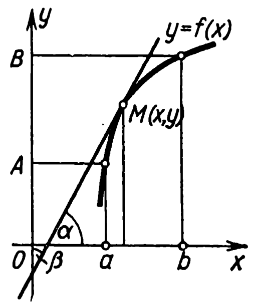


Fig. 69.

The result obtained is clearly illustrated geometrically. Consider the graph of the function $y = f(x)$ (Fig. 69). This curve will also be the graph of the function $x = \varphi(y)$, where x is now regarded as the function and y as the independent variable. Take some point $M(x, y)$ on this curve. Draw a tangent to the curve at this point. Denote by α and β the angles formed by the given tangent and the positive directions of the x- and y-axes. On the basis of the results of Sec. 3 concerning the geometrical meaning of a derivative we have

$$\left.\begin{aligned} f'(x) &= \tan\alpha, \\ \varphi'(y) &= \tan\beta. \end{aligned}\right\} \qquad (3)$$

From Fig. 69 it follows directly that if $\alpha < \frac{\pi}{2}$, then

$$\beta = \frac{\pi}{2} - \alpha.$$

But if $\alpha > \frac{\pi}{2}$, then, as is readily seen, $\beta = \frac{3\pi}{2} - \alpha$. Hence, in any case

$$\tan\beta = \cot\alpha,$$

whence

$$\tan\alpha\tan\beta = \tan\alpha\cot\alpha = 1,$$

or

$$\tan\alpha = \frac{1}{\tan\beta}.$$

Substituting the expressions for $\tan\alpha$ and $\tan\beta$ from formula (3), we get

$$f'(x) = \frac{1}{\varphi'(y)}.$$

SEC. 14. INVERSE TRIGONOMETRIC FUNCTIONS AND THEIR DIFFERENTIATION

1) The function $y = \arcsin x$.
Let us consider the function

$$x = \sin y \tag{1}$$

and construct its graph by directing the y-axis vertically upwards (Fig. 70). This function is defined in the infinite interval $-\infty < y < +\infty$. Over the interval $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, the function $x = \sin y$ is increasing and its values fill the interval $-1 \leqslant x \leqslant 1$. For this reason, the function $x = \sin y$ has an inverse which is denoted by

$$y = \arc \sin x.^{*)}$$

This function is defined on the interval $-1 \leqslant x \leqslant 1$, and its values fill the interval $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$. In Fig. 70, the graph of $y = \arcsin x$ is shown by the heavy line.

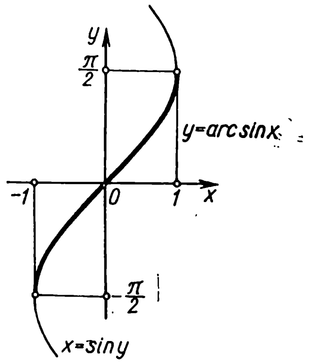


Fig. 70.

Theorem 1. *The derivative of the function* $\arcsin x$ *is equal to* $\frac{1}{\sqrt{1-x^2}}$; i. e.,

$$\text{if } y = \arc \sin x, \text{ then } y' = \frac{1}{\sqrt{1-x^2}}. \tag{XVII}$$

Proof. On the basis of (1) we have

$$x'_y = \cos y.$$

By the rule for differentiating an inverse function,

$$y'_x = \frac{1}{x'_y} = \frac{1}{\cos y}$$

but

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2},$$

---

*) It may be noted that the familiar equation $y = \arc \sin x$ of trigonometry is another way of writing (1). Here (for a given x) y denotes the totality of values of angles whose sine is equal to x.

therefore,

$$y'_x = \frac{1}{\sqrt{1-x^2}};$$

the sign in front of the radical is plus because the function $y = \arcsin x$ takes on values in the interval $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, and, consequently, $\cos y \geqslant 0$.

Example 1. $y = \arcsin e^x$,

$$y' = \frac{1}{\sqrt{1-(e^x)^2}}(e^x)' = \frac{e^x}{\sqrt{1-e^{2x}}}.$$

Example 2.

$$y = \left(\arcsin \frac{1}{x}\right)^2,$$

$$y' = 2 \arcsin \frac{1}{x} \frac{1}{\sqrt{1-\frac{1}{x^2}}} \left(\frac{1}{x}\right)' = -2 \arcsin \frac{1}{x} \frac{1}{x\sqrt{x^2-1}}.$$

2) The function $y = \arccos x$.

As before, we consider the function

$$x = \cos y \qquad (2)$$

and construct its graph with the y-axis extending upwards (Fig. 71). This function is defined on the infinite interval $-\infty < y < +\infty$. On the interval $0 \leqslant y \leqslant \pi$, the function $x = \cos y$ is decreasing and has an inverse that we denote

$$y = \arccos x.$$

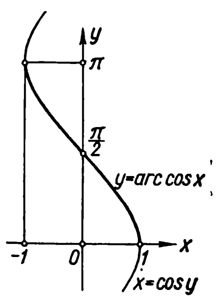


Fig. 71.

This function is defined on the interval $-1 \leqslant x < 1$. The values of the function fill the interval $\pi \geqslant y \geqslant 0$. In Fig. 71, the function $y = \arccos x$ is depicted by the heavy line.

Theorem 2. *The derivative of the function* $\arccos x$ *is* $-\frac{1}{\sqrt{1-x^2}}$; i. e.,

$$\textit{if } y = \arccos x, \textit{ then } y' = -\frac{1}{\sqrt{1-x^2}}. \qquad \text{(XVIII)}$$

Proof. From (2) we have

$$x'_y = -\sin y.$$

Hence

$$y'_x = \frac{1}{x'_y} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}}.$$

But $\cos y = x$, and so

$$y'_x = -\frac{1}{\sqrt{1-x^2}}.$$

In $\sin y = \sqrt{1-\cos^2 y}$ the radical is taken with the plus sign, since the function $y = \arccos x$ is defined on the interval $0 \leqslant y \leqslant \pi$ and, consequently, $\sin y \geqslant 0$.

Example 3. $y = \arccos(\tan x)$,

$$y' = -\frac{1}{\sqrt{1-\tan^2 x}}(\tan x)' = -\frac{1}{\sqrt{1-\tan^2 x}}\frac{1}{\cos^2 x}.$$

3) The function $y = \arctan x$.

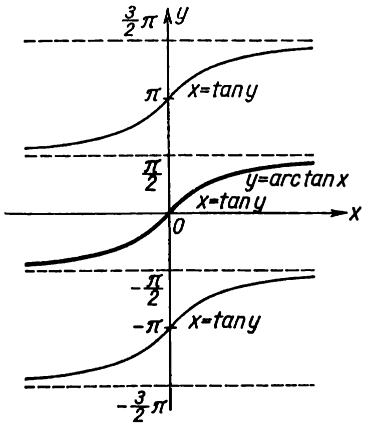


Fig. 72.

We consider the function

$$x = \tan y \qquad (3)$$

and construct its graph (Fig. 72). This function is defined for all values of y except $y = (2k+1)\frac{\pi}{2}$ ($k = 0, \pm 1, \pm 2, \ldots$). On the interval $-\frac{\pi}{2} < y < \frac{\pi}{2}$ the function $x = \tan y$ is increasing and has an inverse:

$$y = \arctan x.$$

This function is defined on the interval $-\infty < x < \infty$. The values of the function fill the interval $-\frac{\pi}{2} < y < \frac{\pi}{2}$. In Fig. 72, the graph of the function $y = \arctan x$ is shown as a heavy line.

Theorem 3. *The derivative of the function* $\arctan x$ *is* $\frac{1}{1+x^2}$; i. e.,

$$\textit{if } y = \arctan x, \textit{ then } y' = \frac{1}{1+x^2}. \qquad \text{(XIX)}$$

Proof. From (3) we have

$$x'_y = \frac{1}{\cos^2 y}.$$

Hence

$$y'_x = \frac{1}{x'_y} = \cos^2 y$$

but

$$\cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y};$$

since $\tan y = x$, we get, finally,

$$y' = \frac{1}{1+x^2}.$$

Example 4. $y = (\text{arc tan}\, x)^4$,

$$y' = 4\,(\text{arc tan}\, x)^3\,(\text{arc tan}\, x)' = 4\,(\text{arc tan}\, x)^3\,\frac{1}{1+x^2}.$$

4) The function $y = \text{arc cot}\, x$. Consider the function

$$x = \cot y. \qquad (4)$$

This function is defined for all values of y except $y = k\pi$ $(k = 0, \pm 1, \pm 2)$. The graph of this function is shown in Fig. 73. On the interval $0 < y < \pi$, the function $x = \cot y$ is decreasing and has an inverse:

$$y = \text{arc cot}\, x.$$

Consequently, this function is defined on the infinite interval $-\infty < x < \infty$, and its values fill the interval $\pi > y > 0$.

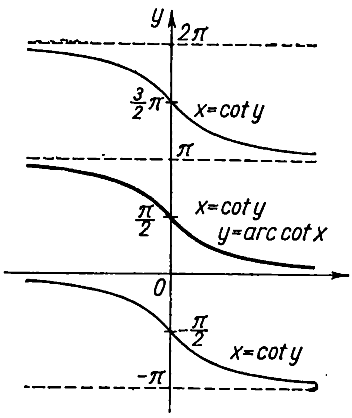


Fig. 73.

Theorem 4. *The derivative of the function* $\text{arc cot}\, x$ *is* $-\frac{1}{1+x^2}$; i. e.,

$$\textit{if } y = \text{arc cot}\, x, \textit{ then } y' = -\frac{1}{1+x^2}. \qquad \text{(XX)}$$

Proof. From (4) we have

$$x'_y = -\frac{1}{\sin^2 y}.$$

Hence

$$y'_x = -\sin^2 y = -\frac{1}{\csc^2 y} = -\frac{1}{1+\cot^2 y}.$$

But

$$\cot y = x.$$

Therefore

$$y'_x = -\frac{1}{1+x^2}.$$

SEC. 15. TABLE OF BASIC DIFFERENTIATION FORMULAS

Let us now bring together into a single table all the basic formulas and rules of differentiation derived in the preceding sections.

$$y = \text{const}, \quad y' = 0.$$

Power function:

$$y = x^{\alpha}, \quad y' = \alpha x^{\alpha-1};$$

particular instances:

$$y = \sqrt{x}, \quad y' = \frac{1}{2\sqrt{x}};$$

$$y = \frac{1}{x}, \quad y' = -\frac{1}{x^2}.$$

Trigonometric functions:

$$y = \sin x, \quad y' = \cos x,$$

$$y = \cos x, \quad y' = -\sin x,$$

$$y = \tan x, \quad y' = \frac{1}{\cos^2 x},$$

$$y = \cot x, \quad y' = -\frac{1}{\sin^2 x}.$$

Inverse trigonometric functions:

$$y = \arcsin x, \quad y' = \frac{1}{\sqrt{1-x^2}},$$

$$y = \arccos x, \quad y' = -\frac{1}{\sqrt{1-x^2}},$$

$$y = \arctan x, \quad y' = \frac{1}{1+x^2},$$

$$y = \operatorname{arc\,cot} x, \quad y' = -\frac{1}{1+x^2}.$$

Exponential function:

$$y = a^x, \quad y' = a^x \ln a;$$

in particular,

$$y=e^x, \; y'=e^x.$$

Logarithmic function:

$$y=\log_a x, \; y'=\frac{1}{x}\log_a e;$$

in particular,

$$y=\ln x, \; y'=\frac{1}{x}.$$

General rules for differentiation:

$$\begin{array}{ll} y=Cu(x), & y'=Cu'(x)\,(C=\text{const}), \\ y=u+v-w, & y'=u'+v'-w', \\ y=u\cdot v, & y'=u'v+uv', \\ y=\dfrac{u}{v}, & y'=\dfrac{u'v-uv'}{v^2}, \\ \left.\begin{array}{l} y=f(u), \\ u=\varphi(x), \end{array}\right\} & y'_x=f'_u(u)\,\varphi'_x(x), \\ y=u^v, & y'=vu^{v-1}u'+u^v v'\ln u. \end{array}$$

If $y=f(x)$, $x=\varphi(y)$, where f and φ are reciprocal functions, then

$$f'(x)=\frac{1}{\varphi'(y)}, \text{ where } y=f(x).$$

SEC. 16. PARAMETRIC REPRESENTATION OF A FUNCTION

Given two equations:

$$\left.\begin{array}{l} x=\varphi(t), \\ y=\psi(t), \end{array}\right\} \tag{1}$$

where t assumes values that lie in the interval $[T_1, T_2]$. To each value of t there correspond values of x and y (the functions φ and ψ are assumed to be single-valued). If one regards the values of x and y as coordinates of a point in a coordinate xy-plane, then to each value of t there will correspond a definite point in the plane. And when t varies from T_1 to T_2, this point will describe a certain curve. Equations (1) are called *parametric equations* of this curve, t is the *parameter*, and *parametric* is the way the curve is represented by equations (1).

Let us further assume that the function $x=\varphi(t)$ has an inverse, $t=\Phi(x)$. Then, obviously, y is a function of x;

$$y=\psi[\Phi(x)]. \tag{2}$$

Thus, equations (1) define y as a function of x, and it is said that the function y of x is represented parametrically.

The explicit expression of the dependence of y on x, $y=f(x)$, is obtained by eliminating the parameter t from equations (1).

Parametric representation of curves is widely used in mechanics. If in the xy-plane there is a certain material point in motion and if we know the laws of motion of the projections of this point on the coordinate axes, then

$$\left.\begin{array}{l} x=\varphi(t), \\ y=\psi(t) \end{array}\right\} \tag{1'}$$

where the parameter t is the time. Then equations (1') are parametric equations of the trajectory of the moving point. Eliminating from these equations the parameter t, we get the equation of the trajectory in the form $y=f(x)$ or $F(x, y)=0$. By way of illustration, let us take the following problem.

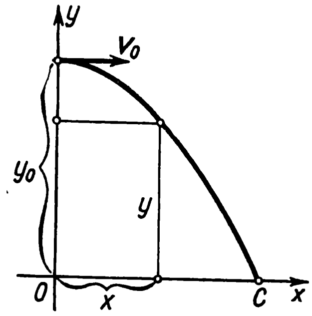


Fig. 74.

Problem. Determine the trajectory and point of impact of a load dropped from an airplane moving horizontally with a velocity v_0 at an altitude y_0 (air resistance is disregarded).

Solution. Taking a coordinate system as shown in Fig. 74, we assume that the airplane drops the load at the instant it cuts the y-axis. It is obvious that the horizontal translation of the load will be uniform and with constant velocity v_0:

$$x=v_0t.$$

Vertical displacement of the falling load due to the force of gravity will be expressed by the formula

$$s=\frac{gt^2}{2}.$$

Hence the distance of the load from the ground at any instant will be

$$y=y_0-\frac{gt^2}{2}.$$

The two equations

$$x=v_0t,$$

$$y=y_0-\frac{gt^2}{2}$$

will be the parametric equations of the trajectory. To eliminate the parameter t, we find the value $t=\frac{x}{v_0}$ from the first equation and substitute it into the second equation. Then we get the equation of the trajectory in the form

$$y=y_0-\frac{g}{2v_0^2}x^2.$$

This is the equation of a parabola with vertex at the point $M(0, y_0)$, the y-axis serving as the axis of symmetry of the parabola.

We determine the length of OC, denote the abscissa of C by X, and note that the ordinate of this point is $y=0$. Putting these values into the preceding formula, we get

$$0=y_0-\frac{g}{2v_0^2}X^2,$$

whence

$$X=v_0\sqrt{\frac{2y_0}{g}}.$$

SEC. 17. THE EQUATIONS OF CERTAIN CURVES IN PARAMETRIC FORM

Circle. Given a circle with centre at the coordinate origin and with radius r (Fig. 75).

Denote by t the angle formed by the x-axis and the radius to some point $M(x, y)$ of the circle. Then the coordinates of any point on the circle will be expressed in terms of the parameter t as follows:

$$\left.\begin{array}{l} x=r\cos t, \\ y=r\sin t, \end{array}\right\} 0\leqslant t\leqslant 2\pi.$$

These are the parametric equations of the circle. If we eliminate the parameter t from these equations, we will have an equation of the circle containing only x and y. Squaring the parametric equations and adding, we get

$$x^2+y^2=r^2(\cos^2 t+\sin^2 t)$$

or

$$x^2+y^2=r^2.$$

Ellipse. Given the equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad (1)$$

Set

$$x=a\cos t. \qquad (2')$$

Putting this expression into equation (1), we get

$$y=b\sin t. \qquad (2'')$$

The equations

$$\left.\begin{array}{l} x=a\cos t, \\ y=b\sin t, \end{array}\right\} 0\leqslant t\leqslant 2\pi \qquad (2)$$

are the parametric equations of the ellipse.

Let us find out the geometrical meaning of the parameter t. Draw two circles with centres at the coordinate origin and with radii a and b (Fig. 76). Let the point M (x, y) lie on the ellipse, and let B be a point of the large

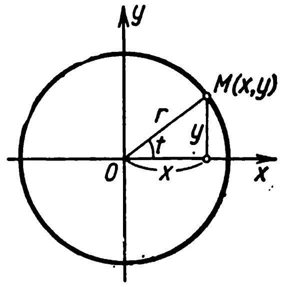


Fig. 75.

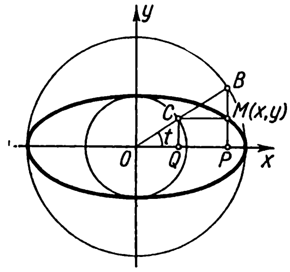


Fig. 76.

circle with the same abscissa as M. Denote by t the angle formed by the radius OB with the x-axis. From the figure it follows directly that

$$x = OP = a\cos t \text{ [this is equation (2')]},$$

$$CQ = b\sin t.$$

From (2″) we conclude that $CQ = y$; in other words, the straight line CM is parallel to the x-axis.

Consequently, in equations (2) t is an angle formed by the radius OB and the axis of abscissas. The angle t is sometimes called an eccentric angle.

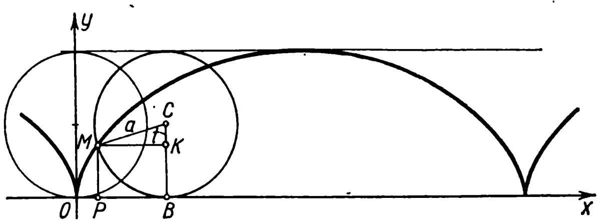


Fig. 77.

Cycloid. The cycloid is a curve described by a point lying on the circumference of a circle if this circle rolls upon a straight line without sliding (Fig. 77). Suppose that when motion began the point M of the rolling circle lay at the origin. Let us determine the coordinates of M after the circle has turned through an angle t. If a is the radius of the rolling circle, it will be seen from Fig. 77 that

$$x = OP = OB - PB,$$

but since the circle rolls without sliding, we have

$$OB = \widehat{MB} = at, \quad PB = MK = a\sin t.$$

Hence, $x = at - a\sin t = a(t - \sin t)$.

Further,

$$y = MP = KB = CB - CK = a - a\cos t = a(1-\cos t).$$

The equations

$$\left.\begin{array}{l} x = a(t-\sin t), \\ y = a(1-\cos t), \end{array}\right\} 0 \leqslant t \leqslant 2\pi \tag{3}$$

are the parametric equations of the cycloid. As t varies between 0 and 2π, the point M will describe one arc of the cycloid.

Eliminating the parameter t from the latter equations, we get x as a function of y directly. In the interval $0 \leqslant t \leqslant \pi$, the function $y = a(1-\cos t)$ has an inverse:

$$t = \arccos\frac{a-y}{a}.$$

Substituting the expression for t into the first of equations (3), we get

$$x = a\arccos\frac{a-y}{a} - a\sin\left(\arccos\frac{a-y}{a}\right)$$

or

$$x = a\arccos\frac{a-y}{a} - \sqrt{2ay-y^2} \text{ when } 0 \leqslant x \leqslant \pi a.$$

Examining the figure we note that when $\pi a \leqslant x \leqslant 2\pi a$

$$x = 2\pi a - \left(a\arccos\frac{a-y}{a} - \sqrt{2ay-y^2}\right).$$

It will be noted that the function

$$x = a(t-\sin t)$$

has an inverse, but it is not expressible in terms of elementary functions. And so the function $y = f(x)$ is not expressible in terms of elementary functions either.

Note 1. The cycloid clearly shows that in certain cases it is more convenient to use the parametric equations for studying functions and curves than the direct relationship of y and x (y as a function of x or x as a function of y).

Astroid. The astroid is a curve represented by the following parametric equations:

$$\left.\begin{array}{l} x = a\cos^3 t, \\ y = a\sin^3 t, \end{array}\right\} 0 \leqslant t \leqslant 2\pi. \tag{4}$$

Raising the terms of both equations to the power 2/3 and adding, we get

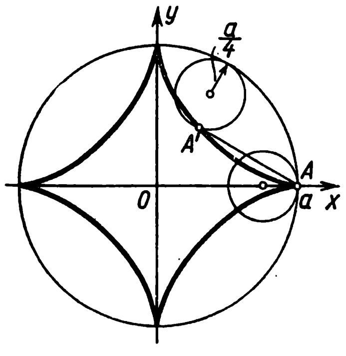


Fig. 78.

the relationship between x and y

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}(\cos^2 t+\sin^2 t),$$

or

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}. \tag{5}$$

Later on (Sec. 12, Ch. V) it will be shown that this curve is of the form shown in Fig. 78. It can be obtained as the trajectory of a certain point on the circumference of a circle of radius $a/4$ rolling (without sliding) upon another circle of radius a (the smaller circle always remains inside the larger one; see Fig. 78).

Note 2. It will be noted that equations (4) and equation (5) define more than one function $y=f(x)$. They define two continuous functions on the intervai $-a\leqslant x\leqslant +a$. One takes on nonnegative values, the other nonpositive values.

SEC. 18. THE DERIVATIVE OF A FUNCTION REPRESENTED PARAMETRICALLY

Let a function y of x be represented by the parametric equations

$$\left.\begin{array}{l} x=\varphi(t), \\ y=\psi(t), \end{array}\right\} t_0\leqslant t\leqslant T. \tag{1}$$

Let us assume that these functions have derivatives and that the function $x=\varphi(t)$ has an inverse $t=\Phi(x)$, which also has a derivative. Then the function $y=f(x)$ defined by the parametric equations may be regarded as a composite function:

$$y=\psi(t), \quad t=\Phi(x),$$

t being the intermediate argument.

By the rule for differentiating a composite function we get

$$y'_x=y'_t\,t'_x=\psi'_t(t)\,\Phi'_x(x). \tag{2}$$

From the theorem for the differentiation of an inverse function, it follows that

$$\Phi'_x(x)=\frac{1}{\varphi'_t(t)}.$$

Putting this expression into (2), we have

$$y'_x=\frac{\psi'(t)}{\varphi'(t)}$$

or

$$y'_x=\frac{y'_t}{x'_t}. \tag{XXI}$$

The derived formula permits finding the derivative y'_x of a function represented parametrically without having to find the expression of y as a direct function of x.

Example 1. The function y of x is given by the parametric equations

$$\left.\begin{array}{l} x=a\cos t, \\ y=a\sin t \end{array}\right\} \quad (0\leqslant t\leqslant \pi).$$

Find the derivative $\frac{dy}{dx}$: 1) for any value of t; 2) for $t=\frac{\pi}{4}$.

Solution.

1) $y'_x=\frac{(a\sin t)'}{(a\cos t)'}=\frac{a\cos t}{-a\sin t}=-\cot t$;

2) $(y'_x)_{t=\frac{\pi}{4}}=-\cot\frac{\pi}{4}=-1$.

Example 2. Find the slope of a tangent to the cycloid

$$x=a(t-\sin t),$$
$$y=a(1-\cos t)$$

at an arbitrary point $(0\leqslant t\leqslant 2\pi)$.

Solution. The slope of a tangent at each point is equal to the value of the derivative y'_x at that point; i. e., it is

$$y'_x=\frac{y'_t}{x'_t}.$$

But

$$x'_t=a(1-\cos t), \quad y'_t=a\sin t.$$

Consequently,

$$y'_x=\frac{a\sin t}{a(1-\cos t)}=\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}}=\cot\frac{t}{2}=\tan\left(\frac{\pi}{2}-\frac{t}{2}\right).$$

Hence, the slope of a tangent to a cycloid at every point is equal to $\tan\left(\frac{\pi}{2}-\frac{t}{2}\right)$, where t is the value of the parameter corresponding to this point. But this means that the angle α of the slope of the tangent to the x-axis is equal to $\frac{\pi}{2}-\frac{t}{2}$ (for values of t lying between $-\pi$ and π)*'.

*' Indeed, the slope is equal to the tangent of the angle of inclination α of the tangent to the x-axis. And so $\tan\alpha=\tan\left(\frac{\pi}{2}-\frac{t}{2}\right)$ and $\alpha=\frac{\pi}{2}-\frac{t}{2}$ for those values of t for which $\frac{\pi}{2}-\frac{t}{2}$ lies between 0 and π.

SEC. 19. HYPERBOLIC FUNCTIONS

In many applications of mathematical analysis we encounter combinations of exponential functions of the form $\frac{1}{2}(e^x-e^{-x})$ and $\frac{1}{2}(e^x+e^{-x})$. These combinations are regarded as new functions and are designated as follows:

$$\left.\begin{aligned}\sinh x&=\frac{e^x-e^{-x}}{2},\\ \cosh x&=\frac{e^x+e^{-x}}{2}.\end{aligned}\right\} \tag{1}$$

The first of these functions is called the *hyperbolic sine*, the second, the *hyperbolic cosine*. These functions may be used to define two more functions: $\tanh x=\frac{\sinh x}{\cosh x}$ and $\coth x=\frac{\cosh x}{\sinh x}$:

$$\left.\begin{aligned}\tanh x&=\frac{e^x-e^{-x}}{e^x+e^{-x}}\text{—the hyperbolic tangent,}\\ \coth x&=\frac{e^x+e^{-x}}{e^x-e^{-x}}\text{—the hyperbolic cotangent.}\end{aligned}\right\} \tag{1'}$$

The functions $\sinh x$, $\cosh x$, $\tanh x$ are obviously defined for all values of x. But the function $\coth x$ is defined everywhere, except the point $x=0$.

The graphs of the hyperbolic functions are given in Figs. 79, 80, 81.

From the definitions of the functions $\sinh x$ and $\cosh x$ [formulas (1)] there follow relationships similar to those between the appropriate trigonometric functions:

$$\cosh^2 x-\sinh^2 x=1, \tag{2}$$

$$\cosh(a+b)=\cosh a\cosh b+\sinh a\sinh b, \tag{3}$$

$$\sinh(a+b)=\sinh a\cosh b+\cosh a\sinh b. \tag{3'}$$

Indeed,

$$\cosh^2 x-\sinh^2 x=\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2=$$
$$=\frac{e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}}{4}=1.$$

Further, noting that

$$\cosh(a+b)=\frac{e^{a+b}+e^{-a-b}}{2},$$

we get

$$\cosh a \cosh b + \sinh a \sinh b = \frac{e^a+e^{-a}}{2}\frac{e^b+e^{-b}}{2} + \frac{e^a-e^{-a}}{2}\frac{e^b-e^{-b}}{2} =$$

$$= \frac{e^{a+b}+e^{-a+b}+e^{a-b}+e^{-a-b}+e^{a+b}-e^{-a+b}-e^{a-b}+e^{-a-b}}{4} =$$

$$= \frac{e^{a+b}+e^{-a-b}}{2} = \cosh(a+b).$$

The prove is similar for relation (3′).

The name "hyperbolic functions" comes from the fact that the functions sinh t and cosh t play the same role in the parametric representation of the hyperbola,

$$x^2 - y^2 = 1,$$

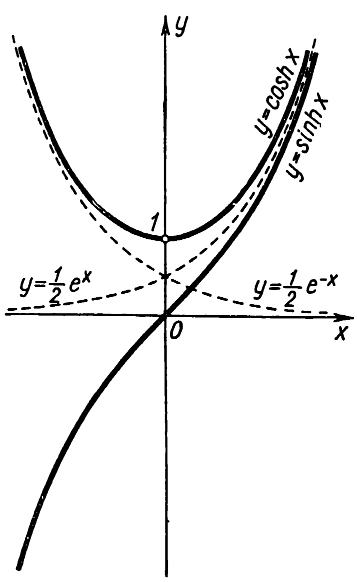


Fig. 79.

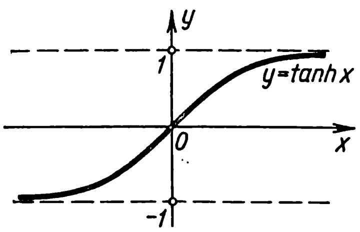


Fig. 80.

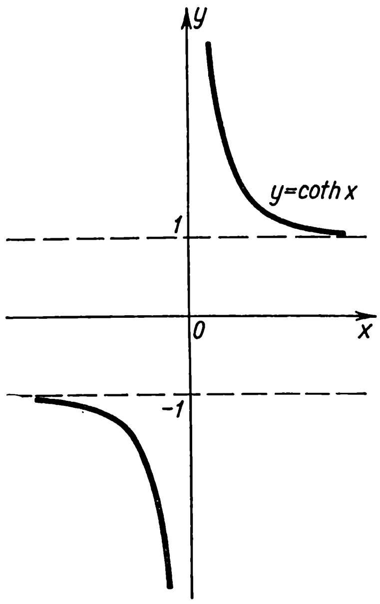


Fig. 81.

as the trigonometric functions sin t and cos t do in the parametric representation of the circle,

$$x^2 + y^2 = 1.$$

Indeed, eliminating the parameter t from the equations

$$x = \cos t, \quad y = \sin t,$$

we get

$$x^2 + y^2 = \cos^2 t + \sin^2 t$$

or

$$x^2 + y^2 = 1 \text{ (the equation of the circle).}$$

Similarly, the equations

$$x = \cosh t,$$
$$y = \sinh t$$

are the parametric equations of the hyperbola.

Indeed, squaring these equations termwise and subtracting the second from the first, we get

$$x^2 - y^2 = \cosh^2 t - \sinh^2 t.$$

Since, on the basis of formula (2), the expression on the right side is equal to unity, we have

$$x^2 - y^2 = 1,$$

which is the equation of the hyperbola.

Let us consider a circle with the equation $x^2 + y^2 = 1$ (Fig. 82). In the equations $x = \cos t$, $y = \sin t$, the parameter t is numerically equal to the central angle AOM or to the doubled area S of the sector AOM, since $t = 2S$.

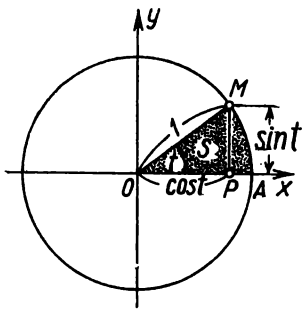


Fig. 82.

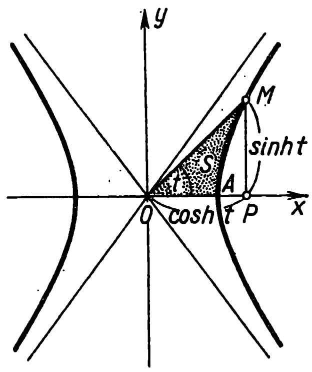


Fig. 83.

Let it be noted, without proof, that in the parametric equations of the hyperbola,

$$x = \cosh t,$$
$$y = \sinh t,$$

the parameter t is also numerically equal to the doubled area of the "hyperbolic sector" AOM (Fig. 83).

The derivatives of the hyperbolic functions are defined by the formulas

$$\left.\begin{aligned}(\sinh x)' = \cosh x, \quad (\tanh x)' = \frac{1}{\cosh^2 x}, \\ (\cosh x)' = \sinh x, \quad (\coth x)' = -\frac{1}{\sinh^2 x}\end{aligned}\right\} \qquad \text{(XXII)}$$

which follow from the very definition of hyperbolic functions; for instance, for the function $\sinh x = \frac{e^x - e^{-x}}{2}$ we have

$$(\sinh x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh x.$$

SEC. 20. THE DIFFERENTIAL

Let the function $y = f(x)$ be differentiable on the interval $[a, b]$. The derivative of this function at some point x of $[a, b]$ is determined by the equality

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x).$$

As $\Delta x \to 0$, the ratio $\frac{\Delta y}{\Delta x}$ approaches a definite number $f'(x)$ and, consequently, differs from the derivative $f'(x)$ by an infinitesimal:

$$\frac{\Delta y}{\Delta x} = f'(x) + \alpha,$$

where $\alpha \to 0$ as $\Delta x \to 0$.

Multiplying all terms of the latter equality by Δx, we get

$$\Delta y = f'(x)\,\Delta x + \alpha \Delta x. \qquad (1)$$

Since in the general case $f'(x) \neq 0$, for a constant x and a variable $\Delta x \to 0$, the product $f'(x)\,\Delta x$ is an infinitesimal of the first order relative to Δx. But the product $\alpha \Delta x$ is always an infinitesimal of higher order relative to Δx, because

$$\lim_{\Delta x \to 0} \frac{\alpha \Delta x}{\Delta x} = \lim_{\Delta x \to 0} \alpha = 0.$$

Thus, the increment Δy of the function consists of two terms, of which the first is [when $f'(x) \neq 0$] the so-called **principal part** of the increment, and is **linear** relative to Δx. The product $f'(x)\,\Delta x$ is called the **differential** of the function and is denoted by dy or $df(x)$ (read, dy or df of x).

And so if a function $y=f(x)$ has a derivative $f'(x)$ at the point x, the product of the derivative $f'(x)$ by the increment Δx of the argument is called the *differential* of the function and is denoted by the symbol dy:

$$dy=f'(x)\,\Delta x. \tag{2}$$

Find the differential of the function $y=x$; here,

$$y'=(x)'=1,$$

and, consequently, $dy=dx=\Delta x$ or $dx=\Delta x$. Thus, **the differential dx of the independent variable x coincides with its increment Δx.** The equality $dx=\Delta x$ might be regarded likewise as a definition of the differential of an independent variable, and then the foregoing example would indicate that this does not contradict the definition of the differential of a function. In any case, we can write formula (2) as

$$dy=f'(x)\,dx.$$

But from this relationship it follows that

$$f'(x)=\frac{dy}{dx}.$$

Hence, **the derivative $f'(x)$ may be regarded as the ratio of the differential of a function to the differential of the independent variable.**

Let us return to expression (1), which, taking (2) into account, may be rewritten thus:

$$\Delta y=dy+\alpha\Delta x. \tag{3}$$

Thus, the increment of a function differs from the differential of a function by an infinitesimal of higher order relative to Δx. If $f'(x)\neq 0$, then $\alpha\Delta x$ is an infinitesimal of higher order relative to dy and

$$\lim_{\Delta x\to 0}\frac{\Delta y}{dy}=1+\lim_{\Delta x\to 0}\frac{\alpha\Delta x}{f'(x)\,\Delta x}=1+\lim_{\Delta x\to 0}\frac{\alpha}{f'(x)}=1.$$

For this reason, in approximate calculations one sometimes uses the approximate equality

$$\Delta y\approx dy \tag{4}$$

or, in expanded form,

$$f(x+\Delta x)-f(x)\approx f'(x)\,\Delta x, \tag{5}$$

thus reducing the volume of computation.

Example 1. Find the differential dy and the increment Δy of the function $y=x^2$:

1) for arbitrary values of x and Δx;
2) for $x=20$, $\Delta x=0.1$.

Solution. 1) $\Delta y=(x+\Delta x)^2-x^2=2x\Delta x+\Delta x^2$,

$$dy=(x^2)'\,\Delta x=2x\Delta x.$$

2) If $x=20$, $\Delta x=0.1$, then $\Delta y=2\cdot 20\cdot 0.1+(0.1)^2=4.01$,

$$dy=2\cdot 20\cdot 0.1=4.00.$$

Replacing Δy by dy yields an error of 0.01. In many cases, it may be considered small compared to $\Delta y=4.01$ and therefore disregarded.

Fig. 84 gives a clear picture of the above problem.

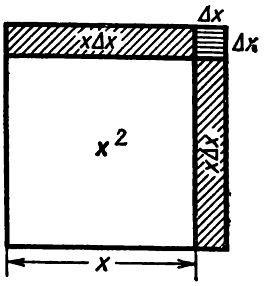


Fig. 84.

In approximate calculations, one also makes use of the following equality, which is obtained from (5):

$$f(x+\Delta x)\approx f(x)+f'(x)\,\Delta x. \qquad (6)$$

Example 2. Let $f(x)=\sin x$, then $f'(x)=\cos x$. In this case the approximate equality (6) takes the form

$$\sin(x+\Delta x)\approx\sin x+\cos x\,\Delta x. \qquad (7)$$

Let us calculate the approximate value of $\sin 46°$.

Put $x=45°=\frac{\pi}{4}$, $\Delta x=1°=\frac{\pi}{180}$, $46°=45°+1°=\frac{\pi}{4}+\frac{\pi}{180}$. Substituting into (7) we get

$$\sin 46°=\sin\left(\frac{\pi}{4}+\frac{\pi}{180}\right)=\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\,\frac{\pi}{180}$$

or

$$\sin 46°=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\,\frac{\pi}{180}=0.7071+0.7071\cdot 0.017=0.7194.$$

Example 3. If in (7) we put $x=0$, $\Delta x=\alpha$, we get the following approximate equality:

$$\sin\alpha\approx\alpha.$$

Example 4. If $f(x)=\tan x$, then by (6), we get the following approximate equality:

$$\tan(x+\Delta x)\approx\tan x+\frac{1}{\cos^2 x}\,\Delta x,$$

for $x=0$, $\Delta x=\alpha$, we get

$$\tan\alpha\approx\alpha.$$

Example 5. If $f(x)=\sqrt{x}$, then (6) yields

$$\sqrt{x+\Delta x}\approx\sqrt{x}+\frac{1}{2\sqrt{x}}\,\Delta x.$$

Putting $x=1$, $\Delta x=\alpha$, we get the approximate equality

$$\sqrt{1+\alpha}\approx 1+\frac{1}{2}\alpha.$$

The problem of finding the differential of a function is equivalent to that of finding the derivative, since, by multiplying the latter into the differential of the argument we get the differential of the function. Consequently, most theorems and formulas pertaining to derivatives are also valid for differentials. Let us illustrate this.

The differential of the sum of two differentiable functions u and v is equal to the sum of the differentials of these functions:

$$d(u+v)=du+dv.$$

The differential of the product of two differentiable functions u and v is determined by the formula

$$d(uv)=u\,dv+v\,du.$$

By way of illustration, let us prove the latter formula. If $y=uv$, then

$$dy=y'\,\Delta x=(uv'+vu')\,\Delta x=uv'\,\Delta x+vu'\,\Delta x,$$

but

$$v'\,\Delta x=dv,\quad u'\,\Delta x=du,$$

therefore

$$dy=u\,dv+v\,du.$$

Other formulas (for instance, the formula defining the differential of a quotient) are proved in similar fashion:

$$\text{if } y=\frac{u}{v},\text{ then } dy=\frac{v\,du-u\,dv}{v^2}.$$

Let us solve some problems dealing with calculating the differential of a function.

Example 6. $y=\tan^2 x$, $dy=2\tan x\,\frac{1}{\cos^2 x}\,dx$.

Example 7. $y=\sqrt{1+\ln x}$, $dy=\frac{1}{2\sqrt{1+\ln x}}\cdot\frac{1}{x}\,dx$.

We find the expression for the differential of a composite function. Let

$$y=f(u),\quad u=\varphi(x),\quad \text{or } y=f[\varphi(x)],$$

Then by the rule for the differentiation of a composite function,

$$\frac{dy}{dx}=f'_u(u)\,\varphi'(x).$$

Hence,

$$dy=f'_u(u)\,\varphi'(x)\,dx,$$

but $\varphi'(x)\,dx=du$, therefore

$$dy=f'(u)\,du.$$

Thus, *the differential of a composite function has the same form as it would have if the intermediate argument were the independent variable*. In other words, *the form of the differential does not depend on whether the argument of a function is an independent variable or a function of another argument*. This important property of a differential, called *invariance of the form of the differential*, will be widely used later on.

Example 8. Given a function $y=\sin\sqrt{x}$. Find dy. **Solution.** Representing the given function as a composite one:

$$y=\sin u,\quad u=\sqrt{x},$$

we find

$$dy=\cos u\,\frac{1}{2\sqrt{x}}\,dx;$$

but $\frac{1}{2\sqrt{x}}\,dx=du$, so we can write

$$dy=\cos u\,du$$

or

$$dy=\cos(\sqrt{x})\,d(\sqrt{x}).$$

SEC. 21. THE GEOMETRIC SIGNIFICANCE OF THE DIFFERENTIAL

Let us consider the function

$$y=f(x)$$

and the curve it represents (Fig. 85).

On the curve $y=f(x)$, take an arbitrary point $M(x, y)$, draw a line tangent to the curve at this point and denote by α the angle *) which the line tangent forms with the positive direction of the x-axis. Increase the independent variable by Δx; then the function

*) Assuming that the function $f(x)$ has a finite derivative at the point x, we get $\alpha\neq\frac{\pi}{2}$.

will change by $\Delta y = NM_1$. To the values $x+\Delta x$, $y+\Delta y$ on the curve $y=f(x)$ there will correspond the point $M_1(x+\Delta x, y+\Delta y)$.
From the triangle MNT we find

$$NT = MN \tan\alpha;$$

since

$$\tan\alpha = f'(x), \quad MN = \Delta x,$$

we get

$$NT = f'(x)\,\Delta x;$$

but by the definition of a differential $f'(x)\,\Delta x = dy$. Thus,

$$NT = dy.$$

The latter equality signifies that *the differential of a function $f(x)$, which corresponds to the given values x and Δx, is equal to the*

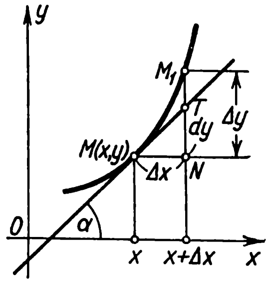


Fig. 85.

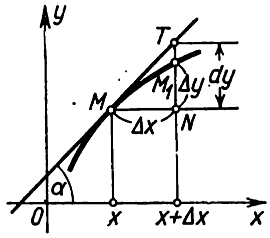


Fig. 86.

increment in the ordinate of the line tangent to the curve $y=f(x)$ at the given point x.
From Fig. 85 it follows directly that

$$M_1T = \Delta y - dy.$$

By what has already been proved, $\frac{M_1T}{NT} \to 0$ as $\Delta x \to 0$.

One should not think that the increment Δy is always greater than dy. For instance, in Fig. 86,

$$\Delta y = M_1N, \quad dy = NT, \quad \text{and} \quad \Delta y < dy.$$

SEC. 22. DERIVATIVES OF DIFFERENT ORDERS

Let a function $y=f(x)$ be differentiable on some interval $[a, b]$. Generally speaking, the values of the derivative $f'(x)$ depend on x, which is to say that the **derivative $f'(x)$ is also a function of x.**

Differentiating this function, we obtain the so-called second derivative of the function $f(x)$.

The derivative of a first derivative is called a *derivative of the second order* or the *second derivative* of the original function and is denoted by the symbol y'' or $f''(x)$:

$$y'' = (y')' = f''(x).$$

For example, if $y = x^5$, then

$$y' = 5x^4;\ y'' = (5x^4)' = 20x^3.$$

The derivative of the second derivative is called a *derivative of the third order* or the *third derivative* and is denoted by y''' or $f'''(x)$.

Generally, a *derivative of the nth order of a function* $f(x)$ is called the derivative (first-order) of the derivative of the $(n-1)$ st order and is denoted by the symbol $y^{(n)}$ or $f^{(n)}(x)$:

$$y^{(n)} = (y^{(n-1)})' = f^{(n)}(x).$$

(The order of the derivative is taken in parentheses so as to avoid confusion with the exponent of a power.)

Derivatives of the fourth, fifth, and higher orders are also denoted by Roman numerals: y^{IV}, y^{V}, y^{VI}, ... Here, the order of the derivative may be written without brackets. For instance, if $y = x^5$, then $y' = 5x^4$, $y'' = 20x^3$, $y''' = 60x^2$, $y^{IV} = y^{(4)} = 120x$, $y^{V} = y^{(5)} = 120$, $y^{(6)} = y^{(7)} = \ldots = 0$.

Example 1. Given a function $y = e^{kx}$ ($k = \text{const}$). Find the expression of its derivative of any order n.

Solution. $y' = ke^{kx}$, $y'' = k^2e^{kx}$, ..., $y^{(n)} = k^ne^{kx}$.

Example 2. $y = \sin x$. Find $y^{(n)}$.

Solution.

$$y' = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$y'' = -\sin x = \sin\left(x + 2\frac{\pi}{2}\right),$$

$$y''' = -\cos x = \sin\left(x + 3\frac{\pi}{2}\right),$$

$$y^{VI} = \sin x = \sin\left(x + 4\frac{\pi}{2}\right),$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$y^{(n)} = \sin\left(x + n\frac{\pi}{2}\right).$$

In similar fashion we can also derive the formulas for the derivatives of any order of certain other elementary functions. The reader himself can find the formulas for derivatives of the nth order of the functions $y=x^n$, $y=\cos x$, $y=\ln x$.

The rules given in theorems 2 and 3, Sec. 7, are readily generalised to the case of derivatives of any order.

In this case we have obvious formulas:

$$(u+v)^{(n)}=u^{(n)}+v^{(n)}, \quad (Cu)^{(n)}=Cu^{(n)}.$$

Let us derive a formula (called the Leibniz rule, or formula) that will enable us to calculate the nth derivative of the product of two functions $u(x)\ v(x)$. To obtain this formula, let us first find several derivatives and then establish the general rule for finding the derivative of any order:

$$\begin{aligned}
y &= uv,\\
y' &= u'v+uv',\\
y'' &= u''v+u'v'+u'v'+uv''=u''v+2u'v'+uv'',\\
y''' &= u'''v+u''v'+2u''v'+2u'v''+u'v''+uv'''=\\
&= u'''v+3u''v'+3u'v''+uv''',\\
y^{IV} &= u^{IV}v+4u'''v'+6u''v''+4u'v'''+uv^{IV}.
\end{aligned}$$

The rule for forming derivatives holds for the derivative of any order and obviously consists in the following.

The expression $(u+v)^n$ is expanded by the binomial theorem, and in the expansion obtained the exponents of the powers of u and v are replaced by indices that are the order of the derivatives, and the zero powers ($u^0=v^0=1$) in the end terms of the expansion are replaced by the functions themselves (that is, "derivatives of zero order"):

$$y^{(n)}=(uv)^{(n)}=u^{(n)}v+nu^{(n-1)}v'+\frac{n(n-1)}{1\cdot 2}u^{(n-2)}v''+\ldots+uv^{(n)}.$$

This is the *Leibniz rule.*

A rigorous proof of this formula may be performed by the method of complete mathematical induction (in other words, to prove that if this formula holds for the nth order it will hold for the order $n+1$).

Example 3. $y=e^{ax}x^2$. Find the derivative of $y^{(n)}$.

Solution.

$$\begin{aligned}
u &= e^{ax}, & v &= x^2,\\
u' &= ae^{ax}, & v' &= 2x,
\end{aligned}$$

$$u'' = a^2e^{ax}, \qquad v'' = 2,$$

$$\cdot \cdot \cdot \cdot \cdot \qquad \cdot \cdot \cdot \cdot$$

$$u^n = a^n e^{ax}, \qquad v''' = v^{IV} = \ldots = 0,$$

$$y^{(n)} = a^n e^{ax} x^2 + na^{n-1}e^{ax} \cdot 2x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}e^{ax} \cdot 2$$

or

$$y^{(n)} = e^{ax}[a^n x^2 + 2na^{n-1}x + n(n-1)a^{n-2}].$$

SEC. 23. DIFFERENTIALS OF VARIOUS ORDERS

Suppose we have a function $y = f(x)$, where x is the independent variable. The differential of this function

$$dy = f'(x)\,dx$$

is some function of x, but only the first factor, $f'(x)$, can depend on x; the second factor, (dx), is an increment of the independent variable x and is independent of the value of this variable. Since dy is a function of x we have the right to speak of the differential of this function.

The differential of the differential of a function is called the *second differential* or the *second-order differential* of this function and is denoted by d^2y:

$$d(dy) = d^2y.$$

Let us find the expression for the second differential. By virtue of the general definition of a differential we have

$$d^2y = [f'(x)\,dx]'\,dx.$$

Since dx is independent of x, dx is taken outside the sign of the derivative upon differentiation, and we get

$$d^2y = f''(x)(dx)^2.$$

When writing the degree of a differential it is common to drop the brackets; in place of $(dx)^2$ we write dx^2 to mean the square of the expression dx; in place of $(dx)^3$ we write dx^3, etc.

The *third differential* or the *third-order differential* of a function is the differential of its second differential:

$$d^3y = d(d^2y) = [f''(x)\,dx^2]'\,dx = f'''(x)\,dx^3.$$

Generally, the *nth differential* is the first differential of a differential of the $(n-1)$st order:

$$d^ny = d(d^{n-1}y) = [f^{(n-1)}(x)\,dx^{n-1}]'\,dx, \quad d^ny = f^{(n)}(x)\,dx^n. \qquad (1)$$

Using differentials of different orders, the derivative of any order may be represented as a ratio of differentials of the appropriate order:

$$f'(x)=\frac{dy}{dx};\quad f''(x)=\frac{d^2y}{dx^2},\ \ldots,\ f^{(n)}(x)=\frac{d^ny}{dx^n}. \tag{2}$$

It should, however, be noted that equalities (1) and (2) (for $n>1$) hold only for the case when x is an independent variable.*)

SEC. 24. DIFFERENT-ORDER DERIVATIVES OF IMPLICIT FUNCTIONS AND OF FUNCTIONS REPRESENTED PARAMETRICALLY

1. An example will illustrate the finding of derivatives of different orders of *implicit functions.*

Let an implicit function y of x be defined by the equality

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0. \tag{1}$$

Differentiate, with respect to x, all terms of the equation and remember that y is a function of x:

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0;$$

from this we get

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}. \tag{2}$$

Again differentiate this equality with respect to x (having in view that y is a function of x):

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\cdot\frac{y-x\dfrac{dy}{dx}}{y^2}.$$

Substituting, in place of the derivative $\frac{dy}{dx}$, its expression from (2), we get

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\cdot\frac{y+x\dfrac{b^2}{a^2}\dfrac{x}{y}}{y^2},$$

or, after simplifying,

$$\frac{d^2y}{dx^2}=-\frac{b^2(a^2y^2+b^2x^2)}{a^4y^3}.$$

* Nevertheless, we shall also write equality (2) when x is not an independent variable; but in this case, the expression $\frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}$ should be regarded as symbols of derivatives.

From equation (1) it follows that

$$a^2y^2+b^2x^2=a^2b^2;$$

therefore the second derivative may be represented as

$$\frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}.$$

Differentiating the latter equation with respect to x, we find $\frac{d^3y}{dx^3}$, etc.

2. Let us now consider the problem of finding the derivatives of higher orders **of a function represented parametrically.**

Let the function y of x be represented by parametric equations

$$\left.\begin{aligned} x&=\varphi(t),\\ y&=\psi(t),\end{aligned}\right\}\ t_0\leqslant t\leqslant T; \tag{3}$$

the function $x=\varphi(t)$ has an inverse function $t=t(x)$ on the interval $[t_0, T]$.

In Sec. 18 it was proved that in this case the derivative $\frac{dy}{dx}$ is defined by the equation

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{4}$$

To find the second derivative, $\frac{d^2y}{dx^2}$, differentiate (4) with respect to x, bearing in mind that t is a function of x:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx}, \tag{5}$$

but

$$\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{\frac{dx}{dt}\frac{d}{dt}\left(\frac{dy}{dt}\right)-\frac{dy}{dt}\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2},$$

$$\frac{dt}{dx}=\frac{1}{\frac{dx}{dt}}.$$

Substituting the latter expressions into (5), we get

$$\frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}.$$

This formula may be written in more compact form as follows:

$$\frac{d^2y}{dx^2}=\frac{\varphi'(t)\psi''(t)-\psi'(t)\varphi''(t)}{[\varphi'(t)]^3}.$$

In similar fashion we can find the derivatives

$$\frac{d^3y}{dx^3},\quad \frac{d^4y}{dx^4}$$

and so forth.

Example. A function y of x is represented parametrically:

$$x=a\cos t,\quad y=b\sin t.$$

Find the derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$.

Solution.

$$\frac{dx}{dt}=-a\sin t;\quad \frac{d^2x}{dt^2}=-a\cos t;$$

$$\frac{dy}{dt}=b\cos t;\quad \frac{d^2y}{dt^2}=-b\sin t;$$

$$\frac{dy}{dx}=\frac{b\cos t}{-a\sin t}=-\frac{b}{a}\cot t;$$

$$\frac{d^2y}{dx^2}=\frac{(-a\sin t)(-b\sin t)-(b\cos t)(-a\cos t)}{(-a\sin t)^3}=-\frac{b}{a^2}\frac{1}{\sin^3 t}.$$

SEC. 25. THE MECHANICAL SIGNIFICANCE OF THE SECOND DERIVATIVE

Let s be the path covered by a body under translation as a function of the time; it is expressed as

$$s=f(t). \tag{1}$$

As we already know (see Sec. 1, Ch. III), the velocity v of a body at any time is equal to the first derivative of the path with respect to time:

$$v=\frac{ds}{dt}. \tag{2}$$

At some time t, let the velocity of the body be v. If the motion is not uniform, then during an interval of time Δt that has elapsed since t the velocity will change by the increment Δv.

The *average acceleration* during time Δt is the ratio of the increment in velocity Δv to the increment in time:

$$a_{av} = \frac{\Delta v}{\Delta t}.$$

Acceleration at a given instant is the limit of the ratio of the increment in velocity to the increment in time as the latter approaches zero:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t};$$

in other words, acceleration (at a given instant) is equal to the derivative of the velocity with respect to time:

$$a = \frac{dv}{dt},$$

but since $v = \frac{ds}{dt}$, consequently,

$$a = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2},$$

or *the acceleration of linear motion is equal to the second derivative of the path covered with respect to time.* Reverting to equation (1), we get

$$a = f''(t).$$

Example. Find the velocity v and the acceleration a of a freely falling body, if the dependence of distance s upon time t is given by the formula

$$s = \frac{1}{2} gt^2 + v_0 t + s_0 \tag{3}$$

where $g = 9.8$ m/sec^2 is the acceleration of gravity, and $s_0 = s_{t=0}$ is the value of s at $t = 0$.

Solution. Differentiating, we find

$$v = \frac{ds}{dt} = gt + v_0; \tag{4}$$

from this formula it follows that $v_0 = (v)_{t=0}$.

Differentiating again, we find

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = g.$$

Let it be noted that, conversely, if the acceleration of some motion is constant and equal to g, the velocity will be expressed by equation (4), and the distance by equation (3) provided that $(v)_{t=0} = v_0$ and $(s)_{t=0} = s_0$.

SEC. 26. THE EQUATIONS OF A TANGENT AND OF A NORMAL. THE LENGTHS OF THE SUBTANGENT AND THE SUBNORMAL

Let us consider a curve whose equation is

$$y=f(x).$$

On this curve take a point $M(x_1, y_1)$ (Fig. 87) and write the equation of the tangent line to the given curve at the point M, assuming that this tangent is not parallel to the axis of ordinates.

The equation of a straight line with slope k passing through the point M is of the form

$$y-y_1=k(x-x_1).$$

For the tangent line (see Sec. 3)

$$k=f'(x_1),$$

and so the **equation of the tangent** is of the form

$$y-y_1=f'(x_1)(x-x_1).$$

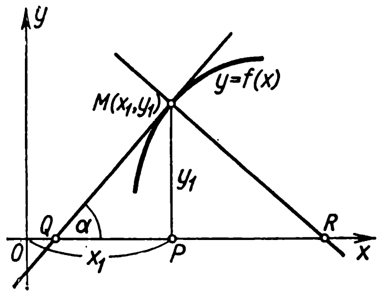


Fig. 87.

In addition to the tangent to a curve at a given point, one often has to consider the normal.

Definition. The *normal* to a curve at a given point is a straight line passing through the given point perpendicular to the tangent at this point.

From the definition of a normal it follows that its slope k_n is connected with the slope k_t of the tangent by the equation

$$k_n=-\frac{1}{k_t}$$

or

$$k_n=-\frac{1}{f'(x_1)}.$$

Hence, the **equation of a normal** to a curve $y=f(x)$ at a point $M(x_1, y_1)$ is of the form

$$y-y_1=-\frac{1}{f'(x_1)}(x-x_1).$$

Example 1. Write the equations of a tangent and a normal to the curve $y=x^3$ at the point M (1, 1).

Solution. Since $y'=3x^2$, the slope of the tangent is $(y')_{x=1}=3$. Therefore, the equation of the tangent is

$$y-1=3(x-1) \text{ or } y=3x-2.$$

The equation of the normal is

$$y-1=-\frac{1}{3}(x-1)$$

or

$$y=-\frac{1}{3}x+\frac{4}{3}$$

(see Fig. 88).

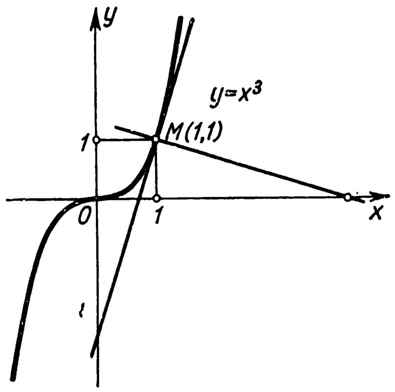


Fig 88.

The length T of the segment QM (Fig. 87) of the tangent between the point of tangency and the x-axis is called the *length of the tangent*. The projection of this segment on the x-axis, that is, QP, is called the *subtangent*; the lenght of the subtangent is denoted by S_T. The length N of the segment MR is called the *length of the normal*, while the projection RP of the segment RM on the x-axis is called the *subnormal*; the length of the subnormal is denoted by S_N.

Let us find the quantities T, S_T, N, S_N for the curve $y=f(x)$ and the point $M(x_1, y_1)$.

From Fig. 87 it will be seen that

$$QP=y_1\cot\alpha=\frac{y_1}{\tan\alpha}=\frac{y_1}{y_1'},$$

therefore

$$S_T=\left|\frac{y_1}{y_1'}\right|,$$

$$T=\sqrt{y_1^2+\frac{y_1^2}{y_1'^2}}=\left|\frac{y_1}{y_1'}\sqrt{y_1'^2+1}\right|.$$

It is further clear from this same figure that

$$PR=y_1\tan\alpha=y_1y_1',$$

and so

$$S_N=|y_1y_1'|,$$

$$N=\sqrt{y_1^2+(y_1y_1')^2}=\left|y_1\sqrt{1+y_1'^2}\right|.$$

These formulas are derived on the assumption that $y_1>0$, $y_1'>0$. However, they hold in the general case as well.

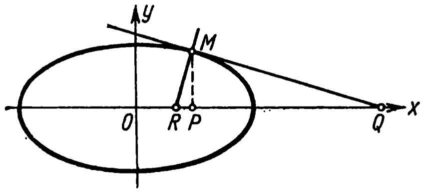


Fig. 89.

Example 2. Find the equations of the tangent and normal, the lengths of the tangent and the subtangent, the lengths of the normal and subnormal for the ellipse

$$x = a\cos t,\ y = b\sin t \qquad (1)$$

at the point $M(x_1, y_1)$ for which $t=\frac{\pi}{4}$ (Fig. 89).

Solution. From equations (1) we find

$$\frac{dx}{dt} = -a\sin t;\quad \frac{dy}{dt} = b\cos t;\quad \frac{dy}{dx} = -\frac{b}{a}\cot t;\quad \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = -\frac{b}{a}\,.$$

We find the coordinates of the point of tangency of M:

$$x_1 = (x)_{t=\frac{\pi}{4}} = \frac{a}{\sqrt{2}},\quad y_1 = (y)_{t=\frac{\pi}{4}} = \frac{b}{\sqrt{2}}.$$

The equation of the tangent is

$$y - \frac{b}{\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right)$$

or

$$bx + ay - ab\sqrt{2} = 0.$$

The equation of the normal is

$$y - \frac{b}{\sqrt{2}} = \frac{a}{b}\left(x - \frac{a}{\sqrt{2}}\right)$$

or

$$(ax - by)\sqrt{2} - a^2 + b^2 = 0.$$

The lengths of the subtangent and subnormal are

$$S_T = \left|\frac{\frac{b}{\sqrt{2}}}{-\frac{b}{a}}\right| = \frac{a}{\sqrt{2}}\,.$$

$$S_N = \left|\frac{b}{\sqrt{2}}\left(-\frac{b}{a}\right)\right| = \frac{b^2}{a\sqrt{2}}\,.$$

The lengths of the tangent and the normal are

$$T = \left|\frac{\frac{b}{\sqrt{2}}}{-\frac{b}{a}}\sqrt{\left(-\frac{b}{a}\right)^2 + 1}\right| = \frac{1}{\sqrt{2}}\sqrt{a^2 + b^2};$$

$$N = \left|\frac{b}{\sqrt{2}}\sqrt{1 + \left(-\frac{b}{a}\right)^2}\right| = \frac{b}{a\sqrt{2}}\sqrt{a^2 + b^2}.$$

SEC. 27. THE GEOMETRIC SIGNIFICANCE OF THE DERIVATIVE OF THE RADIUS VECTOR WITH RESPECT TO THE POLAR ANGLE

We have the following equation of a curve in polar coordinates:

$$\varrho = f(\theta). \qquad (1)$$

Let us write the formulas for changing from polar coordinates to rectangular Cartesian coordinates:

$$x = \varrho \cos \theta, \quad y = \varrho \sin \theta.$$

Substituting, in place of ϱ, its expression in terms of θ from equation (1), we get

$$\left.\begin{aligned} x &= f(\theta) \cos \theta, \\ y &= f(\theta) \sin \theta. \end{aligned}\right\} \qquad (2)$$

Equations (2) are parametric equations of the given curve, the parameter being the polar angle θ (Fig. 90).

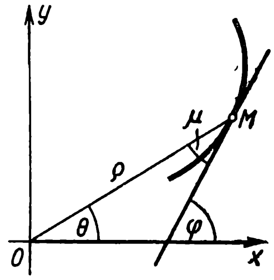


Fig. 90.

If we denote by φ the angle formed by the tangent to the curve at some point $M(\varrho, \theta)$ with the positive direction of the x-axis, we will have

$$\tan \varphi = \frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}$$

or

$$\tan \varphi = \frac{\dfrac{d\varrho}{d\theta} \sin \theta + \varrho \cos \theta}{\dfrac{d\varrho}{d\theta} \cos \theta - \varrho \sin \theta}. \qquad (3)$$

Denote by μ the angle between the direction of the radius vector and the tangent. It is obvious that $\mu = \varphi - \theta$,

$$\tan \mu = \frac{\tan \varphi - \tan \theta}{1 + \tan \varphi \tan \theta}.$$

Substituting, in place of $\tan \varphi$, its expression (3) and making the necessary changes, we get

$$\tan \mu = \frac{(\varrho' \sin \theta + \varrho \cos \theta) \cos \theta - (\varrho' \cos \theta - \varrho \sin \theta) \sin \theta}{(\varrho' \cos \theta - \varrho \sin \theta) \cos \theta + (\varrho' \sin \theta + \varrho \cos \theta) \sin \theta} = \frac{\varrho}{\varrho'}$$

or

$$\varrho'_\theta = \varrho \cot \mu. \qquad (4)$$

Thus, the derivative of the radius vector with respect to the polar angle is equal to the length of the radius vector multiplied by the cotangent of the angle between the radius vector and the tangent to the curve at the given point.

Example. To show that the tangent to the logarithmic spiral

$$\rho = e^{a\theta}$$

intersects the radius vector at a constant angle.

Solution. From the equation of the spiral we get

$$\rho' = ae^{a\theta}.$$

From formula (4) we have

$$\cot \mu = \frac{\rho'}{\rho} = a; \text{ that is, } \mu = \operatorname{arc} \cot a = \text{const.}$$

Exercises on Chapter III

Find the derivatives of functions using the definition of a derivative. **1.** $y = x^3$. *Ans.* $3x^2$. **2.** $y = \frac{1}{x}$. *Ans.* $-\frac{1}{x^2}$. **3.** $y = \sqrt{x}$. *Ans.* $\frac{1}{2\sqrt{x}}$. **4.** $y = \frac{1}{\sqrt{x}}$. *Ans.* $-\frac{1}{2x\sqrt{x}}$. **5.** $y = \sin^2 x$. *Ans.* $2 \sin x \cos x$. **6.** $y = 2x^2 - x$. *Ans.* $4x - 1$.

Determine the tangents of the angles of inclination of tangents to the curves: **7.** $y = x^3$. a) When $x = 1$. *Ans.* 3. b) When $x = -1$. *Ans.* 3. Make a drawing. **8.** $y = \frac{1}{x}$. a) When $x = \frac{1}{2}$. *Ans.* -4. b) When $x = 1$. *Ans.* -1. Make a drawing. **9.** $y = \sqrt{x}$ when $x = 2$. *Ans.* $\frac{1}{2\sqrt{2}}$.

Find the derivatives of the functions: **10.** $y = x^4 + 3x^2 - 6$. *Ans.* $y' = 4x^3 + 6x$. **11.** $y = 6x^3 - x^2$. *Ans.* $y' = 18x^2 - 2x$. **12.** $y = \frac{x^5}{a+b} - \frac{x^2}{a-b} - x$. *Ans.* $y' = \frac{5x^4}{a+b} - \frac{2x}{a-b} - 1$. **13.** $y = \frac{x^3 - x^2 + 1}{5}$. *Ans.* $y' = \frac{3x^2 - 2x}{5}$. **14.** $y = 2ax^3 - \frac{x^2}{b} + c$. *Ans.* $y' = 6ax^2 - \frac{2x}{b}$. **15.** $y = 6x^{\frac{7}{2}} + 4x^{\frac{5}{2}} + 2x$. *Ans.* $y' = 21x^{\frac{5}{2}} + 10x^{\frac{3}{2}} + 2$. **16.** $y = \sqrt{3x} + \sqrt[3]{x} + \frac{1}{x}$. *Ans.* $y' = \frac{\sqrt{3}}{2\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}} - \frac{1}{x^2}$.

17. $y = \frac{(x+1)^3}{x^{\frac{3}{2}}}$. *Ans.* $y' = \frac{3(x+1)^2(x-1)}{2x^{\frac{5}{2}}}$. **18.** $y = \frac{x}{m} + \frac{m}{x} + \frac{x^2}{n^2} + \frac{n^2}{x^2}$.

Ans. $y'=\frac{1}{m}-\frac{m}{x^2}+\frac{2x}{n^2}-\frac{2n^2}{x^3}$. **19.** $y=\sqrt[3]{x^2}-2\sqrt{x}+5$. *Ans.* $y'=\frac{2}{3}\frac{1}{\sqrt[3]{x}}-$ $-\frac{1}{\sqrt{x}}$. **20.** $y=\frac{ax^2}{\sqrt[3]{x}}+\frac{b}{x\sqrt{x}}-\frac{\sqrt[3]{x}}{\sqrt{x}}$. *Ans.* $y'=\frac{5}{3}ax^{\frac{2}{3}}-\frac{3}{2}bx^{-\frac{5}{2}}+$ $+\frac{1}{6}x^{-\frac{7}{6}}$.

21. $y=(1+4x^3)(1+2x^2)$. *Ans.* $y'=4x(1+3x+10x^3)$. **22.** $y=x(2x-1)(3x+2)$. *Ans.* $y'=2(9x^2+x-1)$. **23.** $y=(2x-1)(x^2-6x+3)$. *Ans.* $y'=6x^2-26x+12$. **24.** $y=\frac{2x^4}{b^2-x^2}$. *Ans.* $y'=\frac{4x^3(2b^2-x^2)}{(b^2-x^2)^2}$. **25.** $y=\frac{a-x}{a+x}$. *Ans.* $y'=-\frac{2a}{(a+x)^2}$.

26. $f(t)=\frac{t^3}{1+t^2}$. *Ans.* $f'(t)=\frac{t^2(3+t^2)}{(1+t^2)^2}$. **27.** $f(s)=\frac{(s+4)^2}{s+3}$. *Ans.* $f'(s)=$ $=\frac{(s+2)(s+4)}{(s+3)^2}$. **28.** $y=\frac{x^3+1}{x^2-x-2}$. *Ans.* $y'=\frac{x^4-2x^3-6x^2-2x+1}{(x^2-x-2)^2}$.

29. $y=\frac{x^p}{x^m-a^m}$. *Ans.* $y'=\frac{x^{p-1}[(p-m)x^m-pa^m]}{(x^m-a^m)^2}$. **30.** $y=(2x^2-3)^2$. *Ans.* $y'=8x(2x^2-3)$. **31.** $y=(x^2+a^2)^5$. *Ans.* $y=10x(x^2+a^2)^4$. **32.** $y=$ $=\sqrt{x^2+a^2}$. *Ans.* $y'=\frac{x}{\sqrt{x^2+a^2}}$. **33.** $y=(a+x)\sqrt{a-x}$. *Ans.* $y'=\frac{a-3x}{2\sqrt{a-x}}$.

34. $y=\sqrt{\frac{1+x}{1-x}}$. *Ans.* $y'=\frac{1}{(1-x)\sqrt{1-x^2}}$. **35.** $y=\frac{2x^2-1}{x\sqrt{1+x^2}}$. *Ans.* $y'=$ $=\frac{1+4x^2}{x^2(1+x^2)^{\frac{3}{2}}}$. **36.** $y=\sqrt[3]{x^2+x+1}$. *Ans.* $y'=\frac{2x+1}{3\sqrt[3]{(x^2+x+1)^2}}$. **37.** $y=$ $=(1+\sqrt[3]{x})^3$. *Ans.* $y'=\left(1+\frac{1}{\sqrt[3]{x}}\right)^2$. **38.** $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$. *Ans.* $y'=$ $=\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\left[1+\frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)\right]$. **39.** $y=\sin^2 x$. *Ans.* $y'=\sin 2x$. **40.** $y=2\sin x+\cos 3x$. *Ans.* $y'=2\cos x-3\sin 3x$. **41.** $y=$ $=\tan(ax+b)$. *Ans.* $y'=\frac{a}{\cos^2(ax+b)}$. **42.** $y=\frac{\sin x}{1+\cos x}$. *Ans.* $y'=\frac{1}{1+\cos x}$. **43.** $y=\sin 2x\cdot\cos 3x$. *Ans.* $y'=2\cos 2x\cos 3x-3\sin 2x\sin 3x$. **44.** $y=\cot^2 5x$. *Ans.* $y'=-10\cot 5x\csc^2 5x$. **45.** $y=t\sin t+\cos t$. *Ans.* $y'=t\cos t$. **46.** $y=$ $=\sin^3 t\cos t$. *Ans.* $y'=\sin^2 t(3\cos^2 t-\sin^2 t)$. **47.** $y=a\sqrt{\cos 2x}$. *Ans.* $y'=$ $=-\frac{a\sin 2x}{\sqrt{\cos 2x}}$. **48.** $r=a\sin^3\frac{\varphi}{3}$ *Ans.* $r'_{\varphi}=a\sin^2\frac{\varphi}{3}\cos\frac{\varphi}{3}$. **49.** $y=\frac{\tan\frac{x}{2}+\cot\frac{x}{2}}{x}$.

Ans. $y' = -\frac{2x\cos x + \sin^2 x\left(\tan\frac{x}{2}+\cot\frac{x}{2}\right)}{x^2\sin^2 x}$. 50. $y = a\left(1-\cos^2\frac{x}{2}\right)^2$. *Ans.* $y' = 2a\sin^3\frac{x}{2}\cos\frac{x}{2}$. 51. $y=\frac{1}{2}\tan^2 x$. *Ans.* $y'=\tan x\sec^2 x$. 52. $y=\ln\cos x$. *Ans.* $y'=-\tan x$. 53. $y=\ln\tan x$. *Ans.* $y'=\frac{2}{\sin 2x}$. 54. $y=\ln\sin^2 x$. *Ans.* $y'=2\cot x$. 55. $y=\frac{\tan x-1}{\sec x}$. *Ans.* $y'=\sin x+\cos x$. 56. $y=\ln\sqrt{\frac{1+\sin x}{1-\sin x}}$. *Ans.* $y'=\frac{1}{\cos x}$. 57. $y=\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$. *Ans.* $y'=\frac{1}{\cos x}$. 58. $y=\sin(x+a)\times\cos(x+a)$. *Ans.* $y'=\cos 2(x+a)$. 59. $f(x)=\sin(\ln x)$. *Ans.* $f'(x)=\frac{\cos(\ln x)}{x}$. 60. $f(x)=\tan(\ln x)$. *Ans.* $f'(x)=\frac{\sec^2(\ln x)}{x}$. 61 $f(x)=\sin(\cos x)$. *Ans.* $f'(x)=-\sin x\cos(\cos x)$. 62. $r=\frac{1}{3}\tan^3\varphi-\tan\varphi+\varphi$. *Ans.* $\frac{dr}{d\varphi}=\tan^4\varphi$. 63. $f(x)=(x\cot x)^2$. *Ans.* $f'(x)=2x\cot x(\cot x - x\csc^2 x)$. 64. $y=\ln(ax+b)$. *Ans.* $y'=\frac{a}{ax+b}$. 65. $y=\log_a(x^2+1)$. *Ans.* $y'=\frac{2x}{(x^2+1)\ln a}$. 66. $y=\ln\frac{1+x}{1-x}$. *Ans.* $y'=\frac{2}{1-x^2}$. 67. $y=\log_3(x^2-\sin x)$. *Ans.* $y'=\frac{2x-\cos x}{(x^2-\sin x)\ln 3}$.

68. $y=\ln\frac{1+x^2}{1-x^2}$. *Ans.* $y'=\frac{4x}{1-x^4}$. 69. $y=\ln(x^2+x)$. *Ans.* $y'=\frac{2x+1}{x^2+x}$.

70. $y=\ln(x^3-2x+5)$. *Ans.* $y'=\frac{3x^2-2}{x^3-2x+5}$. 71. $y=x\ln x$. *Ans.* $y'=\ln x+1$

72. $y=\ln^3 x$. *Ans.* $y'=\frac{3\ln^2 x}{x}$. 73. $y=\ln(x+\sqrt{1+x^2})$. *Ans.* $y'=\frac{1}{\sqrt{1+x^2}}$

74. $y=\ln(\ln x)$. *Ans.* $y'=\frac{1}{x\ln x}$. 75. $f(x)=\ln\sqrt{\frac{1+x}{1-x}}$. *Ans.* $f'(x)=\frac{1}{1-x^2}$.

76. $f(x)=\ln\frac{\sqrt{x^2+1}-x}{\sqrt{x^2-1}+x}$. *Ans.* $f'(x)=-\frac{2}{\sqrt{1+x^2}}$. 77. $y=\sqrt{a^2+x^2}-a\ln\frac{a+\sqrt{a^2+x^2}}{x}$. *Ans.* $y'=\frac{\sqrt{a^2+x^2}}{x}$. 78. $y=\ln(x+\sqrt{x^2+a^2})-\frac{\sqrt{x^2+a^2}}{x}$. *Ans.* $y'=\frac{\sqrt{x^2+a^2}}{x^2}$. 79. $y=-\frac{\cos x}{2\sin^2 x}+\frac{1}{2}\ln\tan\frac{x}{2}$. *Ans.* $y'=\frac{1}{\sin^3 x}$.

80. $y=\frac{\sin x}{2\cos^2 x}$. *Ans.* $y'=\frac{1+\sin^2 x}{2\cos^3 x}$. 81. $y=\frac{1}{2}\tan^2 x+\ln\cos x$. *Ans.* $y'=\tan^3 x$. 82. $y=e^{ax}$. *Ans.* $y'=ae^{ax}$. 83. $y=e^{4x+5}$. *Ans.* $y'=4e^{4x+5}$.

84. $y=a^{x^2}$. *Ans.* $2x a^{x^2}\ln a$. 85. $y=7^{x^2+2x}$. *Ans.* $y'=2(x+1)\,7^{x^2+2x}\ln 7$.

86. $y=c^{a^2-x^2}$. *Ans.* $y'=-2xc^{a^2-x^2}\ln c$. 87. $y=ae^{\sqrt{x}}$. *Ans.* $y'=\frac{a}{2\sqrt{x}}e^{\sqrt{x}}$.

88. $r=a^{\theta}$. *Ans.* $r'=a^{\theta}\ln a$. 89. $r=a^{\ln\theta}$. *Ans.* $\frac{dr}{d\theta}=\frac{a^{\ln\theta}\ln a}{\theta}=\theta^{\ln a-1}\ln a$.

90. $y=e^x(1-x^2)$. *Ans.* $y'=e^x(1-2x-x^2)$. 91. $y=\frac{e^x-1}{e^x+1}$. *Ans.* $y'=\frac{2e^x}{(e^x+1)^2}$.

92. $y=\ln\frac{e^x}{1+e^x}$. *Ans.* $y'=\frac{1}{1+e^x}$. 93. $y=\frac{a}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)$. *Ans.* $y'=\frac{1}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$. 94. $y=e^{\sin x}$. *Ans.* $y'=e^{\sin x}\cos x$. 95. $y=a^{\tan nx}$. *Ans.* $y'=na^{\tan nx}\sec^2 nx\ln a$. 96. $y=e^{\cos x}\sin x$. *Ans.* $y'=e^{\cos x}(\cos x-\sin^2 x)$.
97. $y=e^x\ln\sin x$. *Ans.* $y'=e^x(\cot x+\ln\sin x)$. 98. $y=x^n e^{\sin x}$. *Ans.* $y'=x^{n-1}e^{\sin x}(n+x\cos x)$. 99. $y=x^x$. *Ans.* $y'=x^x(\ln x+1)$. 100. $y=x^{\frac{1}{x}}$. *Ans.* $y'=x^{\frac{1}{x}}\left(\frac{1-\ln x}{x^2}\right)$. 101. $y=x^{\ln x}$. *Ans.* $y'=x^{\ln x-1}\ln x^2$. 102. $y=e^{x^x}$. *Ans.* $y'=e^{x^x}(1+\ln x)\,x^x$. 103. $y=\left(\frac{x}{n}\right)^{nx}$. *Ans.* $y'=n\left(\frac{x}{n}\right)^{nx}\left(1+\ln\frac{x}{n}\right)$.

104. $y=x^{\sin x}$. *Ans.* $y'=x^{\sin x}\left(\frac{\sin x}{x}+\ln x\cos x\right)$. 105. $y=(\sin x)^x$. *Ans.* $y'=(\sin x)^x(\ln\sin x+x\cot x)$. 106. $y=(\sin x)^{\tan x}$. *Ans.* $y'=(\sin x)^{\tan x}\times(1+\sec^2 x\ln\sin x)$. 107. $y=\tan\frac{1-e^x}{1+e^x}$. *Ans.* $y'=-\frac{e^{2x}}{(1+e^x)^2}\frac{1}{\cos^2\frac{1-e^x}{1+e^x}}$.

108. $y=\sin\sqrt{1-2^x}$. *Ans.* $y'=-\frac{\cos\sqrt{1-2x}}{2\sqrt{1-2^x}}2^x\ln 2$. 109. $y=10^{x\tan x}$. *Ans.* $y'=10^{x\tan x}\ln 10\left(\tan x+\frac{x}{\cos^2 x}\right)$.

Find the derivatives of the functions after first taking logarithms of these functions:

110. $y=\sqrt[3]{\frac{x(x^2+1)}{(x-1)^2}}$. *Ans.* $y'=\frac{1}{3}\sqrt[3]{\frac{x(x^2+1)}{(x-1)^2}}\left(\frac{1}{x}+\frac{2x}{x^2+1}+\frac{2}{x-1}\right)$.

111. $y=\frac{(x+1)^3\sqrt{(x-2)^3}}{\sqrt[5]{(x-3)^2}}$. *Ans.* $y'=\frac{(x+1)^3\sqrt[4]{(x-2)^3}}{\sqrt[5]{(x-3)^2}}\left(\frac{3}{x+1}+\frac{3}{4(x-2)}-\frac{2}{5(x-3)}\right)$. 112. $y=\frac{(x+1)^2}{(x+2)^3(x+3)^4}$. *Ans.* $y'=-\frac{(x+1)(5x^2+14x+5)}{(x+2)^4(x+3)^5}$.

113. $y=\frac{\sqrt[5]{(x-1)^2}}{\sqrt[4]{(x-2)^3}\sqrt[3]{(x-3)^7}}$. *Ans.* $y'=\frac{-161x^2+480x-271}{60\sqrt[5]{(x-1)^3}\sqrt[4]{(x-2)^7}\sqrt[3]{(x-3)^{10}}}$.

114. $y=\frac{x(1+x^2)}{\sqrt{1-x^2}}$. *Ans.* $y'=\frac{1+3x^2-2x^4}{(1-x^2)^{\frac{3}{2}}}$. 115. $y=x^5(a+3x)^3(a-2x)^2$. *Ans.* $y'=5x^4(a+3x)^2(a-2x)(a^2+2ax-12x^2)$. 116. $y=\arcsin\frac{x}{a}$. *Ans.* $y'=\frac{1}{\sqrt{a^2-x^2}}$.

117. $y=(\arcsin x)^2$. *Ans.* $y'=\frac{2\arcsin x}{\sqrt{1-x^2}}$. 118. $y=\operatorname{arc\,cot}(x^2+1)$. *Ans.* $y'=\frac{2x}{1+(x^2+1)^2}$. 119. $y=\operatorname{arc\,cot}\frac{2x}{1-x^2}$. *Ans.* $y'=\frac{2}{1+x^2}$. 120. $y=\arccos(x^2)$. *Ans.* $y'=\frac{-2x}{\sqrt{1-x^4}}$. 121. $y=\frac{\arccos x}{x}$. *Ans.* $y'=\frac{-(x+\sqrt{1-x^2}\arccos x)}{x^2\sqrt{1-x^2}}$.

122. $y=\arcsin\frac{x+1}{\sqrt{2}}$. *Ans.* $y'=\frac{1}{\sqrt{1-2x-x^2}}$. 123. $y=x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}$.

Ans. $y'=2\sqrt{a^2-x^2}$. 124. $y=\sqrt{a^2-x^2}+a\arcsin\frac{x}{a}$. *Ans.* $y'=\sqrt{\frac{a-x}{a+x}}$. 125. $u=\operatorname{arc\,cot}\frac{v+a}{1-av}$. *Ans.* $\frac{du}{dv}=\frac{1}{1+v^2}$. 126. $y=\frac{1}{\sqrt{3}}\operatorname{arc\,cot}\frac{x\sqrt{3}}{1-x^2}$. *Ans.* $y'=\frac{x^2+1}{x^4+x^2+1}$. 127. $y=x\arcsin x$. *Ans.* $y'=\arcsin x+\frac{x}{\sqrt{1-x^2}}$. 128. $f(x)=\arccos(\ln x)$. *Ans.* $f'(x)=-\frac{1}{x\sqrt{1-\ln^2 x}}$.

129. $f(x)=\arcsin\sqrt{\sin x}$. *Ans.* $f'(x)=\frac{\cos x}{2\sqrt{\sin x-\sin^2 x}}$. 130. $y=\operatorname{arc\,cot}\sqrt{\frac{1-\cos x}{1+\cos x}}$ $(0\leqslant x<\pi)$. *Ans.* $y'=\frac{1}{2}$. 131. $y=e^{\operatorname{arc\,cot} x}$. *Ans.* $y'=\frac{e^{\operatorname{arc\,cot} x}}{1+x^2}$. 132. $y=\operatorname{arc\,cot}\frac{e^x-e^{-x}}{2}$. *Ans.* $y'=\frac{2}{e^x+e^{-x}}$. 133. $y=x^{\arcsin x}$.

Ans. $y'=x^{\arcsin x}\left(\frac{\arcsin x}{x}+\frac{\ln x}{\sqrt{1-x^2}}\right)$. 134. $y=\arcsin(\sin x)$. *Ans.* $y'=\frac{\cos x}{|\cos x|}=\begin{cases}+1 \text{ in 1st and 4th quadrants.}\\ -1 \text{ in 2d and 3d quadrants.}\end{cases}$ 135. $y=\operatorname{arc\,cot}\frac{4\sin x}{3+5\cos x}$. *Ans.* $y'=\frac{4}{5+3\cos x}$. 136. $y=\operatorname{arc\,cot}\frac{a}{x}+\ln\sqrt{\frac{x-a}{x+a}}$. *Ans.* $y'=\frac{2a^3}{x^4-a^4}$. 137. $y=\ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}-\frac{1}{2}\arctan x$. *Ans.* $y'=\frac{x^2}{1-x^4}$. 138. $y=\frac{3x^2-1}{3x^3}+\ln\sqrt{1+x^2}+\arctan x$. *Ans.* $y'=\frac{x^5+1}{x^6+x^4}$. 139. $y=\frac{1}{3}\ln\frac{x+1}{\sqrt{x^2-x+1}}+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$.

Ans. $y'=\frac{1}{x^3-1}$. **140.** $y=\ln\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}+2\arc\tan\frac{x\sqrt{2}}{1-x^2}$. *Ans.* $y'=\frac{4\sqrt{2}}{1+x^4}$.

141. $y=\arccos\frac{x^{2n}-1}{x^{2n}+1}$. *Ans.* $-\frac{2n\,|x|^n}{x(x^{2n}+1)}$.

Differentiation of Implicit Functions

Find $\frac{dy}{dx}$ if: **142.** $y^2=4px$. *Ans.* $\frac{dy}{dx}=\frac{2p}{y}$. **143.** $x^2+y^2=a^2$. *Ans.* $\frac{dy}{dx}=-\frac{x}{y}$. **144.** $b^2x^2+a^2y^2=a^2b^2$. *Ans.* $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$. **145.** $y^3-3y+2ax=0$. *Ans.* $\frac{dy}{dx}=\frac{2a}{3(1-y^2)}$. **146.** $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$. *Ans.* $\frac{dy}{dx}=-\sqrt{\frac{y}{x}}$. **147.** $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. *Ans.* $\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}$. **148.** $y^2-2xy+b^2=0$. *Ans.* $\frac{dy}{dx}=\frac{y}{y-x}$.

149. $x^3+y^3-3axy=0$. *Ans.* $\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$. **150.** $y=\cos(x+y)$. *Ans.* $\frac{dy}{dx}=-\frac{\sin(x+y)}{1+\sin(x+y)}$. **151.** $\cos(xy)=x$. *Ans.* $\frac{dy}{dx}=-\frac{1+y\sin(xy)}{x\sin(xy)}$.

Find $\frac{dy}{dx}$ of functions represented parametrically:

152. $x=a\cos t$; $y=b\sin t$. *Ans.* $\frac{dy}{dx}=-\frac{b}{a}\cot t$. **153.** $x=a(t-\sin t)$; $y=a(1-\cos t)$. *Ans.* $\frac{dy}{dx}=\cot\frac{t}{2}$. **154.** $x=a\cos^3 t$; $y=b\sin^3 t$. *Ans.* $\frac{dy}{dx}=-\frac{b}{a}\tan t$. **155.** $x=\frac{3at}{1+t^2}$; $y=\frac{3at^2}{1+t^2}$. *Ans.* $\frac{dy}{dx}=\frac{2t}{1-t^2}$. **156.** $u=2\ln\cot s$; $v=\tan s+\cot s$. Show that $\frac{du}{dv}=\tan 2s$.

Find the tangents of angles of the slopes of tangent lines to curves:

157. $x=\cos t$, $y=\sin t$ at the point $x=-\frac{1}{2}$, $y=\frac{\sqrt{3}}{2}$. Make a drawing *Ans.* $\frac{1}{\sqrt{3}}$. **158.** $x=2\cos t$, $y=\sin t$ at the point $x=1$, $y=-\frac{\sqrt{3}}{2}$. Make a drawing. *Ans.* $\frac{1}{2\sqrt{3}}$ **159.** $x=a(t-\sin t)$, $y=a(1-\cos t)$ when $t=\frac{\pi}{2}$. Make a drawing. *Ans.* 1. **160.** $x=a\cos^3 t$, $y=a\sin^3 t$ when $t=\frac{\pi}{4}$. Make a drawing. *Ans.* -1. **161.** A body thrown at an angle α to the horizon (in airless space) described a curve, under the force of gravity, whose equations are: $x=$

$=v_0\cos\alpha t$, $y=v_0\sin\alpha t-\frac{gt^2}{2}$ ($g=9.8$ m/sec²). Knowing that $\alpha=60°$, $v_0=$ $=50$ m/sec, determine the direction of motion when: 1) $t=2$ sec; 2) $t=7$ sec. Make a drawing. *Ans.* 1) $\tan\varphi_1=0.948$, $\varphi_1=43°30'$; 2) $\tan\varphi_2=-1.012$, $\varphi_2=+134°7'$.

Find the differentials of the following functions:

162. $y=(a^2-x^2)^5$. *Ans.* $dy=-10x(a^2-x^2)^4\,dx$. **163.** $y=\sqrt{1+x^2}$. *Ans.* $dy=$ $=\frac{x\,dx}{\sqrt{1+x^2}}$. **164.** $y=\frac{1}{3}\tan^3 x+\tan x$. *Ans.* $dy=\sec^4 x\,dx$. **165.** $y=\frac{x\ln x}{1-x}+$ $+\ln(1-x)$. *Ans.* $dy=\frac{\ln x\,dx}{(1-x)^2}$.

Calculate the increments and differentials of the functions:

166. $y=2x^2-x$ when $x=1$, $\Delta x=0.01$. *Ans.* $\Delta y=0.0302$, $dy=0.03$. **167.** Given $y=x^3+2x$. Find Δy and dy when $x=-1$, $\Delta x=0.02$. *Ans.* $\Delta y=0.098808$, $dy=0.1$. **168.** Given $y=\sin x$. Find dy when $x=\frac{\pi}{3}$, $\Delta x=\frac{\pi}{18}$. *Ans.* $dy=$ $=\frac{\pi}{36}=0.00873$. **169.** Knowing that $\sin 60°=\frac{\sqrt{3}}{2}=0.866025$; $\cos 60°=\frac{1}{2}$, find the approximate values of $\sin 60°3'$ and $\sin 60°18'$. Compare the results with tabular data. *Ans.* $\sin 60°3'\approx 0.866461$; $\sin 60°18'\approx 0.868643$. **170.** Find the approximate value of $\tan 45°4'30''$. *Ans.* 1.00262. **171.** Knowing that $\log_{10}200=2.30103$ find the approximate value of $\log_{10}200.2$. *Ans.* 2.30146. Derivatives of different orders. **172.** $y=3x^3-2x^2+5x-1$. Find y''. *Ans.* $18x-4$. **173.** $y=\sqrt[5]{x^3}$. Find y'''. *Ans.* $\frac{42}{125}x^{-\frac{12}{5}}$. **174.** $y=x^6$. Find $y^{(6)}$. *Ans.* 6!. **175.** $y=\frac{C}{x^n}$. Find y''. *Ans.* $\frac{n(n+1)C}{x^{n+2}}$. **176.** $y=\sqrt{a^2-x^2}$. Find y''. *Ans.* $-\frac{a^2}{(a^2-x^2)\sqrt{a^2-x^2}}$. **177.** $y=2\sqrt{x}$. Find $y^{(4)}$. *Ans.* $-\frac{15}{8\sqrt{x^7}}$. **178.** $y=$ $=ax^2+bx+c$. Find y'''. *Ans.* 0. **179.** $f(x)=\ln(x+1)$. Find $f^{IV}(x)$. *Ans.* $-\frac{6}{(x+1)^4}$. **180.** $y=\tan x$. Find y'''. *Ans.* $6\sec^4 x-4\sec^2 x$. **181.** $y=\ln\sin x$. Find. y'''. *Ans.* $2\cot x\csc^2 x$. **182.** $f(x)=\sqrt{\sec 2x}$. Find. $f''(x)$. *Ans.* $f''(x)=$ $=3[f(x)]^5-f(x)$. **183.** $y=\frac{x^4}{1-x}$. Find $f^{(4)}(x)$. *Ans.* $\frac{4!}{(1-x)^5}$. **184.** $p=$ $=(q^2+a^2)\arctan\frac{q}{a}$. Find $\frac{d^3p}{dq^3}$. *Ans.* $\frac{4a^3}{(a^2+g^2)^2}$. **185.** $y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})$. Find $\frac{d^2y}{dx^2}$. *Ans.* $\frac{y}{a^2}$. **186.** $y=\cos ax$. Find $y^{(n)}$. *Ans.* $a^n\cos\left(ax+n\frac{\pi}{2}\right)$. **187.** $y=a^x$. Find $y^{(n)}$. *Ans.* $(\ln a^n)a^x$. **188.** $y=\ln(1+x)$. Find $y^{(n)}$. *Ans.* $(-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$. **189.** $y=\frac{1-x}{1+x}$. Find $y^{(n)}$. *Ans.* $2(-1)^n\frac{n!}{(1+x)^{n+1}}$. **190.** $y=e^x x$. Find $y^{(n)}$. *Ans.* $e^x(x+n)$. **191.** $y=x^{n-1}\ln x$. Find $y^{(n)}$.

Ans. $\frac{(n-1)!}{x}$. **192.** $y=\sin^2 x$. Find $y^{(n)}$. *Ans.* $-2^{n-1}\cos\left(2x+\frac{\pi}{2}n\right)$. **193.** $y=x\sin x$. Find $y^{(n)}$. *Ans.* $x\sin\left(x+\frac{\pi}{2}n\right)-n\cos\left(x+\frac{\pi}{2}n\right)$. **194.** If $y=e^x\sin x$, prove that $y''-2y'+2y=0$. **195.** $y^2=4ax$. Find $\frac{d^2y}{dx^2}$. *Ans.* $-\frac{4a^2}{y^3}$. **196.** $b^2x^2+a^2y^2=a^2b^2$. Find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$. *Ans.* $-\frac{b^4}{a^2y^3}$; $-\frac{3b^6x}{a^4y^5}$. **197.** $x^2+y^2=r^2$. Find $\frac{d^2y}{dx^2}$. *Ans.* $-\frac{r^2}{y^3}$. **198.** $y^2-2xy=0$. Find $\frac{d^3y}{dx^3}$. *Ans.* 0. **199.** $\varrho=\tan(\varphi+\varrho)$. Find $\frac{d^3\varrho}{d\varphi^3}$. *Ans.* $-\frac{2(5+8\varrho^2+3\varrho^4)}{\varrho^8}$. **200.** $\sec\varphi\cos\varrho=C$. Find $\frac{d^2\varrho}{d\varphi^2}$. *Ans.* $\frac{\tan^2\varrho-\tan^2\varphi}{\tan^3\varrho}$. **201.** $e^x+x=e^y+y$. Find $\frac{d^2y}{dx^2}$. *Ans.* $\frac{(1-e^{x+y})(e^x-e^y)}{(e^y+1)^3}$.

202. $y^3+x^3-3axy=0$. Find $\frac{d^2y}{dx^2}$. *Ans.* $-\frac{2a^3xy}{(y^2-ax)^3}$. **203.** $x=a(t-\sin t)$, $y=a(1-\cos t)$. Find $\frac{d^2y}{dx^2}$. *Ans.* $-\frac{1}{4a\sin^4\left(\frac{t}{2}\right)}$. **204.** $x=a\cos 2t$, $y=b\sin^2 t$. Show that $\frac{d^2y}{dx^2}=0$. **205.** $x=a\cos t$, $y=a\sin t$. Find $\frac{d^3y}{dx^3}$. *Ans.* $-\frac{3\cos t}{a^2\sin^5 t}$.

206. Show that $\frac{d^{2n}}{dx^{2n}}(\sinh x)=\sinh x$; $\frac{d^{2n+1}}{dx^{2n+1}}(\sinh x)=\cosh x$.

Equations of a Tangent and Normal. Lengths of a Subtangent and a Subnormal

207. Write the equations of the tangent and normal to the curve $y=x^3-3x^2-x+5$ at the point $M(3, 2)$. *Ans.* The tangent is $8x-y-22=0$; the normal, $x+8y-19=0$. **208.** Find the equations of the tangent and normal of the length of the subtangent and subnormal of the circle $x^2+y^2=r^2$ at the point $M(x_1, y_1)$. *Ans.* The tangent is $xx_1+yy_1=r^2$; the normal is $x_1y-y_1x=0$; $s_T=\left|-\frac{y_1^2}{x_1}\right|$; $s_N=|-x_1|$.

209. Show that the subtangent of the parabola $y^2=4px$ at any point is divided into two by the vertex, and the subnormal is constant and equal to $2p$. Make a drawing.

210. Find the equation of a tangent at the point $M(x_1, y_1)$:

a) To the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. *Ans.* $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

b) To the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. *Ans.* $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

211. Find the equations of the tangent and normal to the Witch of Agnesi $y=\frac{8a^3}{4a^2+x^2}$ at the point where $x=2a$. *Ans.* The tangent is $x+2y=4a$; the normal is $y=2x-3a$.

212. Show that the normal to the curve $3y=6x-5x^3$ drawn to the point $M\left(1, \frac{1}{3}\right)$ passes through the coordinate origin.

213. Show that the tangent to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at the point $M(a, b)$ is $\frac{x}{a}+\frac{y}{b}=2$.

214. Find the equation of that tangent to the parabola, $y^2=20x$, which forms an angle of 45° with the x-axis. *Ans.* $y=x+5$ [at the point (5, 10)].

215. Find the equations of those tangents to the circle $x^2+y^2=52$, which are parallel to the straight line $2x+3y=6$. *Ans.* $2x+3y\pm 26=0$.

216. Find the equations of those tangents to the hyperbola $4x^2-9y^2=36$, which are perpendicular to the straight line $2y+5x=10$. *Ans.* There are no such tangents.

217. Show that the segment (lying between the coordinate axes) of the tangent to the hyperbola $xy=m$ is divided into two by the point of tangency.

218. Prove that the segment (between the coordinate axes) of a tangent to the asteroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is of constant length.

219. At what angle α do the curves $y=a^x$ and $y=b^x$ intersect? *Ans.* $\tan\alpha=\frac{\ln a-\ln b}{1+\ln a\cdot\ln b}$.

220. Find the lengths of the subtangent, subnormal, tangent and normal of the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ at the point at which $\theta=\frac{\pi}{2}$. *Ans.* $s_T=a$; $s_N=a$; $T=a\sqrt{2}$; $N=a\sqrt{2}$.

221. Find the quantities s_T, s_N, T and N for the hypocycloid $x=4a\cos^3 t$, $y=4a\sin^3 t$. *Ans.* $s_T=-4a\sin^2 t\cos t$; $s_N=-4a\frac{\sin^4 t}{\cos t}$; $T=4a\sin^2 t$; $N=$ $=4a\sin^2 t\tan t$.

Miscellaneous Problems

Find the derivatives of the following functions: 222. $y=\frac{\sin x}{2\cos^2 x}-\frac{1}{2}\times$ $\times\ln\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$. *Ans.* $y'=\frac{1}{\cos^3 x}$. 223. $y=\arcsin\frac{1}{x}$. *Ans.* $y'=-\frac{1}{|x|\sqrt{x^2-1}}$.

224. $y=\arcsin(\sin x)$. *Ans.* $y'=\frac{\cos x}{|\cos x|}$. 225. $y=\frac{2}{\sqrt{a^2-b^2}}\times$ $\times\arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$ $(a>0,\ b>0)$. *Ans.* $y'=\frac{1}{a+b\cos x}$. 226. $y=|x|$. *Ans.* $y'=\frac{x}{|x|}$. 227. $y=\arcsin\sqrt{1-x^2}$. *Ans.* $y'=-\frac{x}{|x|}\frac{1}{\sqrt{1-x^2}}$.

228. From the formulas for the volume and surface of a sphere,

$$v=\frac{4}{3}\pi r^3 \text{ and } s=4\pi r^2$$

it follows that $\frac{dv}{dr}=s$. Explain the geometric significance of this result. Find a similar relationship between the area of a circle and the length of the circumference.

229. In a triangle ABC, the side a is expressed in terms of the other two sides b, c and the angle A between them by the formula

$$a=\sqrt{b^2+c^2-2bc\cos A}.$$

For b and c constant, side a is a function of the angle A. Show that $\frac{da}{dA}=h_a$, where h_a is the altitude of the triangle corresponding to the base a. Interpret this result geometrically.

230. Using the differential concept, determine the origin of the approximate formulas

$$\sqrt{a^2+b^2}\approx a+\frac{b}{2a}, \quad \sqrt[3]{a^3+b}\approx a+\frac{b}{3a^2}$$

where $|b|$ is a number small compared with a.

231. The period of oscillation of a pendulum is computed by the formula

$$T=\pi\sqrt{\frac{l}{g}}.$$

In calculating the period T, how will the error be affected by an error of 1% in the measurement of: 1) the length of the pendulum l; 2) the acceleration of gravity g? *Ans.* 1) $\approx 1/2\%$; 2) $\approx 1/2\%$.

232. The tractrix has the property that for any point of it, the segment of the tangent T remains constant in length. Prove this on the basis of: 1) the equation of the tractrix in the form

$$x=\sqrt{a^2-y^2}+\frac{a}{2}\ln\frac{a-\sqrt{a^2-y^2}}{a+\sqrt{a^2-y^2}} \quad (a>0);$$

2) the parametric equations of the curve

$$x=a\left(\ln\tan\frac{t}{2}+\cos t\right),$$
$$y=a\sin t.$$

233. Prove that the function $y=C_1e^{x}+C_2e^{-2x}$ satisfies the equation $y''+3y'+2y=0$ (here C_1 and C_2 are constants).

234. Putting $y=e^x\sin x$, $z=e^x\cos x$ prove the equalities $y''=2z$, $z''=-2y$.

235. Prove that the function $y=\sin(m\arcsin x)$ satisfies the equation $(1-x^2)y''-xy'+m^2y=0$.

236. Prove that if $(a+bx)e^{\frac{y}{x}}=x$, then $x^3\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2$.

CHAPTER IV

SOME THEOREMS ON DIFFERENTIABLE FUNCTIONS

SEC. 1. A THEOREM ON THE ROOTS OF A DERIVATIVE (ROLLE'S THEOREM)

Rolle's Theorem. *If a function $f(x)$ is continuous on an interval $[a, b]$ and is differentiable at all interior points of this interval, and vanishes $[f(a)=f(b)=0]$ at the end points $x=a$ and $x=b$, then inside $[a, b]$ there exists at least one point $x=c$, $a<c<b$, at which the derivative $f'(x)$ vanishes, that is, $f'(c)=0$.* *)

Proof. Since the function $f(x)$ is continuous on the interval $[a, b]$, it has a maximum M and a minimum m on this interval.

If $M=m$ the function $f(x)$ is constant, which means that for all values of x it has a constant value $f(x)=m$. But then at any point of the interval $f'(x)=0$, and the theorem is proved.

Suppose $M \neq m$. Then at least one of these numbers is not equal to zero.

For the sake of definiteness, let us assume that $M>0$ and that the function takes on its maximum value at $x=c$, so that $f(c)=M$. Let it be noted that, here, c is not equal either to a or to b, since it is given that $f(a)=0$, $f(b)=0$. Since $f(c)$ is the maximum value of the function, $f(c+\Delta x)-f(c) \leqslant 0$, both when $\Delta x>0$ and when $\Delta x<0$. Whence it follows that

$$\frac{f(c+\Delta x)-f(c)}{\Delta x} \leqslant 0 \quad \text{when} \quad \Delta x>0; \tag{1'}$$

$$\frac{f(c+\Delta x)-f(c)}{\Delta x} \geqslant 0 \quad \text{when} \quad \Delta x<0. \tag{1''}$$

Since it is given in the theorem that the derivative at $x=c$ exists, we get, upon passing to the limit as $\Delta x \rightarrow 0$,

$$\lim_{\Delta x \rightarrow 0} \frac{f(c+\Delta x)-f(c)}{\Delta x}=f'(c) \leqslant 0 \quad \text{when} \quad \Delta x>0;$$

$$\lim_{\Delta x \rightarrow 0} \frac{f(c+\Delta x)-f(c)}{\Delta x}=f'(c) \geqslant 0 \quad \text{when} \quad \Delta x<0.$$

But the relations $f'(c) \leqslant 0$ and $f'(c) \geqslant 0$ are compatible only if $f'(c)=0$. Consequently, there is a point c inside the interval $[a, b]$ at which the derivative $f'(x)$ is equal to zero.

---

*) The number c is called the root of the function $\varphi(x)$ if $\varphi(c)=0$.

The theorem about the roots of a derivative has a simple geometric interpretation: if a continuous curve, which at each point has a tangent, intersects the x-axis at points with abscissas a and b, then on this curve there will be at least one point with abscissa c, $a < c < b$, at which the tangent is parallel to the x-axis.

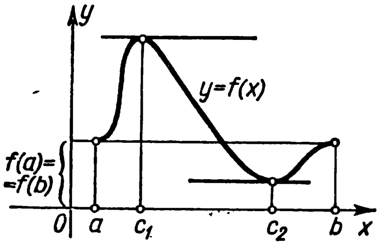


Fig. 91.

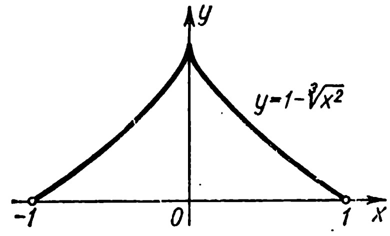


Fig. 92.

Note 1. The theorem that has just been proved also holds for a differentiable function such that does not vanish at the end points of the interval $[a, b]$, but takes on equal values $f(a) = f(b)$ (Fig. 91). The proof in this case in exactly the same as before.

Note 2. If the function $f(x)$ is such that the derivative does not exist at all points within the interval $[a, b]$, the assertion of the theorem may prove erroneous (in this case there might not be a point c in the interval $[a, b]$, at which the derivative $f'(x)$ vanishes).

For example, the function

$$y = f(x) = 1 - \sqrt[3]{x^2}$$

(Fig. 92) is continuous on the interval $[-1, 1]$ and vanishes at the end points of the interval, yet the derivative

$$f'(x) = -\frac{2}{3\sqrt[3]{x}}$$

within the interval does not vanish. This is because there is a point $x = 0$ inside the interval at which the derivative does not exist (becomes infinite).

The graph shown in Fig. 93 is another instance of a function whose derivative does not vanish in the interval $[0, 2]$.

The conditions of the Rolle theorem are not fulfilled for this function either, because at the point $x = 1$ the function has no derivative.

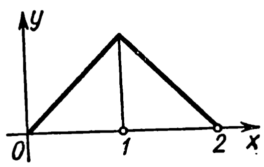


Fig. 93.

SEC. 2. A THEOREM ON FINITE INCREMENTS (LAGRANGE'S THEOREM)

Lagrange's Theorem. *If a function $f(x)$ is continuous on the interval $[a, b]$ and differentiable at all interior points of this interval, there will be, within $[a, b]$, at least one point c, $a<c<b$, such that*

$$f(b)-f(a)=f'(c)(b-a). \tag{1}$$

Proof. Let us denote by Q the number $\frac{f(b)-f(a)}{b-a}$:

$$Q=\frac{f(b)-f(a)}{b-a}, \tag{2}$$

and let us consider the auxiliary function $F(x)$ defined by the equation

$$F(x)=f(x)-f(a)-(x-a)Q. \tag{3}$$

What is the geometric significance of the function $F(x)$? First write the equation of the chord AB (Fig. 94), taking into account that its slope is $\frac{f(b)-f(a)}{b-a}=Q$ and that it passes through the point $(a, f(a))$:

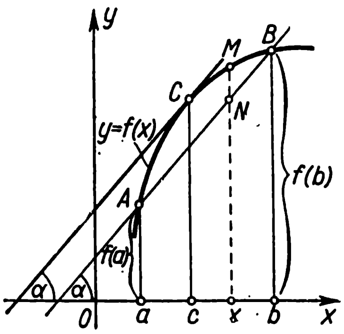


Fig. 94.

$$y-f(a)=Q(x-a);$$

whence

$$y=f(a)+Q(x-a).$$

But $F(x)=f(x)-[f(a)+Q(x-a)]$. Thus, for each value of x, $F(x)$ is equal to the difference of the ordinates of the curve $y=f(x)$ and the chord $y=f(a)+Q(x-a)$ for points with the same abscissa.

It will be readily seen that $F(x)$ is continuous on the interval $[a, b]$, is differentiable within this interval, and vanishes at the end points of the interval; in other words, $F(a)=0$, $F(b)=0$. Hence, the Rolle theorem is applicable to the function $F(x)$. By this theorem, there exists within the interval a point $x=c$ such that

$$F'(c)=0.$$

But

$$F'(x)=f'(x)-Q.$$

And so

$$F'(c)=f'(c)-Q=0,$$

whence

$$Q = f'(c).$$

Substituting the value of Q in (2), we get

$$\frac{f(b)-f(a)}{b-a} = f'(c), \qquad (1')$$

whence follows formula (1) directly. The theorem is thus proved.

See Fig. 94 for an explanation of the geometric significance of the Lagrange theorem. From the figure it is immediately clear that the quantity $\frac{f(b)-f(a)}{b-a}$ is the tangent of the angle of inclination α of the chord passing through the points A and B of the graph with abscissas a and b.

On the other hand, $f'(c)$ is the tangent of the angle of inclination of the tangent line to the curve at the point with abscissa c. Thus, the geometric significance of (1') or its equivalent (1) consists in the following: if at all points of the arc AB there is a tangent line, then there will be, on this arc, a point C between A and B at which the **tangent is parallel to the chord** connecting points A and B.

Now note the following. Since the value of c satisfies the condition $a < c < b$, it follows that $c-a < b-a$, or

$$c-a = \theta(b-a),$$

where θ is a certain number between 0 and 1, that is,

$$0 < \theta < 1.$$

But then

$$c = a + \theta(b-a),$$

and formula (1) may be written as follows:

$$f(b)-f(a) = (b-a)f'[a+\theta(b-a)], \quad 0<\theta<1. \qquad (1'')$$

SEC. 3. A THEOREM ON THE RATIO OF THE INCREMENTS OF TWO FUNCTIONS (CAUCHY'S THEOREM)

Cauchy's Theorem. *If $f(x)$ and $\varphi(x)$ are two functions continuous on the interval $[a, b]$ and differentiable within it, and $\varphi'(x)$ does not vanish anywhere inside the interval, there will be found, in $[a, b]$, some point $x = c$, $a < c < b$, such that*

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)} = \frac{f'(c)}{\varphi'(c)}. \qquad (1)$$

Proof. Let us define the number Q by the equation

$$Q=\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}. \tag{2}$$

It will be noted that $\varphi(b)-\varphi(a)\neq 0$, since otherwise $\varphi(b)$ would be equal to $\varphi(a)$, and then, by the Rolle theorem, the derivative $\varphi'(x)$ would vanish in the interval; but this contradicts the statement of the theorem.

Let us construct an auxiliary function

$$F(x)=f(x)-f(a)-Q[\varphi(x)-\varphi(a)].$$

It is obvious that $F(a)=0$ and $F(b)=0$ (this follows from the definition of the function $F(x)$ and the definition of the number Q). Noting that the function $F(x)$ satisfies all the hypotheses of the Rolle theorem on the interval $[a, b]$, we conclude that there exists between a and b a value $x=c$ $(a<c<b)$ such that $F'(c)=0$. But $F'(x)=f'(x)-Q\varphi'(x)$, hence

$$F'(c)=f'(c)-Q\varphi'(c)=0,$$

whence

$$Q=\frac{f'(c)}{\varphi'(c)}.$$

Substituting the value of Q into (2) we get (1).

Note. The Cauchy theorem cannot be proved (as it might appear at first glance) by applying the Lagrange theorem to the numerator and denominator of the fraction

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}.$$

Indeed, in this case we would (after cancelling out $b-a$) get the formula

$$\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f'(c_1)}{\varphi'(c_2)}$$

in which $a<c_1<b$, $a<c_2<b$. But since, generally, $c_1\neq c_2$, the result obtained obviously does not yet yield the Cauchy theorem.

SEC. 4. THE LIMIT OF A RATIO OF TWO INFINITESIMALS (EVALUATION OF INDETERMINATE FORMS OF THE TYPE $\frac{0}{0}$)

Let the functions $f(x)$ and $\varphi(x)$, on a certain interval $[a, b]$, satisfy the Cauchy theorem and vanish at the point $x=a$ of this interval; $f(a)=0$ and $\varphi(a)=0$.

The ratio $\frac{f(x)}{\varphi(x)}$ is not defined for $x=a$, but has a very definite meaning for the values $x \neq a$. Hence, we can raise the question of searching for the limit of this ratio as $x \to a$. Evaluating limits of this type is usually known as evaluating indeterminate forms of the type $\frac{0}{0}$.

We have already encountered such problems, for instance when considering the limit $\lim\limits_{x \to 0} \frac{\sin x}{x}$ and when finding derivatives of elementary functions. For $x=0$, the expression $\frac{\sin x}{x}$ is meaningless; the function $F(x)=\frac{\sin x}{x}$ is not defined for $x=0$, but we have seen that the limit of the expression $\frac{\sin x}{x}$ as $x \to 0$ exists and is equal to unity.

L'Hospital's Theorem (Rule). *Let the functions $f(x)$ and $\varphi(x)$, in some interval, satisfy the Cauchy theorem and vanish at some point $x=a$: $f(a)=\varphi(a)=0$; then, if the ratio $\frac{f'(x)}{\varphi'(x)}$ has a limit as $x \to a$, there also exists $\lim\limits_{x \to a} \frac{f(x)}{\varphi(x)}$, and*

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = \lim_{x \to a} \frac{f'(x)}{\varphi'(x)}.$$

Proof. On the interval $[\alpha, \beta]$ take some point $x \neq a$. Applying the Cauchy formula we have

$$\frac{f(x)-f(a)}{\varphi(x)-\varphi(a)} = \frac{f'(\xi)}{\varphi'(\xi)}$$

where ξ lies between a and x. But it is given that $f(a)=\varphi(a)=0$, and so

$$\frac{f(x)}{\varphi(x)} = \frac{f'(\xi)}{\varphi'(\xi)}. \tag{1}$$

If $x \to a$, then $\xi \to a$ also, since ξ lies between x and a. And if $\lim\limits_{x \to a} \frac{f'(x)}{\varphi'(x)} = A$, then $\lim\limits_{\xi \to a} \frac{f'(\xi)}{\varphi'(\xi)}$ exists and is equal to A. Whence it is clear that

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = \lim_{\xi \to a} \frac{f'(\xi)}{\varphi'(\xi)} = \lim_{x \to a} \frac{f'(x)}{\varphi'(x)} = A,$$

and, finally,

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = \lim_{x \to a} \frac{f'(x)}{\varphi'(x)}.$$

Note 1. The theorem holds also for the case when the functions $f(x)$ or $\varphi(x)$ **are not defined for $x=a$**, but

$$\lim_{x\to a} f(x)=0, \quad \lim_{x\to a}\varphi(x)=0.$$

In order to reduce this case to the earlier considered case, we **redefine** the functions $f(x)$ and $\varphi(x)$ at the point $x=a$ so that they become **continuous at the point a**. To do this, it is sufficient to put

$$f(a)=\lim_{x\to a} f(x)=0; \quad \varphi(a)=\lim_{x\to a}\varphi(x)=0,$$

since it is obvious that the limit of the ratio $\frac{f(x)}{\varphi(x)}$ as $x\to a$ does not depend on whether the functions $f(x)$ and $\varphi(x)$ are defined at $x=a$.

Note 2. If $f'(a)=\varphi'(a)=0$ and the derivatives $f'(x)$ and $\varphi'(x)$ satisfy the conditions that were imposed by the theorem on the functions $f(x)$ and $\varphi(x)$, then applying the L'Hospital rule to the ratio $\frac{f'(x)}{\varphi'(x)}$, we arrive at the formula $\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}=\lim_{x\to a}\frac{f''(x)}{\varphi''(x)}$, and so forth.

Example 1.

$$\lim_{x\to 0}\frac{\sin 5x}{3x}=\lim_{x\to 0}\frac{(\sin 5x)'}{(3x)'}=\lim_{x\to 0}\frac{5\cos 5x}{3}=\frac{5}{3}.$$

Example 2.

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{\frac{1}{1+x}}{1}=\frac{1}{1}=1.$$

Example 3.

$$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x}=\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}=\frac{2}{1}=2.$$

Here, we had to apply the L'Hospital rule three times because the ratios of the first, second and third derivatives at $x=0$ yield the indeterminate form $\frac{0}{0}$.

Note 3. The L'Hospital rule is also applicable if

$$\lim_{x\to\infty} f(x)=0 \quad\text{and}\quad \lim_{x\to\infty}\varphi(x)=0.$$

Indeed, putting $x=\frac{1}{z}$, we see that $z\to 0$ as $x\to\infty$ and therefore

$$\lim_{z\to 0} f\left(\frac{1}{z}\right)=0, \quad \lim_{z\to 0}\varphi\left(\frac{1}{z}\right)=0.$$

Applying the L'Hospital rule to the ratio $\dfrac{f\left(\frac{1}{z}\right)}{\varphi\left(\frac{1}{z}\right)}$ we find

$$\lim_{x \to \infty} \frac{f(x)}{\varphi(x)} = \lim_{z \to 0} \frac{f\left(\frac{1}{z}\right)}{\varphi\left(\frac{1}{z}\right)} = \lim_{z \to 0} \frac{f'\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)}{\varphi'\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)} =$$

$$= \lim_{z \to 0} \frac{f'\left(\frac{1}{z}\right)}{\varphi'\left(\frac{1}{z}\right)} = \lim_{x \to \infty} \frac{f'(x)}{\varphi'(x)},$$

which is what we wanted to prove.

Example 4.

$$\lim_{x \to \infty} \frac{\sin \frac{k}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{k \cos \frac{k}{x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \to \infty} k \cos \frac{k}{x} = k.$$

SEC. 5. THE LIMIT OF A RATIO OF TWO INFINITELY LARGE QUANTITIES (EVALUATION OF INDETERMINATE FORMS OF THE TYPE $\frac{\infty}{\infty}$)

Let us now consider the question of the limit of a ratio of two functions $f(x)$ and $\varphi(x)$ approaching infinity as $x \to a$ (or as $x \to \infty$).

Theorem. *Let the functions $f(x)$ and $\varphi(x)$ be continuous and differentiable for all $x \neq a$ in the neighbourhood of the point a: the derivative $\varphi'(x)$ does not vanish; further, let*

$$\lim_{x \to a} f(x) = \infty, \quad \lim_{x \to a} \varphi(x) = \infty$$

and let there be a limit

$$\lim_{x \to a} \frac{f'(x)}{\varphi'(x)} = A. \tag{1}$$

Then there is a limit $\lim\limits_{x \to a} \dfrac{f(x)}{\varphi(x)}$ and

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = \lim_{x \to a} \frac{f'(x)}{\varphi'(x)} = A. \tag{2}$$

Proof. In the given neighbourhood of the point a, take two points α and x such that $\alpha < x < a$ (or $a > x > \alpha$). By Cauchy's theorem we have

$$\frac{f(x) - f(\alpha)}{\varphi(x) - \varphi(\alpha)} = \frac{f'(c)}{\varphi'(c)}, \tag{3}$$

where $\alpha < c < x$. We transform the left side of (3) as follows:

$$\frac{f(x)-f(\alpha)}{\varphi(x)-\varphi(\alpha)} = \frac{f(x)}{\varphi(x)} \frac{1-\frac{f(\alpha)}{f(x)}}{1-\frac{\varphi(\alpha)}{\varphi(x)}}. \tag{4}$$

From relations (3) and (4) we have

$$\frac{f'(c)}{\varphi'(c)} = \frac{f(x)}{\varphi(x)} \frac{1-\frac{f(\alpha)}{f(x)}}{1-\frac{\varphi(\alpha)}{\varphi(x)}}.$$

Whence we find

$$\frac{f(x)}{\varphi(x)} = \frac{f'(c)}{\varphi'(c)} \frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}}. \tag{5}$$

From the condition (1) it follows that for an arbitrarily small $\varepsilon > 0$, α may be chosen so close to a that for all $x = c$ where $\alpha < c < a$, the following inequality will be fulfilled:

$$\left| \frac{f'(c)}{\varphi'(c)} - A \right| < \varepsilon$$

or

$$A - \varepsilon < \frac{f'(c)}{\varphi'(c)} < A + \varepsilon. \tag{6}$$

Let us further consider the fraction

$$\frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}}.$$

Fixing α in such manner that the inequality (6) will be fulfilled, we allow x to approach a. Since $f(x) \to \infty$ and $\varphi(x) \to \infty$ as $x \to a$, we have

$$\lim_{x \to a} \frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}} = 1$$

and, consequently, for the earlier chosen $\varepsilon > 0$ (for x sufficiently close to a) we will have

$$\left| 1 - \frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}} \right| < \varepsilon$$

or

$$1-\varepsilon<\frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}}<1+\varepsilon. \tag{7}$$

Multiplying together the appropriate terms of inequalities (6) and (7), we get

$$(A-\varepsilon)(1-\varepsilon)<\frac{f'(c)}{\varphi'(c)}\,\frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}}<(A+\varepsilon)(1+\varepsilon)$$

or, from (5),

$$(A-\varepsilon)(1-\varepsilon)<\frac{f(x)}{\varphi(x)}<(A+\varepsilon)(1+\varepsilon).$$

Since ε is an arbitrarily small number for x sufficiently close to a, it follows from the latter inequalities that

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=A$$

or, by (1),

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}=A,$$

which is what had to be proved.

Note 1. If in premise (1) $A=\infty$, that is,

$$\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}=\infty,$$

then equality (2) holds in this case as well. Indeed, from the preceding expression it follows that

$$\lim_{x\to a}\frac{\varphi'(x)}{f'(x)}=0.$$

Then by the theorem just proved

$$\lim_{x\to a}\frac{\varphi(x)}{f(x)}=\lim_{x\to a}\frac{\varphi'(x)}{f'(x)}=0,$$

whence

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\infty.$$

Note 2. The theorem just proved is readily extended to the case where $x \to \infty$. If $\lim\limits_{x\to\infty} f(x) = \infty$, $\lim\limits_{x\to\infty} \varphi(x) = \infty$ and $\lim\limits_{x\to\infty} \frac{f'(x)}{\varphi'(x)}$ exists, then

$$\lim_{x\to\infty} \frac{f(x)}{\varphi(x)} = \lim_{x\to\infty} \frac{f'(x)}{\varphi'(x)}. \qquad (8)$$

The proof is performed by replacing $x = \frac{1}{z}$, as was done under similar conditions in the case of the indeterminate form $\frac{0}{0}$ (see Sec. 4, Note 3).

Example 1.

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{(e^x)'}{(x)'} = \lim_{x\to\infty} \frac{e^x}{1} = \infty.$$

Note 3. Once again note that formulas (2) and (8) hold only if the limit on the right (finite or infinite) exists. It may happen that the limit on the left exists while there is no limit on the right. To illustrate, let it be required to find

$$\lim_{x\to\infty} \frac{x+\sin x}{x}.$$

This limit exists and is equal to 1. Indeed,

$$\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} \left(1 + \frac{\sin x}{x}\right) = 1.$$

But the ratio of derivatives

$$\frac{(x+\sin x)'}{(x)'} = \frac{1+\cos x}{1} = 1 + \cos x$$

as $x \to \infty$ does not approach any limit, it oscillates between 0 and 2.

Example 2.

$$\lim_{x\to\infty} \frac{ax^2+b}{cx^2-d} = \lim_{x\to\infty} \frac{2ax}{2cx} = \frac{a}{c}.$$

Example 3.

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\tan 3x} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\cos^2 x}}{\frac{3}{\cos^2 3x}} = \lim_{x\to\frac{\pi}{2}} \frac{1}{3} \frac{\cos^2 3x}{\cos^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{1}{3} \frac{2\cdot 3\cos 3x \sin 3x}{2\cos x \sin x} =$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos 3x}{\cos x} \lim_{x\to\frac{\pi}{2}} \frac{\sin 3x}{\sin x} = \lim_{x\to\frac{\pi}{2}} \frac{3\sin 3x}{\sin x} \cdot \frac{(-1)}{(1)} = 3\frac{(-1)}{(1)} \cdot \frac{(-1)}{(1)} = 3.$$

Example 4.

$$\lim_{x\to\infty}\frac{x}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}=0.$$

Generally, for any integral $n>0$,

$$\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}=\ldots=\lim_{x\to\infty}\frac{n(n-1)\ldots 1}{e^x}=0.$$

The other indeterminate forms reduce to the foregoing cases. These forms may be written symbolically as follows:

a) $0\cdot\infty$, b) 0°, c) ∞°, d) 1^∞, e) $\infty-\infty$. They have the following meaning.

a) Let $\lim\limits_{x\to a} f(x)=0$; $\lim\limits_{x\to a}\varphi(x)=\infty$; it is required to find

$$\lim_{x\to a}[f(x)\varphi(x)].$$

This indeterminate form is of the type $0\cdot\infty$.

If the required expression is rewritten as follows:

$$\lim_{x\to a}[f(x)\varphi(x)]=\lim_{x\to a}\frac{f(x)}{\frac{1}{\varphi(x)}}$$

or in the form

$$\lim_{x\to a}[f(x)\varphi(x)]=\lim_{x\to a}\frac{\varphi(x)}{\frac{1}{f(x)}},$$

then as $x\to a$ we obtain the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 5.

$$\lim_{x\to 0}x^n\ln x=\lim_{x\to 0}\frac{\ln x}{\frac{1}{x^n}}=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{n}{x^{n+1}}}=-\lim_{x\to 0}\frac{x^n}{n}=0.$$

b) Let

$$\lim_{x\to a}f(x)=0,\quad \lim_{x\to a}\varphi(x)=0;$$

it is required to find

$$\lim[f(x)]^{\varphi(x)}$$

or, as we say, to evaluate the indeterminate form 0°.

Putting

$$y=[f(x)]^{\varphi(x)},$$

take logarithms of both sides of the equality:

$$\ln y = \varphi(x)\,[\ln f(x)].$$

As $x \to a$ we obtain (on the right) the indeterminate form $0\cdot\infty$. Finding $\lim\limits_{x\to a} \ln y$, it is easy to get $\lim\limits_{x\to a} y$. Indeed, by virtue of the continuity of the logarithmic function, $\lim\limits_{x\to a} \ln y = \ln \lim\limits_{x\to a} y$ and if $\ln \lim\limits_{x\to a} y = b$, it is obvious that $\lim\limits_{x\to a} y = e^b$. If, in particular, $b = +\infty$ or $-\infty$, then we will have $\lim y = +\infty$ or 0, respectively.

Example 6. It is required to find $\lim\limits_{x\to 0} x^x$. Putting $y = x^x$ we find $\ln \lim y = \lim \ln y = \lim \ln (x^x) = \lim (x \ln x)$;

$$\lim_{x\to 0}(x \ln x) = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0} x = 0,$$

consequently, $\ln \lim y = 0$, whence $\lim y = e^0 = 1$, or

$$\lim_{x\to 0} x^x = 1.$$

The technique is similar for finding limits in other cases.

SEC. 6. TAYLOR'S FORMULA

Let us assume that the function $y = f(x)$ has all the derivatives up to the $(n+1)$th order, inclusive, in some interval containing the point $x = a$. Let us find a polynomial $y = P_n(x)$ of degree not above n, the value of which at $x = a$ is equal to the value of the function $f(x)$ at this point, and the values of its derivatives up to the nth order at $x = a$ are equal to the values of the corresponding derivatives of the function $f(x)$ at this point:

$$P_n(a) = f(a),\ P'_n(a) = f'(a),\ P''_n(a) = f''(a),\ \ldots,\ P_n^{(n)}(a) = f^{(n)}(a). \quad (1)$$

It is natural to expect that, in a certain sense, such a polynomial is "close" to the function $f(x)$.

Let us look for this polynomial in the form of a polynomial in degrees of $(x-a)$ with undetermined coefficients:

$$P_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \\ + \ldots + C_n(x-a)^n. \quad (2)$$

We difine the undetermined coefficients $C_1, C_2, \ldots, C_n$ so that they will satisfy conditions (1).

Let us first find the derivatives of $P_n(x)$:

$$\left.\begin{array}{l} P'_n(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \ldots + nC_n(x-a)^{n-1}, \\ P''_n(x) = 2C_2 + 3\cdot 2C_3(x-a) + \ldots + n(n-1)C_n(x-a)^{n-2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P_n^{(n)}(x) = \qquad\qquad\qquad\qquad n(n-1)\ \ldots\ 2\cdot 1\cdot C_n. \end{array}\right\} \quad (3)$$

Substituting, into the left and right sides of (2) and (3), the value of a in place of x and replacing, by equalities (1), $P_n(a)$ by $f(a)$, $P'_n(a) = f'(a)$, etc., we get

$$\begin{aligned} &f(a) = C_0, \\ &f'(a) = C_1, \\ &f''(a) = 2\cdot 1C_2, \\ &f'''(a) = 3\cdot 2\cdot 1C_3, \\ &\ldots\ldots\ldots\ldots\ldots \\ &f^{(n)}(a) = n(n-1)(n-2)\ \ldots\ 2\cdot 1C_n, \end{aligned}$$

whence we find

$$C_0 = f(a), \quad C_1 = f'(a), \quad C_2 = \frac{1}{1\cdot 2}f''(a),$$

$$C_3 = \frac{1}{1\cdot 2\cdot 3}f'''(a), \ \ldots, \ C_n = \frac{1}{1\cdot 2\ldots n}f^{(n)}(a). \quad (4)$$

Substituting into (2) the values of C_1, C_2, C_n that have been found, we get the required polynomial:

$$P_n(x) = f(a) + \frac{x-a}{1}f'(a) + \frac{(x-a)^2}{1\cdot 2}f''(a) + \frac{(x-a)^3}{1\cdot 2\cdot 3}f'''(a) + \ldots$$
$$\ldots + \frac{(x-a)^n}{1\cdot 2\ldots n}f^{(n)}(a). \quad (5)$$

Designate by $R_n(x)$ the difference of the values of the given function $f(x)$ and of the constructed polynomial $P_n(x)$ (Fig. 95):

$$R_n(x) = f(x) - P_n(x),$$

whence

$$f(x) = P_n(x) + R_n(x)$$

or, in expanded form,

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$$
$$\ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x). \quad (6)$$

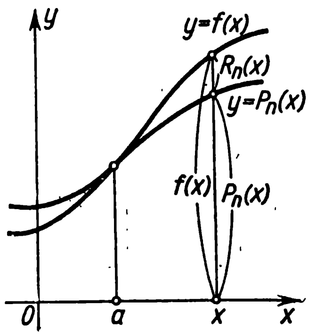


Fig. 95.

$R_n(x)$ is called the *remainder*. For those values of x, for which the remainder $R_n(x)$ is small, the polynomial $P_n(x)$ yields an approximate representation of the function $f(x)$.

Thus, formula (6) enables one to replace the function $y=f(x)$ by the polynomial $y=P_n(x)$ to an appropriate degree of accuracy equal to the value of the remainder $R_n(x)$.

Our next problem is to evaluate the quantity $R_n(x)$ for various values of x.

Let us write the remainder in the form

$$R_n(x)=\frac{(x-a)^{n+1}}{(n+1)!}Q(x), \tag{7}$$

where $Q(x)$ is a certain function to be defined, and accordingly rewrite (6):

$$f(x)=f(a)+\frac{x-a}{1}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\dots$$
$$\dots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\frac{(x-a)^{n+1}}{(n+1)!}Q(x). \tag{6'}$$

For fixed x and a, the function $Q(x)$ has a definite value; denote it by Q.

Let us further examine the auxiliary function of t (t lying between a and x):

$$F(t)=f(x)-f(t)-\frac{x-t}{1}f'(t)-\frac{(x-t)^2}{2!}f''(t)-\dots$$
$$\dots-\frac{(x-t)^n}{n!}f^{(n)}(t)-\frac{(x-t)^{n+1}}{(n+1)!}Q,$$

where Q has the value defined by the relationship (6'); here we consider a and x to be definite numbers.

We find the derivative $F'(t)$:

$$F'(t)=-f'(t)+f'(t)-\frac{x-t}{1}f''(t)+\frac{2(x-t)}{2!}f''(t)-$$
$$-\frac{(x-t)^2}{2!}f'''(t)+\dots+\frac{(x-t)^{n-1}}{(n-1)!}f^{(n)}(t)+\frac{n(x-t)^{n-1}}{n!}f^{(n)}(t)-$$
$$-\frac{(x-t)^n}{n!}f^{(n+1)}(t)+\frac{(n+1)(x-t)^n}{(n+1)!}Q,$$

or, on cancelling,

$$F'(t) = -\frac{(x-t)^n}{n!} f^{(n+1)}(t) + \frac{(x-t)^n}{n!} Q. \tag{8}$$

Thus, the function $F(t)$ has a derivative at all points t lying near the point with abscissa a.

It will further be noted that, on the basis of (6'),

$$F(x) = 0, \quad F(a) = 0.$$

Therefore, the Rolle theorem is applicable to the function $F(t)$ and, consequently, there exists a value $t = \xi$ lying between a and x such that $F'(\xi) = 0$. Whence, on the basis of relation (8), we get

$$-\frac{(x-\xi)^n}{n!} f^{(n+1)}(\xi) + \frac{(x-\xi)^n}{n!} Q = 0,$$

and from this

$$Q = f^{(n+1)}(\xi).$$

Substituting this expression into (7), we get

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi).$$

This is the so-called *Lagrange form* of the remainder.

Since ξ lies between x and a, it may be represented in the form *)

$$\xi = a + \theta(x-a)$$

where θ is a number lying between 0 and 1, that is, $0 < \theta < 1$; then the formula of the remainder takes the form

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}[a + \theta(x-a)].$$

The following formula

$$f(x) = f(a) + \frac{x-a}{1} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$
$$\dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}[a + \theta(x-a)] \tag{9}$$

is called *Taylor's formula* of the function $f(x)$.

If in the Taylor formula we put $a = 0$ we will have

$$f(x) = f(0) + \frac{x}{1} f'(0) + \frac{x^2}{2!} f''(0) + \dots$$
$$\dots + \frac{x^n}{n!} f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x) \tag{10}$$

where θ lies between 0 and 1. This special case of the Taylor formula is sometimes called *Maclaurin's formula.*

*) See end of Sec. 2 of this chapter.

SEC. 7. EXPANSION OF THE FUNCTIONS e^x, SIN x, AND COS x IN A TAYLOR SERIES

1. Expansion of the function $f(x) = e^x$.

Finding the successive derivatives of $f(x)$, we have

$$f(x) = e^x, \quad f(0) = 1,$$
$$f'(x) = e^x, \quad f'(0) = 1,$$
$$\dots\dots\dots\dots$$
$$f^{(n)}(x) = e^x, \quad f^{(n)}(0) = 1.$$

Substituting the expressions obtained into formula (10), Sec. 6, we get

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\theta x}, \quad 0 < \theta < 1.$$

If $|x| \leqslant 1$, then, taking $n = 8$, we obtain an evaluation of the remainder:

$$R_8 < \frac{1}{9!}3.$$

For $x = 1$ we get a formula that permits approximating the number e:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{8!};$$

evaluating to the fifth decimal place, we have

$$e = 2.71827.$$

Here there are four significant digits, since the error does not exceed $\frac{3}{9!}$, or 0.00001.

Observe that no matter what x is, the remainder

$$R_n = \frac{x^{n+1}}{(n+1)!}e^{\theta x} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Indeed, since $\theta < 1$, the guantity $e^{\theta x}$ for fixed x is bounded (it is less than e^x for $x > 0$, and less than 1 for $x < 0$).

We shall prove that, no matter what the fixed number x,

$$\frac{x^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Indeed,

$$\left|\frac{x^{n+1}}{(n+1)!}\right| = \left|\frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \dots \frac{x}{n} \cdot \frac{x}{n+1}\right|.$$

If x is a fixed number, there will be a positive integer N such that

$$|x| < N.$$

We introduce the notation $\frac{|x|}{N}=q$; then, noting that $0<q<1$, we can write (for $n=N+1$, $N+2$, $N+3$, etc.):

$$\left|\frac{x^{n+1}}{(n+1)!}\right|=\left|\frac{x}{1}\cdot\frac{x}{2}\cdot\frac{x}{3}\cdots\frac{x}{n}\cdot\frac{x}{n+1}\right|=$$
$$=\left|\frac{x}{1}\right|\left|\frac{x}{2}\right|\left|\frac{x}{3}\right|\cdots\left|\frac{x}{N-1}\right|\cdot\left|\frac{x}{N}\right|\cdots\left|\frac{x}{n}\right|\cdot\left|\frac{x}{n+1}\right|<$$
$$<\frac{x}{1}\cdot\frac{x}{2}\cdot\frac{x}{3}\cdots\frac{x}{N-1}\cdot q\cdot q\cdots q=\frac{x^{N-1}}{(N-1)!}q^{n-N+2},$$

for the reason that

$$\left|\frac{x}{N}\right|=q;\quad \left|\frac{x}{N+1}\right|<q;\quad \ldots;\quad \left|\frac{x}{n+1}\right|<q.$$

But $\frac{x^{N-1}}{(N-1)!}$ is a constant quantity; that is to say, it is independent of n, while q^{n-N+2} approaches zero as $n\to\infty$. And so

$$\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)!}=0. \tag{1}$$

Consequently, $R_n(x)=e^{\theta x}\frac{x^{n+1}}{(n+1)!}$ also approaches zero as n approaches infinity.

From the foregoing it follows that for any x (if a sufficient number of terms is taken) we can evaluate e^x to any degree of accuracy.

2. Expansion of the function $f(x)=\sin x$.

We find the successive derivatives of $f(x)=\sin x$:

$$\begin{aligned}
f(x) &= \sin x, & f(0)&=0,\\
f'(x) &= \cos x=\sin\left(x+\frac{\pi}{2}\right), & f'(0)&=1,\\
f''(x) &= -\sin x=\sin\left(x+2\frac{\pi}{2}\right), & f''(0)&=0,\\
f'''(x) &= -\cos x=\sin\left(x+3\frac{\pi}{2}\right), & f'''(0)&=-1,\\
f^{\text{IV}}(x) &= \sin x=\sin\left(x+4\frac{\pi}{2}\right), & f^{\text{IV}}(0)&=0,\\
&\cdots\cdots & &\cdots\cdots\\
f^{(n)}(x) &= \sin\left(x+n\frac{\pi}{2}\right), & f^{(n)}(0)&=\sin n\frac{\pi}{2},\\
f^{(n+1)}(x) &= \sin\left(x+(n+1)\frac{\pi}{2}\right), & f^{(n+1)}(\xi)&=\sin\left[\xi+(n+1)\frac{\pi}{2}\right].
\end{aligned}$$

Substituting the values obtained into (10), Sec. 6, we get an expansion of the function $f(x)=\sin x$ by the Taylor formula:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\dots + \frac{x^n}{n!}\sin n\frac{\pi}{2} + \frac{x^{n+1}}{(n+1)!}\sin\left[\xi + (n+1)\frac{\pi}{2}\right].$$

Since $\left|\sin\left[\xi+(n+1)\frac{\pi}{2}\right]\right| \leqslant 1$, we have $\lim\limits_{n\to\infty} R_n(x)=0$ for all values of x.

Let us apply the formula obtained for an approximate evaluation of $\sin 20°$. Put $n=3$, thus restricting ourselves to the first two terms of the expansion:

$$\sin 20° = \sin\frac{\pi}{9} \approx \frac{\pi}{9} - \frac{1}{3!}\left(\frac{\pi}{9}\right)^3 = 0.343.$$

Evaluate the error, which is equal to the remainder:

$$|R_3| = \left|\left(\frac{\pi}{9}\right)^4 \frac{1}{4!}\sin(\xi+2\pi)\right| \leqslant \left(\frac{\pi}{9}\right)^4\frac{1}{4!} = 0.0006 < 0.001.$$

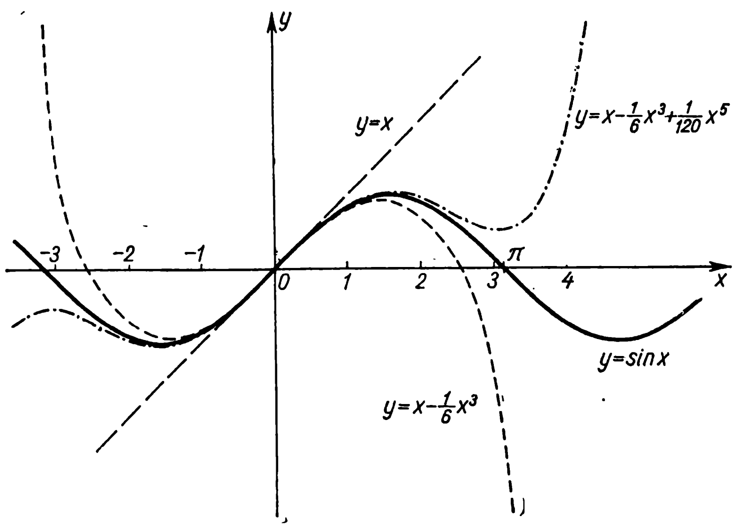


Fig. 96.

Hence, the error is less than 0.001, and so $\sin 20° = 0.343$ to three places of decimals.

Fig. 96 shows the graphs of the function $f(x)=\sin x$ and the

first three approximations: $S_1(x)=x$; $S_2(x)=x-\frac{x^3}{3!}$; $S_3(x)=x-\frac{x^3}{3!}=\frac{x^5}{5!}$.

3. Expansion of the function $f(x)=\cos x$.

Finding the values of the successive derivatives for $x=0$ of the function $f(x)=\cos x$ and substituting them into the Maclaurin formula, we get the expansion:

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+\frac{x^n}{n!}\cos\left(n\frac{\pi}{2}\right)+$$
$$+\frac{x^{n+1}}{(n+1)!}\cos\left[\xi+(n+1)\frac{\pi}{2}\right],$$
$$|\xi|<|x|.$$

Here again, $\lim\limits_{n\to\infty} R_n(x)=0$ for all values of x.

Exercises on Chapter IV

Verify the truth of Rolle's theorem for the functions: **1.** $y=x^2-3x+2$ on the interval $[1, 2]$. **2.** $y=x^3+5x^2-6x$ on the interval $[0, 1]$. **3.** $y=(x-1)(x-2)(x-3)$ on the interval $[1, 3]$. **4.** $y=\sin^2 x$ on the interval $[0, \pi]$.

5. The function $f(x)=4x^3+x^2-4x-1$ has roots 1 and -1. Find the root of the derivative $f'(x)$ mentioned in Rolle's theorem.

6. Verify that between the roots of the function $y=\sqrt[3]{x^2-5x+6}$ lies the root of its derivative.

7. Verify the truth of Rolle's theorem for the function $y=\cos^2 x$ on the interval $\left[-\frac{\pi}{4}, +\frac{\pi}{4}\right]$.

8. The function $y=1-\sqrt[5]{x^4}$ becomes zero at the end points of the interval $[-1, 1]$. Make it clear that the derivative of this function does not vanish anywhere in the interval $(-1, 1)$. Explain why Rolle's theorem is not applicable here.

9. Form Lagrange's formula for the function $y=\sin x$ on the interval $[x_1, x_2]$. *Ans.* $\sin x_2-\sin x_1=(x_2-x_1)\cos c$, $x_1<c<x_2$.

10. Verify the truth of the Lagrange formula for the function $y=2x-x^2$ on the interval $[0, 1]$.

11. At what point is the tangent to the curve $y=x^n$ parallel to the chord from point $M_1(0, 0)$ to $M_2(a, a^n)$? *Ans.* At the point with abscissa $c=\frac{a}{\sqrt[n-1]{n}}$.

12. At what point is the tangent to the curve $y=\ln x$ parallel to the chord linking the points $M_1(1, 0)$ and $M_2(e, 1)$? *Ans.* At the point with abscissa $c=e-1$.

Applying the Lagrange theorem, prove the inequalities: **13.** $e^x>1+x$. **14.** $\ln(1+x)<x$ $(x>0)$. **15.** $b^n-a^n<nb^{n-1}(b-2)$ for $b>a$. **16.** $\arctan x<x$.

17. Write the Cauchy formula for the functions $f(x)=x^2$, $\varphi(x)=x^3$ on the interval $[1, 2]$ and find c. *Ans.* $c=\frac{14}{9}$.

Evaluate the following limits: 18. $\lim\limits_{x\to 1}\frac{x-1}{x^n-1}$. *Ans.* $\frac{1}{n}$. 19. $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$. *Ans.* 2. 20. $\lim\limits_{x\to 0}\frac{\tan x-x}{x-\sin x}$. *Ans.* 2. 21. $\lim\limits_{x\to 0}\frac{e^{x^2}-1}{\cos x-1}$. *Ans.* -2. 22. $\lim\limits_{x\to 0}\frac{\sin x}{\sqrt{1-\cos x}}$. *Ans.* There is no limit ($\sqrt{2}$ as $x\to+0$, $-\sqrt{2}$ as $x\to-0$). 23. $\lim\limits_{x\to\frac{\pi}{2}}\frac{\ln\sin x}{(\pi-2x)^2}$. *Ans.* $-\frac{1}{8}$. 24. $\lim\limits_{x\to 0}\frac{a^x-b^x}{x}$. *Ans.* $\ln\frac{a}{b}$.

25. $\lim\limits_{x\to 0}\frac{x-\arcsin x}{\sin^3 x}$. *Ans.* $-\frac{1}{6}$. 26. $\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a}$. *Ans.* $\cos a$.

27. $\lim\limits_{y\to 0}\frac{e^y+\sin y-1}{\ln(1+y)}$. *Ans.* 2. 28. $\lim\limits_{x\to 0}\frac{e^x\sin x-x}{3x^2+x^5}$. *Ans.* $\frac{1}{3}$. 29. $\lim\limits_{x\to\infty}\frac{3x-1}{2x+5}$. *Ans.* $\frac{3}{2}$. 30. $\lim\limits_{x\to\infty}\frac{\ln x}{x^n}$ (where $n>0$). *Ans.* 0. 31. $\lim\limits_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\operatorname{arc\,tan} x}$. *Ans.* 1.

32. $\lim\limits_{x\to\infty}\frac{\ln\frac{x+1}{x}}{\ln\frac{x-1}{x}}$. *Ans.* -1. 33. $\lim\limits_{y\to+\infty}\frac{y}{e^{ay}}$. *Ans.* 0 for $a>0$; ∞ for $a\leqslant 0$.

34. $\lim\limits_{x\to+\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}$. *Ans.* 1. 35. $\lim\limits_{x\to 0}\frac{\ln\sin 3x}{\ln\sin x}$. *Ans.* 1. 36. $\lim\limits_{x\to 0}\frac{\ln\tan 7x}{\ln\tan 2x}$. *Ans.* 1. 37. $\lim\limits_{x\to 1}\frac{\ln(x-1)-x}{\tan\frac{\pi}{2x}}$. *Ans.* 0. 38. $\lim\limits_{x\to 1}(1-x)\tan\frac{\pi x}{2}$. *Ans.* $\frac{2}{\pi}$.

39. $\lim\limits_{x\to 1}\left[\frac{2}{x^2-1}-\frac{1}{x-1}\right]$. *Ans.* $-\frac{1}{2}$. 40. $\lim\limits_{x\to 1}\left[\frac{1}{\ln x}-\frac{x}{\ln x}\right]$. *Ans.* -1.

41. $\lim\limits_{\varphi\to\frac{\pi}{2}}(\sec\varphi-\tan\varphi)$. *Ans.* 0. 42. $\lim\limits_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\ln x}\right]$. *Ans.* $\frac{1}{2}$.

43. $\lim\limits_{x\to 0}x\cot 2x$. *Ans.* $\frac{1}{2}$. 44. $\lim\limits_{x\to 0}x^2e^{\frac{1}{x^2}}$. *Ans.* ∞. 45. $\lim\limits_{x\to 1}x^{\frac{1}{1-x}}$. *Ans.* $\frac{1}{e}$.

46. $\lim\limits_{t\to\infty}\sqrt[t]{t^2}$. *Ans.* 1. 47. $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$. *Ans.* 1. 48. $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^x$. *Ans.* e^a. 49. $\lim\limits_{x\to\infty}(\cot x)^{\frac{1}{\ln x}}$. *Ans.* $\frac{1}{e}$. 50. $\lim\limits_{x\to\frac{\pi}{2}}(\cos x)^{\frac{\pi}{2}-x}$. *Ans.* 1.

51. $\lim\limits_{\varphi\to 0}\left(\frac{\sin\varphi}{\varphi}\right)^{\frac{1}{\varphi^2}}$. *Ans.* $\frac{1}{\sqrt[6]{e}}$. 52. $\lim\limits_{x\to 1}\left(\tan\frac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$. *Ans.* $\frac{1}{e}$.

53. Expand, in powers of $x-2$, the polynomial $x^4-5x^3+5x^2+x+2$. *Ans.* $2-7(x-2)-(x-2)^2+3(x-2)^3+(x-2)^4$.

54. Expand, in powers of $x+1$, the polynomial $x^5+2x^4-x^2+x+1$. *Ans.* $(x+1)^2+2(x+1)^3-3(x+1)^4+(x+1)^5$.

55. Write Taylor's formula for the function $y=\sqrt{x}$ when $a=1$, $n=3$. *Ans.* $\sqrt{x}=1+\frac{x-1}{1}\cdot\frac{1}{2}-\frac{(x-1)^2}{1\cdot 2}\cdot\frac{1}{4}+\frac{(x-1)^3}{1\cdot 2\cdot 3}\cdot\frac{3}{8}-\frac{(x-1)^4}{4!}\cdot\frac{5}{16}\cdot[1+\theta\times$ $\times(x-1)]^{-\frac{7}{2}}$, $0<\theta<1$.

56. Write the Maclaurin formula for the function $y=\sqrt{1+x}$ when $n=2$. *Ans.* $\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{x^3}{16(1+\theta x)^{\frac{5}{2}}}$, $0<\theta<1$.

57. Using the results of the preceding exercise, evaluate the error of the approximate equality $\sqrt{1+x}\approx 1+\frac{1}{2}x-\frac{1}{8}x^2$ when $x=0.2$. *Ans.* Less than $\frac{1}{2\cdot 10^3}$.

Determine the origin of the approximate equalities for small values of x and evaluate the errors of these equalities: 58. $\ln\cos x\approx-\frac{x^2}{2}-\frac{x^4}{12}$. 59. $\tan x\approx x+\frac{x^3}{3}+\frac{2x^5}{15}$. 60. $\arcsin x\approx x+\frac{x^3}{6}$. 61. $\arctan x\approx x-\frac{x^3}{3}$. 62. $\frac{e^x+e^{-x}}{2}\approx 1+\frac{x^2}{2}+\frac{x^4}{24}$. 63. $\ln(x+\sqrt{1-x^2})\approx x-\frac{x^3}{3!}$.

Using Taylor's formula, compute the limits of the following expressions:

64. $\lim\limits_{x\to 0}\frac{x-\sin x}{e^x-1-x-\frac{x^2}{2}}$. *Ans.* 1. 65. $\lim\limits_{x\to 0}\frac{\ln^2(1+x)-\sin^2 x}{1-e^{-x^2}}$. *Ans.* 0.

66. $\lim\limits_{x\to 0}\frac{2(\tan x-\sin x)-x^3}{x^5}$. *Ans.* 1. 67. $\lim\limits_{x\to\infty}\left[x-x^2\ln\left(1+\frac{1}{x}\right)\right]$. *Ans.* $\frac{1}{2}$.

68. $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\frac{\cot x}{x}\right)$. *Ans.* $\frac{1}{3}$. 69. $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$. *Ans.* $\frac{2}{3}$.

CHAPTER V

INVESTIGATING THE BEHAVIOUR OF FUNCTIONS

SEC. 1. STATEMENT OF THE PROBLEM

A study of the quantitative aspect of natural phenomena leads to the establishment and study of functional relations between the variables involved. If such a functional relationship can be expressed analytically, that is, in the form of one or more formulas, we are then in a position to investigate it with the tools of mathematical analysis. For instance, a study of the flight of a shell in empty space yields a formula that gives the dependence of the range R upon the angle of elevation α and the initial velocity v_0:

$$R = \frac{v_0^2 \sin 2\alpha}{g}$$

(g is the acceleration of gravity).

With this formula we can determine at what angle α the range R will be greatest, or least, and what the conditions must be for the range to increase as the angle α is increased, etc.

Let us consider another instance. Studies of oscillations of a load on a spring (of a tank or automobile) yielded a formula showing how the deviation y of the load from a position of equilibrium depends on the time t:

$$y = e^{-kt}(A \cos \omega t + B \sin \omega t).$$

The quantities k, A, B, ω that enter into this formula have a very definite significance for a given oscillatory system (they depend upon the elasticity of the spring, the load, etc., but do not change with time t) and for this reason are considered constant.

On the basis of this formula we can find out at what values of t the deviation y will increase with increasing t, how the maximum deviation varies as a function of time, for what values of t we observe these maximum deviations, for what values of t we obtain maximum velocities of motion of the load, and a number of other things.

All these questions are embraced by the concept "investigating the behaviour of a function". It is obviously very difficult to determine all these questions by calculating the values of a function at specific points (like we did in Chapter II). The purpose of this chapter is to establish more general techniques for investigating the behaviour of functions.

SEC. 2. INCREASE AND DECREASE OF A FUNCTION

In Sec. 6 of Ch. I we gave a difinition of an increasing and a decreasing function. We will now apply the concept of the derivative to investigate the increase and decrease of a function.

Theorem. *If a function $f(x)$, which has a derivative on the interval $[a, b]$, increases on this interval, then its derivative on $[a, b]$ is not negative, that is, $f'(x) \geqslant 0$.*

2) *If the function $f(x)$ is continuous on the interval $[a, b]$ and is differentiable on (a, b), where $f'(x) > 0$ for $a < x < b$, then this function increases on the interval $[a, b]$.*

Proof. Let us first prove the first part of the theorem. Let $f(x)$ increase on the interval $[a, b]$. Increase the argument x by Δx and consider the relation

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}. \tag{1}$$

Since $f(x)$ is an increasing function,

$$f(x+\Delta x) > f(x) \quad \text{for} \quad \Delta x > 0$$

and

$$f(x+\Delta x) < f(x) \quad \text{for} \quad \Delta x < 0.$$

In both cases

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} > 0, \tag{2}$$

and consequently

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \geqslant 0$$

which means $f'(x) \geqslant 0$, which is what we set out to prove. [If we had $f'(x) < 0$, then for sufficiently small values of Δx, relation (1) would be negative, but this would contradict relationship (2).]

Let us now prove the second part of the theorem. Let $f'(x) > 0$ for all values of x on the interval (a, b).

Let us consider any two values x_1 and x_2, $x_1 < x_2$, on the interval $[a, b]$.

By the Lagrange theorem on finite increments we have

$$f(x_2)-f(x_1)=f'(\xi)(x_2-x_1), \quad x_1 < \xi < x_2.$$

It is given that $f'(\xi) > 0$, hence $f(x_2)-f(x_1) > 0$, and this means that $f(x)$ is an increasing function.

There is a similar theorem for a decreasing (differentiable) function as well, namely:

If $f(x)$ decreases on an interval $[a, b]$, then $f'(x) \leqslant 0$ on this interval. If $f'(x) < 0$ on (a, b), then $f(x)$ decreases on $[a, b]$. [Of

course, we again assume that the function is continuous at all points of $[a, b]$ and is differentiable everywhere on (a, b).]

Note. The foregoing theorem expresses the following geometric fact. If on an interval $[a, b]$ a function $f(x)$ increases, then the tangent to the curve $y=f(x)$ at each point on this interval forms

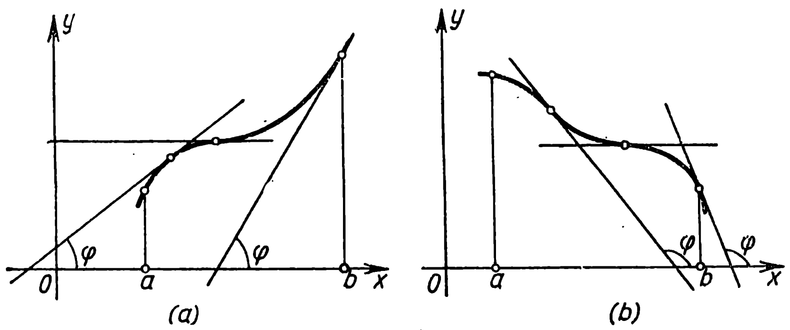


Fig. 97.

an **acute** angle φ with the x-axis or (at certain points) is horizontal; the tangent of this angle is not negative: $f'(x)=\tan\varphi \geqslant 0$ (Fig. 97, a). If the function $f(x)$ decreases on the interval $[a, b]$, then the angle of inclination of the tangent forms an obtuse angle (or, at some points, the tangent is horizontal); the tangent of this angle is not positive (Fig. 97, b). We can illustrate the second part of the theorem in similar fashion. This theorem permits judging the increase or decrease of a function by the sign of its derivative.

Example. Determine the domains of increase and decrease of the function

$$y=x^4.$$

Solution. The derivative is equal to

$$y'=4x^3;$$

for $x>0$ we have $y'>0$ and the function increases;
for $x<0$ we have $y'<0$ and the function decreases (Fig. 98).

SEC. 3. MAXIMA AND MINIMA OF FUNCTIONS

Definition of a maximum. A function $f(x)$ has a *maximum* at the point x_1 if the value of the function $f(x)$ at the point x_1 is greater than its values at all points of a certain interval containing the point x_1. In other words, the function $f(x)$ has a *maxi-*

mum when $x=x_1$ if $f(x_1+\Delta x)<f(x_1)$ for any Δx (positive and negative) that are sufficiently small in absolute value.*)

For example, the function $y=f(x)$, whose graph is given in Fig. 99, has a maximum at $x=x_1$.

Definition of a minimum. A function $f(x)$ has a *minimum* at $x=x_2$ if

$$f(x_2+\Delta x)>f(x_2)$$

for any Δx (positive and negative) that are sufficiently small in absolute value (Fig. 99).

For instance, the function $y=x^4$ considered at the end of the preceding section (see Fig. 98) has a minimum for $x=0$, since $y=0$ when $x=0$ and $y>0$ for all other values of x.

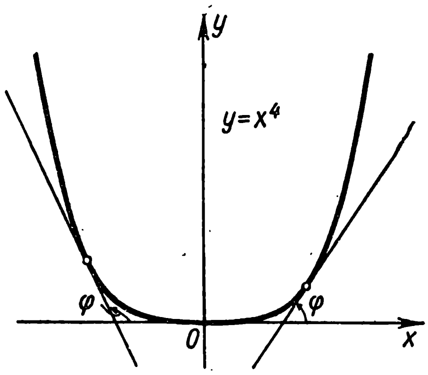


Fig. 98.

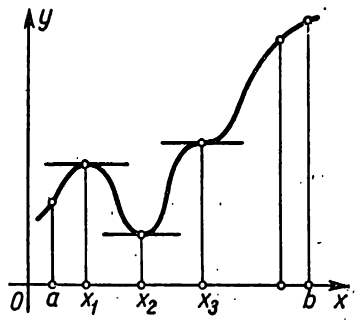


Fig. 99.

In connection with the definitions of maximum and minimum, note the following.

1. A function defined on an interval can reach maximum and minimum values only for values of x that lie *within* the given interval.

2. One should not think that the maximum and minimum of a function are its respective largest and smallest values over a given interval: at a point of maximum, a function has the largest value only in comparison with those values that it has at all points *sufficiently close* to the point of maximum, and the smallest value

*) This definition is sometimes formulated as follows: the function $f(x)$ has a *maximum* at x_1 if it is possible to find a neighbourhood (α, β) of x_1 $(\alpha<x_1<\beta)$ such that for all points of this neighbourhood different from x_1 the inequality $f(x)<f(x_1)$ is fulfilled.

only in comparison with those that it has at all points *sufficiently close* to the minimum point.

To illustrate, take Fig. 100, which shows a function defined on the interval $[a, b]$, which

at $x=x_1$ and $x=x_3$ has a maximum;
at $x=x_2$ and $x=x_4$ has a minimum,

but the minimum of the function at $x=x_4$ is greater than the maximum of the function at $x=x_1$. At $x=b$, the value of the function is greater than any maximum of the function on the interval under consideration.

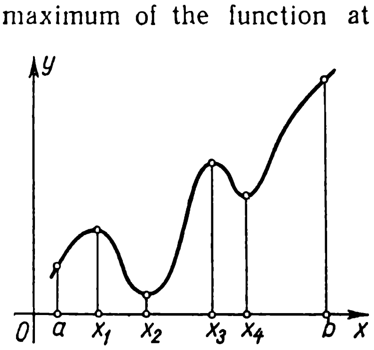


Fig. 100.

The generic terms for maxima and minima of a function are *extremum* (pl. *extrema*) or *extreme values* of the function.

To some extent, the extrema of a function and their positions on the interval $[a, b]$ characterise the variation of the function versus changes in the argument.

Below we give a method for finding extrema.

Theorem 1. (A necessary condition for the existence of an extremum). *If at the point $x=x_1$ a differentiable function $y=f(x)$ has a maximum or minimum, its derivative vanishes at this point:* $f'(x_1)=0$.

Proof. For definiteness, let us assume that at the point $x=x_1$ the function has a maximum. Then, for sufficiently small (in absolute value) increments $\Delta x\, (\Delta x \neq 0)$ we have

$$f(x_1+\Delta x) < f(x_1),$$

that is,

$$f(x_1+\Delta x)-f(x_1) < 0.$$

But in this case the sign of the ratio

$$\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$$

is determined by the sign of Δx, namely:

$$\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x} > 0 \text{ when } \Delta x < 0$$

$$\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x} < 0 \text{ when } \Delta x > 0.$$

By the definition of a derivative we have

$$f'(x_1)=\lim_{\Delta x\to 0}\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}.$$

If $f(x_1)$ has a derivative at $x=x_1$, the limit on the right is independent of how Δx approaches zero (remaining positive or negative).

But if $\Delta x\to 0$ and remains negative, then

$$f'(x_1)\leqslant 0.$$

But if $\Delta x\to 0$ and remains positive, then

$$f'(x_1)\geqslant 0.$$

Since $f'(x_1)$ is a definite number that is independent of the way in which Δx approaches zero, the latter two inequalities are compatible only if

$$f'(x_1)=0.$$

The proof is similar for the case of a minimum of a function.

Corresponding to this theorem is the following obvious geometric fact: if at points of maximum and minimum, a function $f(x)$ has a derivative, the tangent line to the curve $y=f(x)$ at these points is parallel to the x-axis. Indeed, from the fact that $f'(x_1)=\tan\varphi=0$, where φ is the angle between the tangent line and the x-axis, it follows that $\varphi=0$ (Fig. 99).

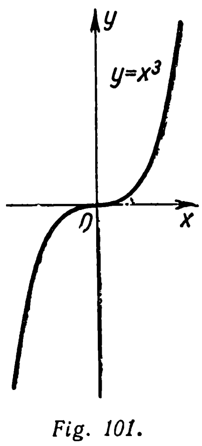


Fig. 101.

From Theorem 1 it follows straightway that *if for all considered values of the argument x the function f(x) has a derivative, then it can have an extremum (maximum or minimum) only at those values for which the derivative vanishes.* The converse does not hold: *it cannot be said that there definitely exists a maximum or minimum for every value at which the derivative vanishes.* For instance, in Fig. 99 we have a function for which the derivative at $x=x_3$ vanishes (the tangent line is horizontal), yet the function at this point is neither a maximum nor a minimum.

In exactly the same way, the function $y=x^3$ (Fig. 101) at $x=0$ has a derivative equal to zero:

$$(y')_{x=0}=(3x^2)_{x=0}=0,$$

but at this point the function has neither a maximum nor a minimum. Indeed, no matter how close the point x is to O, we

will always have

$$x^3 < 0 \text{ when } x < 0$$

and

$$x^3 > 0 \text{ when } x > 0.$$

We have investigated the case when a function has a derivative at all points on some closed interval. Now what about those points at which there is no derivative? The following examples will show that at these points there can only be a maximum or a minimum, but there may not be either one or the other.

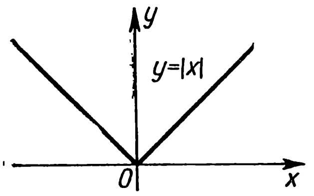


Fig. 102.

Example 1. The function $y = |x|$ has no derivative at the point $x = 0$ (at this point the curve does not have a definite tangent line), but the function has a minimum at this point. $y = 0$ when $x = 0$, whereas for any other point x different from zero, we have $y > 0$ (Fig. 102).

Example 2. The function $y = (1 - x^{\frac{2}{3}})^{3/2}$ has no derivative at $x = 0$, since $y' = -(1 - x^{2/3})^{\frac{1}{2}} x^{-\frac{1}{3}}$ becomes infinite at $x = 0$, but the function has a maximum at this point: $f(0) = 1$, $f(x) < 1$ at x different from zero (Fig. 103).

Example 3. The function $y = \sqrt[3]{x}$ has no derivative at $x = 0$ ($y' \to \infty$ as $x \to 0$). At this point the function does not have either a maximum or a minimum: $f(0) = 0$; $f(x) < 0$ for $x < 0$; $f(x) > 0$ for $x > 0$ (Fig. 104).

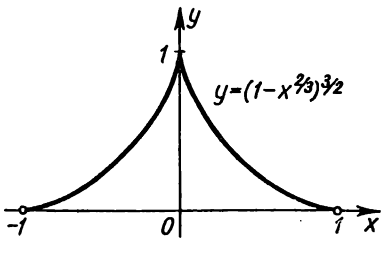


Fig. 103.

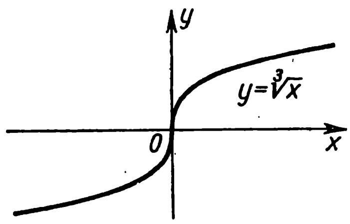


Fig. 104.

Thus, a function can have an extremum only in two cases: either at points where the derivative exists and is zero; or at points where the derivative does not exist.

It must be noted that if the derivative does not exist at some point (but exists at close-lying points), then at this point the derivative is **discontinuous**.

The values of the argument for which the derivative vanishes or is discontinuous are called *critical points* or *critical values*.

From what has been said it follows that not for every critical value does a function have a maximum or a minimum. However, if at some point the function attains a maximum or a minimum, this point is definitely critical. And so to find the extrema of a function do as follows: find all the critical points, and then, investigating separately each critical point, find out whether the function will have a maximum or a minimum at this point, or whether there will be neither maximum nor minimum.

Investigations of functions at critical points is based on the following theorem.

Theorem 2. (Sufficient conditions for the existence of an extremum). *Let there be a function $f(x)$ continuous on some interval containing a critical point x_1 and differentiable at all points of this interval (with the exception, possibly, of the point x_1 itself). If in moving from left to right through this point the derivative changes sign from plus to minus, then at $x = x_1$ the function has a maximum. But if in moving through the point x_1 from left to right the derivative changes sign from minus to plus, the function has a minimum at this point.*

And so

$$\text{if a)} \begin{cases} f'(x) > 0 \text{ when } x < x_1, \\ f'(x) < 0 \text{ when } x > x_1 \end{cases}$$

then at x_1 the function has a *maximum*;

$$\text{if b)} \begin{cases} f'(x) < 0 \text{ when } x < x_1, \\ f'(x) > 0 \text{ when } x > x_1, \end{cases}$$

then at x_1 the function has a *minimum*. Note here that the conditions a) or b) must be fulfilled for all values of x that are sufficiently close to x_1, that is, at all points of some sufficiently small neighbourhood of the critical point x_1.

Proof. Let us first assume that the derivative changes sign from plus to minus, in other words, that for all x sufficiently close to x_1 we have

$$f'(x) > 0 \text{ when } x < x_1,$$
$$f'(x) < 0 \text{ when } x > x_1.$$

Applying the Lagrange theorem to the difference $f(x) - f(x_1)$ we have

$$f(x) - f(x_1) = f'(\xi)(x - x_1)$$

where ξ is a point lying between x and x_1.

1) Let $x < x_1$; then

$$\xi < x_1, \quad f'(\xi) > 0, \quad f'(\xi)(x - x_1) < 0$$

and, consequently,

$$f(x) - f(x_1) < 0,$$

or

$$f(x) < f(x_1). \tag{1}$$

2) Let $x > x_1$; then

$$\xi > x_1, \quad f'(\xi) < 0, \quad f'(\xi)(x - x_1) < 0$$

and, consequently,

$$f(x) - f(x_1) < 0$$

or

$$f(x) < f(x_1). \tag{2}$$

The relations (1) and (2) show that for all values of x sufficiently close to x_1 the values of the function are less than those at x_1. Hence, the function $f(x)$ has a maximum at the point x_1.

The second part of the theorem on the sufficient condition for a minimum is proved in similar fashion.

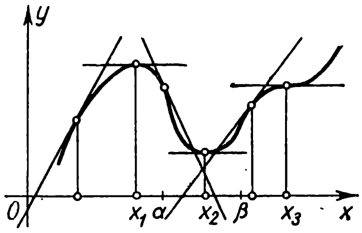


Fig. 105.

Fig. 105 illustrates the meaning of Theorem 2.

At $x = x_1$, let there be $f'(x_1) = 0$ and let the following inequalities be fulfilled for all x sufficiently close to x_1:

$$f'(x) > 0 \text{ when } x < x_1,$$
$$f'(x) < 0 \text{ when } x > x_1.$$

Then when $x < x_1$ the tangent to the curve forms with the x-axis an acute angle, and the function increases, but when $x > x_1$ the tangent forms with the x-axis an obtuse angle, and the function decreases; at $x = x_1$ the function passes from increasing to decreasing, which means it has a maximum.

If at x_2 we have $f'(x_2) = 0$ and for all values of x sufficiently close to x_2 the following inequalities are fulfilled:

$$f'(x) < 0 \text{ when } x < x_2,$$
$$f'(x) > 0 \text{ when } x > x_2,$$

then at $x < x_2$ the tangent to the curve forms with the x-axis an obtuse angle, the function decreases, and at $x > x_2$ the tangent to the curve forms an acute angle, and the function increases. At $x = x_2$ the function passes from decreasing to increasing, which means it has a minimum.

If at $x = x_3$ we have $f'(x_3) = 0$ and for all values of x sufficiently close to x_3 the following inequalities are fulfilled:

$$f'(x) > 0 \text{ when } x < x_3,$$

$$f'(x) > 0 \text{ when } x > x_3,$$

then the function increases both for $x < x_3$ and for $x > x_3$. Therefore, at $x = x_3$ the function has neither a maximum nor a minimum. Such is the case with the function $y = x^3$ at $x = 0$.

Indeed, the derivative $y' = 3x^2$, hence,

$$(y')_{x=0} = 0,$$

$$(y')_{x<0} > 0,$$

$$(y')_{x>0} > 0,$$

and this means that at $x = 0$ the function has neither a maximum nor a minimum (see above, Fig. 101).

SEC. 4. TESTING A DIFFERENTIABLE FUNCTION FOR MAXIMUM AND MINIMUM WITH A FIRST DERIVATIVE

The preceding section permits us to formulate a rule for testing a differentiable function, $y = f(x)$, for maximum and minimum:

1. Find the first derivative of the function, i. e., $f'(x)$.

2. Find the critical values of the argument x; to do this:

a) equate the first derivative to zero and find the real roots of the equation $f'(x) = 0$ obtained;

b) find the values of x at which the derivative $f'(x)$ becomes discontinuous.

3. Investigate the sign of the derivative on the left and right of the critical point. Since the sign of the derivative remains constant on the interval between two critical points, it is sufficient, for investigating the sign of the derivative on the left and right of, say, the critical point x_2 (Fig. 105), to determine the sign of the derivative at the points α and β ($x_1 < \alpha < x_2$, $x_2 < \beta < x_3$, where x_1 and x_3 are the closest critical points).

4. Evaluate the function $f(x)$ for every critical value of the argument.

This gives us the following diagram of possible cases:

Signs of derivative $f'(x)$ when passing through critical point x_1:			Character of critical point
$x<x_1$	$x=x_1$	$x>x_1$	
$+$	$f'(x_1)=0$ or is discontinuous	$-$	Maximum point
$-$	$f'(x_1)=0$ or is discontinuous	$+$	Minimum point
$+$	$f'(x_1)=0$ or is discontinuous	$+$	Neither maximum nor minimum (function increases)
$-$	$f'(x_1)=0$ or is discontinuous	$-$	Neither maximum nor minimum (function decreases)

Example 1. Test the following function for maximum and minimum:

$$y=\frac{x^3}{3}-2x^2+3x+1.$$

Solution. 1) Find the first derivative:

$$y'=x^2-4x+3.$$

2) Find the real roots of the derivative:

$$x^2-4x+3=0.$$

Consequently,

$$x_1=1, \quad x_2=3.$$

The derivative is everywhere continuous and so there are no other critical points.

3) Investigate the critical values and record the results in Fig. 106.

Investigate the first critical point $x_1=1$. Since $y'=(x-1)(x-3)$,

for $x<1$ we have $y'=(-)\cdot(-)>0$,

for $x>1$ we have $y'=(+)\cdot(-)<0$.

Thus, when passing (from left to right) through the value $x_1=1$ the derivative changes sign from plus to minus. Hence, at $x=1$ the function has a maximum, namely

$$(y)_{x=1}=\frac{7}{3}.$$

Investigate the second critical point $x_2=3$:

when $x<3$ we have $y'=(+)\cdot(-)<0$,

when $x>3$ we have $y'=(+)\cdot(+)>0$.

Thus, when passing through the value $x=3$ the derivative changes sign from minus to plus. Therefore, at $x=3$ the function has a minimum, namely:

$$(y)_{x=3}=1.$$

This investigation yields the graph of the function (Fig. 106).

Example 2. Test for maximum and minimum the function

$$y=(x-1)\sqrt[3]{x^2}.$$

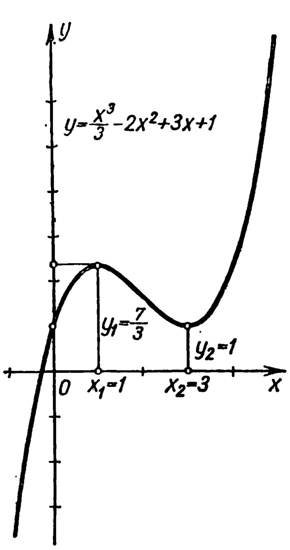


Fig. 106.

Solution. 1) Find the first derivative:

$$y'=\sqrt[3]{x^2}+\frac{2(x-1)}{3\sqrt[3]{x}}=\frac{5x-2}{3\sqrt[3]{x}}.$$

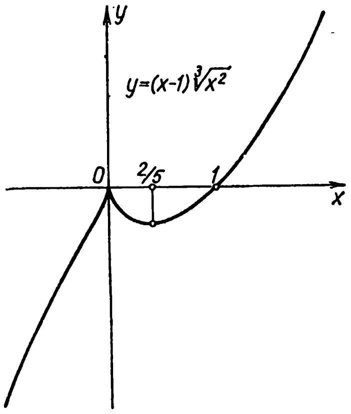


Fig. 107.

2) Find the critical values of the argument: a) find the points at which the derivative vanishes:

$$y'=\frac{5x-2}{3\sqrt[3]{x}}=0, \qquad x_1=\frac{2}{5};$$

b) find the points at which the derivative becomes discontinuous (in this instance, it becomes infinite). Obviously, that point is

$$x_2=0.$$

(It will be noted that for $x_2=0$ the function is defined and continuous.)

There are no other critical points.

3) Investigate the character of the critical points obtained. Investigate the point $x_1=\frac{2}{5}$. Noting that

$$(y')_{x<\frac{2}{5}}<0, \quad (y')_{x>\frac{2}{5}}>0,$$

we conclude that at $x=\frac{2}{5}$ the function has a minimum. The value of the function at the minimum point is

$$(y)_{x=\frac{2}{5}}=\left(\frac{2}{5}-1\right)\sqrt[3]{\frac{4}{25}}=-\frac{2}{5}\sqrt[3]{\frac{4}{25}}.$$

Investigate the second critical point $x=0$. Noting that

$$(y)_{x<0}>0, \quad (y')_{x>0}<0$$

we conclude that at $x=0$ the function has a maximum, and $(y)_{x=0}=0$. The graph of the investigated function is shown in Fig. 107.

SEC. 5. TESTING A FUNCTION FOR MAXIMUM AND MINIMUM WITH A SECOND DERIVATIVE

Let the derivative of the function $y=f(x)$ vanish at $x=x_1$; we have $f'(x_1)=0$. Also, let the second derivative $f''(x)$ exist and be continuous in some neighbourhood of the point x_1. Then the following theorem holds.

Theorem. *Let $f'(x_1)=0$; then at $x=x_1$ the function has a maximum if $f''(x_1)<0$, and a minimum if $f''(x_1)>0$.*

Proof. Let us first prove the first part of the theorem. Let

$$f'(x_1)=0 \text{ and } f''(x_1)<0.$$

Since it is given that $f''(x)$ is continuous in some small interval about the point $x=x_1$, there will obviously be some small closed interval about the point $x=x_1$, at all points of which the second derivative $f''(x)$ will be negative.

Since $f''(x)$ is the first derivative of the first derivative, $f''(x)= =(f'(x))'$, it follows from the condition $(f'(x))'<0$ that $f'(x)$ decreases on the closed interval containing $x=x_1$ (Sec. 2, Ch. V). But $f'(x_1)=0$, and so on this interval we have $f'(x)>0$ when $x<x_1$, and when $x>x_1$ we have $f'(x)<0$; in other words, the derivative $f'(x)$ changes sign from plus to minus when passing through the point $x=x_1$, and this means that at the point x_1 the function $f(x)$ has a maximum. The first part of the theorem is proved.

The second part of the theorem is proved in similar fashion: if $f''(x_1)>0$ then $f''(x)>0$ at all points of some closed interval about the point x_1, but then on this interval $f''(x)=(f'(x))'>0$ and, hence, $f'(x)$ increases. Since $f'(x_1)=0$ the derivative $f'(x)$ changes sign from minus to plus when passing through the point x_1, i. e., the function $f(x)$ has a minimum at $x=x_1$.

If at the critical point $f''(x_1)=0$, then at this point there may be either a maximum or a minimum or neither maximum nor

minimum. In this case, investigate by the first method (see Sec. 4, Ch. V).

The scheme for investigating extrema with a second derivative is shown in the following table.

$f'(x_1)$	$f''(x_1)$	Character of critical point
0	$-$	Maximum point
0	$+$	Minimum point
0	0	Unknown

Example 1. Examine the following function for maximum and minimum

$$y = 2\sin x + \cos 2x.$$

Solution. Since the function is periodic with a period of 2π, it is sufficient to investigate the function in the interval $[0, 2\pi]$.

1) Find the derivative:

$$y' = 2\cos x - 2\sin 2x = 2(\cos x - 2\sin x\cos x) = 2\cos x(1 - 2\sin x).$$

2) Find the critical values of the argument:

$$2\cos x(1 - 2\sin x) = 0,$$

$$x_1 = \frac{\pi}{6}; \quad x_2 = \frac{\pi}{2}; \quad x_3 = \frac{5\pi}{6}; \quad x_4 = \frac{3\pi}{2}.$$

3) Find the second derivative:

$$y'' = -2\sin x - 4\cos 2x.$$

4) Investigate the character of each critical point:

$$(y'')_{x=\frac{\pi}{6}} = -2\cdot\frac{1}{2} - 4\cdot\frac{1}{2} = -3 < 0.$$

Hence, at the point $x_1 = \frac{\pi}{6}$ we have a maximum:

$$(y)_{x=\frac{\pi}{6}} = 2\cdot\frac{1}{2} + \frac{1}{2} = \frac{3}{2}.$$

Further,

$$(y'')_{x=\frac{\pi}{2}} = -2\cdot 1 + 4\cdot 1 = 2 > 0.$$

And so at the point $x_2 = \frac{\pi}{2}$ we have a minimum:

$$(y)_{x=\frac{\pi}{2}} = 2\cdot 1 - 1 = 1.$$

At $x_3=\frac{5\pi}{6}$ we have

$$(y'')_{x=\frac{5\pi}{6}}=-2\cdot\frac{1}{2}-4\cdot\frac{1}{2}=-3<0.$$

Thus, at $x_3=\frac{5\pi}{6}$ the function has a maximum:

$$(y)_{x_3=\frac{5\pi}{6}}=2\cdot\frac{1}{2}+\frac{1}{2}=\frac{3}{2}.$$

Finally,

$$(y'')_{x=\frac{3\pi}{2}}=-2(-1)-4(-1)=6>0.$$

Consequently, at $x_4=\frac{3\pi}{2}$ we have a minimum:

$$(y)_{x=\frac{3\pi}{2}}=2(-1)-1=-3.$$

The graph of the function under investigation is shown in Fig. 108.

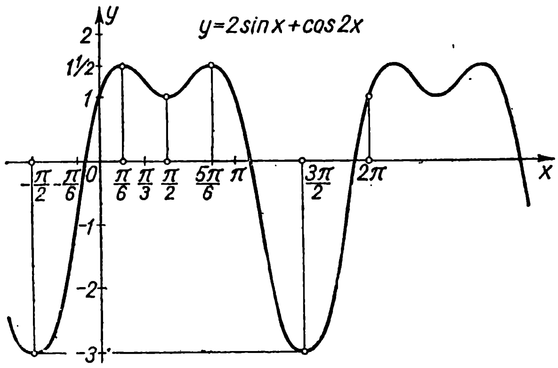


Fig. 108.

The following examples will show that if at a certain point $x=x_1$ we have $f'(x_1)=0$ and $f''(x_1)=0$, then at this point the function $f(x)$ can have either a maximum or a minimum or neither.

Example 2. Test the following function for maximum and minimum:

$$y=1-x^4.$$

Solution. 1) Find the critical points:

$$y'=-4x^3,\quad -4x^3=0,\quad x=0.$$

2) Determine the sign of the second derivative at $x=0$:

$$y''=-12x^2,\quad (y'')_{x=0}=0.$$

It is thus impossible here to determine the character of the critical point by means of the sign of the second derivative.

3) Investigate the character of the critical point by the first method (see Sec. 4., Ch. V):

$$(y')_{x<0} > 0, \quad (y')_{x>0} < 0.$$

Consequently, at $x=0$ the function has a maximum, namely

$$(y)_{x=0} = 1.$$

The graph of this function is given in Fig. 109.

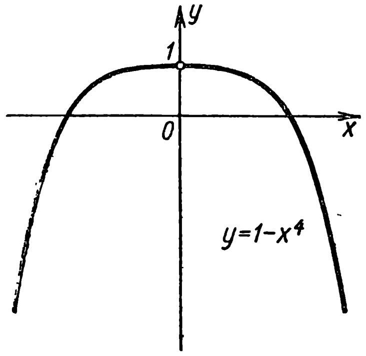


Fig. 109.

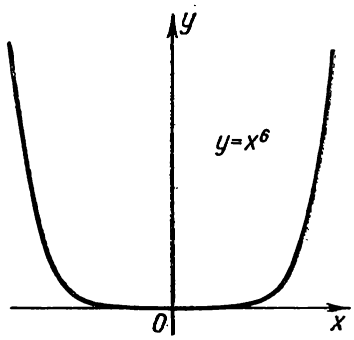


Fig. 110.

Example 3. Test for maximum and minimum the function

$$y = x^6.$$

Solution. By the second method we find

$$1)\ y' = 6x^5, \quad y' = 6x^5 = 0, \quad x = 0;$$

$$2)\ y'' = 30x^4, \quad (y'')_{x=0} = 0.$$

Thus, the second method does not yield anything. Resorting to the first method we get

$$(y')_{x<0} < 0, \quad (y')_{x>0} > 0.$$

Therefore, at $x=0$ the function has a minimum (Fig. 110).

Example 4. Test for maximum and minimum the function

$$y = (x-1)^3.$$

Solution. Second method:

$$y' = 3(x-1)^2, \quad 3(x-1)^2 = 0, \quad x = 1;$$

$$y'' = 6(x-1), \quad (y'')_{x=1} = 0.$$

Thus, the second method does not yield an answer. By the first method we get

$$(y')_{x<1} > 0, \quad (y')_{x>1} > 0.$$

Consequently, at $x=1$ the function does not have either a maximum or a minimum (Fig. 111).

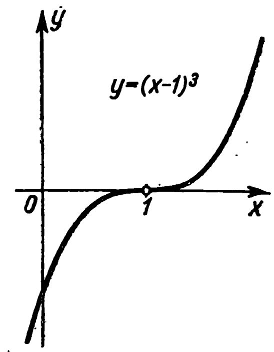


Fig. 111.

SEC. 6. MAXIMA AND MINIMA OF A FUNCTION ON AN INTERVAL

Let the function $y=f(x)$ be continuous on the interval $[a, b]$. Then the function on this interval will have a maximum (see Sec. 10, Ch. II). We will assume that on the given interval the function $f(x)$ has a finite number of critical points. If the maximum is reached within the interval $[a, b]$, it is obvious that this value will be one of the maxima of the function (if there are several maxima), namely, the greatest maximum. But it may happen that the maximum value is reached at one of the end points of the interval.

To summarise, then, on the interval $[a, b]$ the function reaches its greatest value either at one of the end points of the interval, or at such an interior point as is the maximum point.

The same may be said about the minimum value of the function: it is attained either at one of the end points of the interval or at an interior point such that the latter is the minimum point.

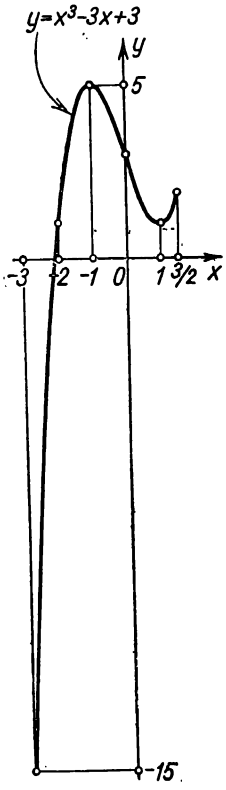


Fig. 112.

From the foregoing we get the following rule: if it is required to find the maximum of a continuous function on an interval $[a, b]$, do the following:

1) Find all maxima of the function on the interval.

2) Determine the values of the function at the end points of the interval; that is, evaluate $f(a)$ and $f(b)$.

3) Of all the values of the function obtained choose the greatest; it will be the maximum value of the function on the interval.

The minimum value of a function on an interval is found in similar fashion.

Example. Determine the maximum and minimum of the function $y=x^3-3x+3$ on the interval $\left[-3, \frac{3}{2}\right]$.

Solution. 1) Find the maxima and minima of the function on the interval $\left[-3, \frac{3}{2}\right]$:

$$y'=3x^2-3, \quad 3x^2-3=0, \quad x_1=1, \quad x_2=-1,$$
$$y''=6x, \quad (y'')_{x=1}=6>0.$$

Thus, at $x=1$ there is a minimum:

$$(y)_{x=1}=1.$$

Further,

$$(y'')_{x=-1}=-6<0.$$

And so at $x=-1$ we have a maximum:

$$(y)_{x=-1}=5.$$

2) Determine the value of the function at the end points of the interval:

$$(y)_{x=\frac{3}{2}}=\frac{15}{8}, \quad (y)_{x=-3}=-15.$$

Thus, the greatest value of this function on the interval $\left[-3, \frac{3}{2}\right]$ is

$$(y)_{x=-1}=5,$$

and the smallest value is

$$(y)_{x=-3}=-15.$$

The graph of the function is shown in Fig. 112.

SEC. 7. APPLYING THE THEORY OF MAXIMA AND MINIMA OF FUNCTIONS TO THE SOLUTION OF PROBLEMS

The theory of maxima and minima is applied in the solution of many problems of geometry, mechanics, and so forth. Let us examine a few.

Problem 1. The range $R=OA$ (Fig. 113) of a shell (in empty space) fired with an initial velocity v_0 from a gun inclined to the horizon at an angle φ, is determined by the formula

$$R=\frac{v_0^2 \sin 2\varphi}{g}$$

(g is the acceleration of gravity). Determine the angle φ at which the range R will be a maximum for a given initial velocity v_0.

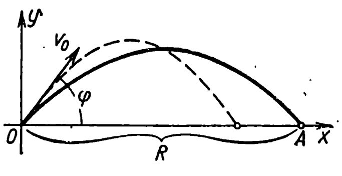


Fig. 113.

Solution. The quantity R is a function of the variable angle φ. Test this function for a maximum on the interval $0 \leqslant \varphi \leqslant \frac{\pi}{2}$:

$$\frac{dR}{d\varphi}=\frac{2v_0^2 \cos 2\varphi}{g}; \quad \frac{2v_0^2 \cos 2\varphi}{g}=0; \text{ critical value } \varphi=\frac{\pi}{4};$$

$$\frac{d^2R}{d\varphi^2}=-\frac{4v_0^2 \sin 2\varphi}{g}; \left(\frac{d^2R}{d\varphi^2}\right)_{\varphi=\frac{\pi}{4}}=-\frac{4v_0^2}{g}<0.$$

Hence, for the value $\varphi=\frac{\pi}{4}$ the function R has a maximum

$$(R)_{\varphi=\frac{\pi}{4}}=\frac{v_0^2}{g}.$$

The values of the function R at the end points of the interval $\left[0, \frac{\pi}{2}\right]$ are

$$(R)_{\varphi=0}=0, \quad (R)_{\varphi=\frac{\pi}{2}}=0.$$

Thus, the maximum obtained is the sought-for greatest value of R.

Problem 2. What should the dimensions be of a cylinder so that for a given volume v its total surface S is a minimum?

Solution. Denoting by r the radius of the base of the cylinder and by h the altitude, we have

$$S=2\pi r^2+2\pi rh.$$

Since the volume of the cylinder is given, for a given r the quantity h is determined by the formula

$$v=\pi r^2 h,$$

whence

$$h=\frac{v}{\pi r^2}.$$

Substituting this expression of h into the formula for S, we have

$$S=2\pi r^2+2\pi r\frac{v}{\pi r^2}$$

or

$$S=2\left(\pi r^2+\frac{v}{r}\right).$$

Here, v is given, so we have represented S as a function of a single independent variable r.

Find the minimum value of this function on the interval $0<r<\infty$:

$$\frac{dS}{dr}=2\left(2\pi r-\frac{v}{r^2}\right),$$

$$2\pi r-\frac{v}{r^2}=0, \qquad r_1=\sqrt[3]{\frac{v}{2\pi}},$$

$$\left(\frac{d^2S}{dr^2}\right)_{r=r_1}=2\left(2\pi+\frac{2v}{r^3}\right)_{r=r_1}>0.$$

Thus, at the point $r=r_1$ the function S has a minimum. Noticing that $\lim\limits_{r\to 0}S=\infty$ and $\lim\limits_{r\to\infty}S=\infty$; that is, that as r approaches zero or infinity the surface S increases without bound, we arrive at the conclusion that at $r=r_1$ the function S has a **minimum.**

But if $r=\sqrt[3]{\frac{v}{2\pi}}$ then

$$h=\frac{v}{\pi r^2}=2\sqrt[3]{\frac{v}{2\pi}}=2r.$$

Therefore, for the total surface S of a cylinder to be a minimum for a given volume v, the altitude of the cylinder must be equal to its diameter.

SEC. 8. TESTING A FUNCTION FOR MAXIMUM AND MINIMUM BY MEANS OF TAYLOR'S FORMULA

In Sec. 5, Ch. V, it was noted that if at a certain point $x=a$ we have $f'(a)=0$ and $f''(a)=0$, then at this point there may be either a maximum or a minimum or neither. And it was noted that in this instance the problem is solved by investigating by the first method; in other words, by testing the sign the first derivative on the left and on the right of the point $x=a$.

Now we will show that it is possible in this case to investigate by means of Taylor's formula, which was derived in Sec. 6, Ch. IV.

For greater generality, we assume that not only $f''(x)$, but also all derivatives up to the nth order inclusive of the functions $f(x)$ vanish at $x=a$:

$$f'(a)=f''(a)=\ldots=f^{(n)}(a)=0 \qquad (1)$$

and

$$f^{(n+1)}(a)\neq 0.$$

Further, assume that $f(x)$ has continuous derivatives up to the $(n+1)$ st order inclusive in the neighbourhood of the point $x=a$.

Write the Taylor formula for $f(x)$, taking account of equality (1):

$$f(x)=f(a)+\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \qquad (2)$$

where ξ is a number that lies between a and x.

Since $f^{(n+1)}(x)$ is continuous in the neighbourhood of the point a and $f^{(n+1)}(a)\neq 0$, there will be a small positive number h such that for any x that satisfies the inequality $|x-a|<h$, there will be $f^{(n+1)}(x)\neq 0$. And if $f^{(n+1)}(a)>0$, then at all points of the interval $(a-h,\ a+h)$ we will have $f^{(n+1)}(x)>0$; if $f^{(n+1)}(a)<0$, then at all points of this interval we will have $f^{(n+1)}(x)<0$.

Rewrite formula (2) in the form

$$f(x)-f(a)=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) \qquad (2')$$

and consider various special cases.

Case 1. n is odd.

a) Let $f^{(n+1)}(a)<0$. Then there will be an interval $(a-h, a+h)$ at all points of which the $(n+1)$st derivative is negative. If x is a point of this interval then ξ likewise lies between $a-h$ and $a+h$ and, consequently, $f^{(n+1)}(\xi)<0$. Since $n+1$ is an even number, $(x-a)^{n+1}>0$ for $x\neq a$, and therefore the right side of formula (2′) is negative.

Thus, for $x\neq a$ at all points of the interval $(a-h, a+h)$ we have

$$f(x)-f(a)<0,$$

and this means that at $x=a$ the function has a maximum.

b) Let $f^{(n+1)}(a)>0$. Then we have $f^{(n+1)}(\xi)>0$ for a sufficiently small value of h at all points x of the interval $(a-h, a+h)$. Hence, the right side of formula (2′) will be positive; in other words, for $x\neq a$ we will have the following at all points in the given interval:

$$f(x)-f(a)>0,$$

and this means that at $x=a$ the function has a minimum.

Case 2. n is even.

Then $n+1$ is odd and the quantity $(x-a)^{n+1}$ has different signs for $x<a$ and $x>a$.

If h is sufficiently small in absolute value, then the $(n+1)$st derivative retains the same sign as at the point a at all points of the interval $(a-h, a+h)$. Thus, $f(x)-f(a)$ has different signs for $x<a$ and $x>a$. But this means that there is neither maximum nor minimum at $x=a$.

It will be noted that if $f^{(n+1)}(a)>0$ when n is even, then $f(x)<f(a)$ for $x<a$ and $f(x)>f(a)$ for $x>a$.

But if $f^{(n+1)}(a)<0$ when n is even, then $f(x)>f(a)$ for $x<a$ and $f(x)<f(a)$ for $x>a$.

The results obtained may be formulated as follows.

If at $x=a$ we have

$$f'(a)=f''(a)=\ldots=f^{(n)}(a)=0$$

and the first nonvanishing derivative $f^{(n+1)}(a)$ is a derivative of **even** order, then at the point a

$$f(x) \text{ has a } \mathbf{maximum} \text{ if } f^{(n+1)}(a)<0,$$
$$f(x) \text{ has a } \mathbf{minimum} \text{ if } f^{(n+1)}(a)>0.$$

But if the first nonvanishing derivative $f^{(n+1)}(a)$ is a derivative of **odd** order, then the function has neither maximum nor minimum at the point a. Here,

$$f(x) \textbf{ increases} \text{ if } f^{(n+1)}(a)>0,$$
$$f(x) \textbf{ decreases} \text{ if } f^{(n+1)}(a)<0.$$

Example. Test the following function for maximum and minimum:

$$f(x)=x^4-4x^3+6x^2-4x+1.$$

Solution. Let us find the critical values of the function

$$f'(x)=4x^3-12x^2+12x-4=4(x^3-3x^2+3x-1).$$

From equation

$$4(x^3-3x^2+3x-1)=0$$

we obtain the only critical point

$$x=1$$

(since this equation has only one real root).

Investigate the character of the critical point $x=1$:

$$f''(x)=12x^2-24x+12=0 \quad \text{for } x=1,$$
$$f'''(x)=24x-24=0 \quad \text{for } x=1,$$
$$f^{IV}(x)=24>0 \quad \text{for any } x.$$

Consequently, for $x=1$ the function $f(x)$ has a minimum.

SEC. 9. CONVEXITY AND CONCAVITY OF A CURVE. POINTS OF INFLECTION

Let us consider, in a plane, the curve $y=f(x)$, which is the graph of a single-valued differentiable function $f(x)$.

Definition 1. We say that a curve is *convex upwards* on the interval (a, b) if all points of the curve lie below any tangent to it on this interval.

We say that the curve is *convex downwards* on the interval (b, c) if all points of the curve lie above any tangent to it on this interval.

We shall call a curve convex up, a *convex curve*, and a curve convex down, a *concave curve*.

Fig. 114 shows a curve convex on the interval (a, b) and concave on the interval (b, c).

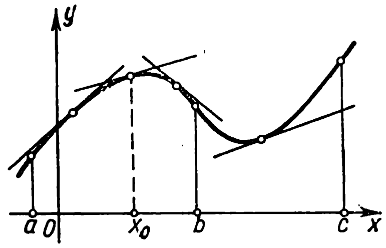


Fig. 114.

An important characteristic of the shape of a curve is its convexity or concavity. This section will be devoted to establishing the characteristics by which, when investigating a function $y=f(x)$, one can judge of the convexity or concavity (direction of bulge) on various intervals.

We shall prove the following theorem.

Theorem 1. *If at all points of an interval (a, b) the second derivative of the function $f(x)$ is negative, i. e., $f''(x)<0$, the curve $y=f(x)$ on this interval is convex upwards (the curve is convex).*

Proof. In the interval (a, b) take an arbitrary point $x=x_0$ (Fig. 114) and draw a tangent to the curve at the point with abscissa $x=x_0$. The theorem will be proved provided we establish that all the points of the curve on the interval (a, b) lie below this tangent; that is, that the ordinate of any point of the curve $y=f(x)$ is less than the ordinate $\bar{y}$ of the tangent line for one and the same value of x.

The equation of the curve is of the form

$$y=f(x). \tag{1}$$

But the equation of the tangent to the curve at this point $x=x_0$ is of the form

$$\bar{y}-f(x_0)=f'(x_0)(x-x_0)$$

or

$$\bar{y}=f(x_0)+f'(x_0)(x-x_0). \tag{2}$$

From equations (1) and (2) it follows that the difference of the ordinates of the curve and the tangent for the same value of x is

$$y-\bar{y}=f(x)-f(x_0)-f'(x_0)(x-x_0).$$

Applying the Lagrange theorem to the difference $f(x)-f(x_0)$, we get

$$y-\bar{y}=f'(c)(x-x_0)-f'(x_0)(x-x_0)$$

(where c lies between x_0 and x) or

$$y-\bar{y}=[f'(c)-f'(x_0)](x-x_0).$$

We again apply the Lagrange theorem to the expression in the square brackets; then

$$y-\bar{y}=f''(c_1)(c-x_0)(x-x_0) \tag{3}$$

(where c_1 lies between x_0 and c).

Let us first examine the case when $x>x_0$. In this case, $x_0<c<x$; since

$$x-x_0>0, \quad c-x_0>0$$

and since, in addition, it is given that

$$f''(c_1)<0,$$

it follows from equality (3) that $y-\bar{y}<0$.

Now let us consider the case when $x<x_0$. In this case $x<c<c_1<x_0$ and $x-x_0<0$, $c-x_0<0$, and since it is given that

$f''(c_1)<0$, then it follows from (3) that

$$y-\bar{y}<0.$$

We have thus proved that every point of the curve lies below the tangent to the curve, no matter what values x and x_0 have on the interval (a, b). And this signifies that the curve is convex. The theorem is proved.

The following theorem is proved in similar fashion.

Theorem 1′. *If at all points of the interval (b, c), the second derivative of the function $f(x)$ is positive, that is, $f''(x)>0$, then the curve $y=f(x)$ on this interval is convex downwards (the curve is concave).*

Note. The content of Theorems 1 and 1′ may be illustrated geometrically. Consider the curve $y=f(x)$, convex upwards on the interval (a, b) (Fig. 115). The derivative $f'(x)$ is equal to the

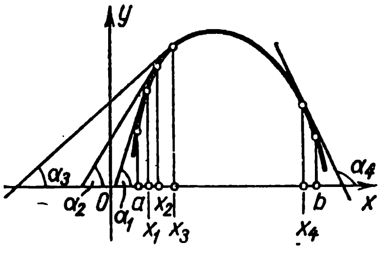


Fig. 115.

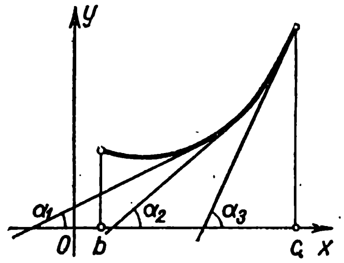


Fig. 116.

tangent of the angle of inclination α of the tangent line at the point with abscissa x, or $f'(x)=\tan\alpha$. For this reason, $f''(x)=[\tan\alpha]'_x$. If $f''(x)<0$ for all x on the interval (a, b), this means that $\tan\alpha$ decreases with increasing x. It is geometrically obvious that if $\tan\alpha$ decreases with increasing x, then the corresponding curve is convex. Theorem 1 is an analytic proof of this fact.

Theorem 1′ is illustrated geometrically in similar fashion (Fig. 116).

Example 1. Establish the intervals of convexity and concavity of a curve represented by the equation

$$y=2-x^2.$$

Solution. The second derivative

$$y''=-2<0$$

for all values of x. Hence, the curve is everywhere convex upwards (Fig. 117).

Example 2. The curve is given by the equation

$$y = e^x.$$

Since

$$y'' = e^x > 0$$

for all values of x, the curve is therefore everywhere concave (bulges, or is convex, downwards) (Fig. 118).

Example 3. A curve is defined by the equation

$$y = x^3.$$

Since

$$y'' = 6x,$$

$y'' < 0$ for $x < 0$ and $y'' > 0$ for $x > 0$. Hence, for $x < 0$ the curve is convex upwards, and for $x > 0$, convex down (Fig. 119).

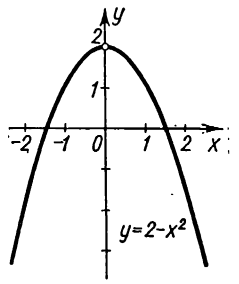


Fig. 117.

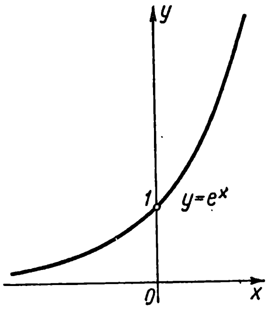


Fig. 118.

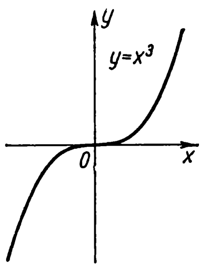


Fig. 119.

Definition 2. The point that separates the convex part of a continuous curve from the concave part is called the *point of inflection* of the curve.

In Figs. 119 and 120 the points O and B are points of inflection.

It is obvious that at the point of inflection the tangent **cuts** the curve, because on one side the curve lies **under** the tangent and on the other side, **above** it.

Let us now establish the sufficient conditions for a given point of a curve to be a point of inflection.

Theorem 2. *Let a curve be defined by the equation $y = f(x)$. If $f''(a) = 0$ or $f''(a)$ does not exist and if the derivative $f''(x)$ changes sign when passing through $x = a$, then the point of the curve with abscissa $x = a$ is the point of inflection.*

Proof. 1) Let $f''(x) < 0$ for $x < a$ and $f''(x) > 0$ for $x > a$.

Then for $x < a$ the curve is convex up and for $x > a$, it is convex down. Hence, the point A of the curve with abscissa $x = a$ is the point of inflection (Fig. 120).

2) If $f''(x)>0$ for $x<b$ and $f''(x)<0$ for $x>b$, then for $x<b$ the curve is convex down, and for $x>b$, it is convex up. Hence, the point B of the curve with abscissa $x=b$ is the point of inflection (see Fig. 121).

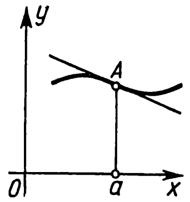


Fig. 120.

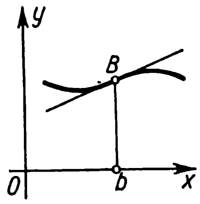


Fig. 121.

Example 4. Find the points of inflection and determine the intervals of convexity and concavity of the curve

$$y=e^{-x^2} \quad \text{(Gaussian curve).}$$

Solution. 1) Find the first and second derivatives:

$$y'=-2xe^{-x^2},$$
$$y''=2e^{-x^2}(2x^2-1).$$

2) The second derivative exists everywhere. Find the values of x for which $y''=0$:

$$2e^{-x^2}(2x^2-1)=0,$$
$$x_1=-\frac{1}{\sqrt{2}}, \quad x_2=\frac{1}{\sqrt{2}}.$$

3) Investigate the values obtained:

$$\text{for } x<-\frac{1}{\sqrt{2}} \text{ we have } y''>0,$$
$$\text{for } x>-\frac{1}{\sqrt{2}} \text{ we have } y''<0;$$

the second derivative changes sign when passing through the point x_1. Hence, for $x_1=-\frac{1}{\sqrt{2}}$, there is a point of inflection on the curve; its coordinates are: $\left(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$.

$$\text{For } x<\frac{1}{\sqrt{2}} \quad y''<0,$$
$$\text{for } x>\frac{1}{\sqrt{2}} \quad y''>0.$$

Thus, there is also a point of inflection on the curve for $x_2=\frac{1}{\sqrt{2}}$; its coordinates are $\left(\frac{1}{\sqrt{2}},\ e^{-\frac{1}{2}}\right)$. Incidentally, the existence of the second point of inflection follows directly from the symmetry of the curve about the y-axis.

4) From the foregoing it follows that

for $-\infty<x<-\frac{1}{\sqrt{2}}$ the curve is concave;

for $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$ the curve is convex;

for $\frac{1}{\sqrt{2}}<x<\infty$ the curve is concave.

5) From the expression of the first derivative

$$y'=-2xe^{-x^2}$$

it follows that

for $x<0$ $y'>0$, the function increases;
for $x>0$ $y'<0$, the function decreases;
for $x=0$ $y'=0$.

At this point the function has a maximum, namely, $y=1$. The foregoing analysis makes it easy to construct a graph of the curve (Fig. 122).

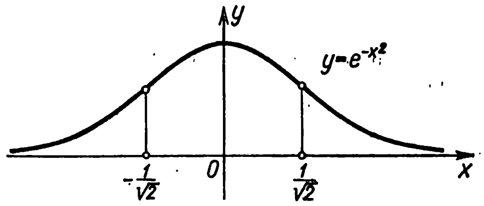


Fig. 122.

Example 5. Test the curve $y=x^4$ for points of inflection.

Solution. 1) Find the second derivative:

$$y''=12x^2.$$

2) Determine the points at which $y''=0$: $12x^2=0$; $x=0$.

3) Investigate the value $x=0$ obtained:

for $x<0$ $y''>0$, the curve is concave;
for $x>0$ $y''>0$, the curve is concave.

Thus, the curve has no points of inflection (Fig. 123).

Example 6. Investigate the following curve for points of inflection:

$$y=(x-1)^{\frac{1}{3}}.$$

Solution. 1) Find the first and second derivatives:

$$y' = \frac{1}{3}(x-1)^{-\frac{2}{3}}; \quad y'' = -\frac{2}{9}(x-1)^{-\frac{5}{3}}.$$

2) The second derivative does not vanish anywhere, but at $x=1$ it does not exist ($y'' = \pm\infty$).

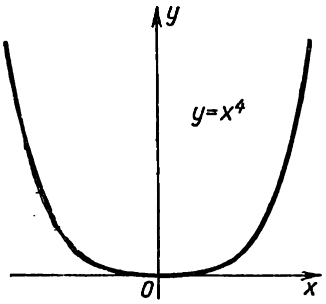


Fig. 123.

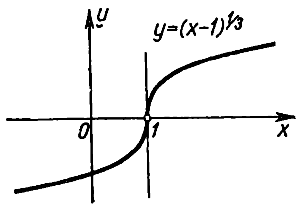


Fig. 124.

3) Investigate the value $x=1$:

for $x<1$ $y''>0$, the curve is concave;
for $x>1$ $y''<0$, the curve is convex.

Consequently, at $x=1$ there is a point of inflection (1, 0).

It will be noted that for $x=1$ $y'=\infty$; the curve at this point has a vertical tangent (Fig. 124).

SEC. 10. ASYMPTOTES

Very frequently one has to investigate the shape of a curve $y=f(x)$ and, consequently, the type of variation of the corresponding function **in the case of an unlimited increase** (in absolute value) of the abscissa or ordinate of a variable point of the curve, or of the abscissa and ordinate simultaneously. Here, an important special case is when the curve under study approaches a given line without bound as the variable point of the curve recedes to infinity. *)

Definition. The straight line A is called an *asymptote* to a curve, if the distance δ from the variable point M of the curve to this straight line approaches zero as the point M recedes to infinity (Figs. 125 and 126).

*) We say the variable point M moves along a curve to infinity if the distance of the point from the origin increases without bound.

In future we shall differentiate between *vertical asymptotes* (parallel to the axis of ordinates) and *inclined asymptotes* (not parallel to the axis of ordinates).

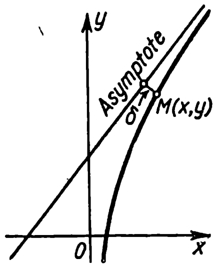


Fig. 125.

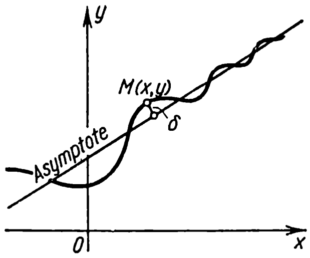


Fig. 126.

I. Vertical asymptotes.

From the definition of an asymptote it follows that

if $\lim\limits_{x\to a+0} f(x)=\infty$ or $\lim\limits_{x\to a-0} f(x)=\infty$ or $\lim\limits_{x\to a} f(x)=\infty$,

then the straight line $x=a$ is an asymptote to the curve $y=f(x)$; and, conversely, if the straight line $x=a$ is an asymptote, then one of the foregoing equalities is fulfilled.

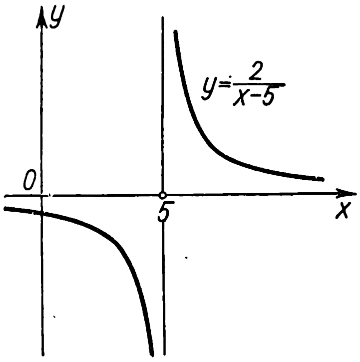


Fig. 127.

Consequently, to find vertical asymptotes one has to find values of $x=a$ such that when they are approached by the function $y=f(x)$ the latter approaches infinity. Then the straight line $x=a$ will be a vertical asymptote.

Example 1. The curve $y=\frac{2}{x-5}$ has a vertical asymptote $x=5$, since $y\to\infty$ as $x\to 5$ (Fig. 127).

Example 2. The curve $y=\tan x$ has an infinite number of vertical asymptotes:

$$x=\pm\frac{\pi}{2};\quad x=\pm\frac{3\pi}{2};\quad x=\pm\frac{5\pi}{2};\ \ldots$$

This follows from the fact that $\tan x\to\infty$ as x approaches the values $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$, or $-\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, \ldots$ (Fig. 128).

Example 3. The curve $y=e^{\frac{1}{x}}$ has a vertical asymptote $x=0$, since $\lim\limits_{x\to+0} e^{\frac{1}{x}} = \infty$ (Fig. 129).

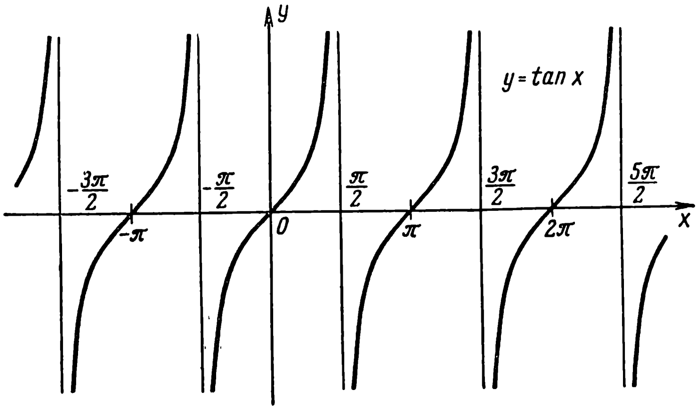


Fig. 128.

II. Inclined asymptotes.

Let the curve $y=f(x)$ have an inclined asymptote whose equation is

$$y = kx + b. \quad (1)$$

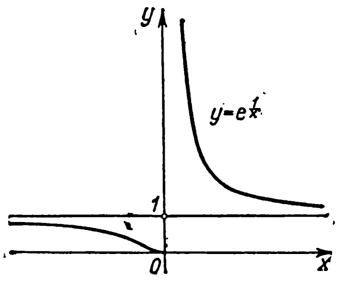


Fig. 129.

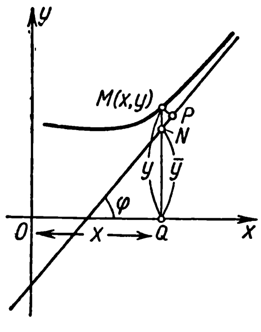


Fig. 130.

Determine the numbers k and b (Fig. 130). Let $M(x, y)$ be a point lying on the curve and $N(x, \bar{y})$, a point lying on the asymptote. The length of MP is equal to the distance from the point M to

the asymptote. It is given that

$$\lim_{x \to +\infty} MP = 0. \tag{2}$$

Designating the angle of inclination of the asymptote to the x-axis by φ, we find from $\triangle NMP$ that

$$NM = \frac{MP}{\cos \varphi}.$$

Since φ is a constant angle $\left(\text{not equal to } \frac{\pi}{2}\right)$, by virtue of the foregoing equation

$$\lim_{x \to +\infty} NM = 0 \tag{2'}$$

and, conversely, from (2') we get (2). But

$$NM = |QM - QN| = |y - \bar{y}| = |f(x) - (kx + b)|,$$

and (2') takes the form

$$\lim_{x \to +\infty} [f(x) - kx - b] = 0. \tag{3}$$

To summarise: if the straight line (1) is an asymptote, then (3) is fulfilled; and conversely, if, given constants k and b, equation (3) is fulfilled, then the straight line $y = kx + b$ is an asymptote. Let us now define k and b. Taking x outside the brackets in (3), we get

$$\lim_{x \to +\infty} x\left[\frac{f(x)}{x} - k - \frac{b}{x}\right] = 0.$$

Since $x \to +\infty$, the following equation must be fulfilled:

$$\lim_{x \to +\infty} \left[\frac{f(x)}{x} - k - \frac{b}{x}\right] = 0.$$

For b constant, $\lim\limits_{x \to \infty} \frac{b}{x} = 0$. Hence,

$$\lim_{x \to +\infty} \left[\frac{f(x)}{x} - k\right] = 0,$$

or

$$k = \lim_{x \to +\infty} \frac{f(x)}{x}. \tag{4}$$

Knowing k, we find b from (3):

$$b = \lim_{x \to +\infty} [f(x) - kx]. \tag{5}$$

Thus, if the straight line $y=kx+b$ is an asymptote, then k and b may be found from (4) and (5). Conversely, if the limits (4) and (5) exist, then (3) is fulfilled and the straight line $y=kx+b$ is an asymptote. If even one of the limits (4) or (5) does not exist, then the curve does not have an asymptote.

It should be noted that we carried out our investigation as applied to Fig. 130, as $x \to +\infty$, but all the arguments hold also for the case $x \to -\infty$.

Example 4. Find the asymptotes of the curve

$$y=\frac{x^2+2x-1}{x}.$$

Solution. 1) Look for vertical asymptotes:

when $x \to -0$ $y \to +\infty$;
when $x \to +0$ $y \to -\infty$.

Therefore, the straight line $x=0$ is a vertical asymptote.

2) Look for inclined asymptotes:

$$k=\lim_{x\to\pm\infty}\frac{y}{x}=\lim_{x\to\pm\infty}\frac{x^2+2x-1}{x^2}=$$
$$=\lim_{x\to\pm\infty}\left[1+\frac{2}{x}-\frac{1}{x^2}\right]=1,$$

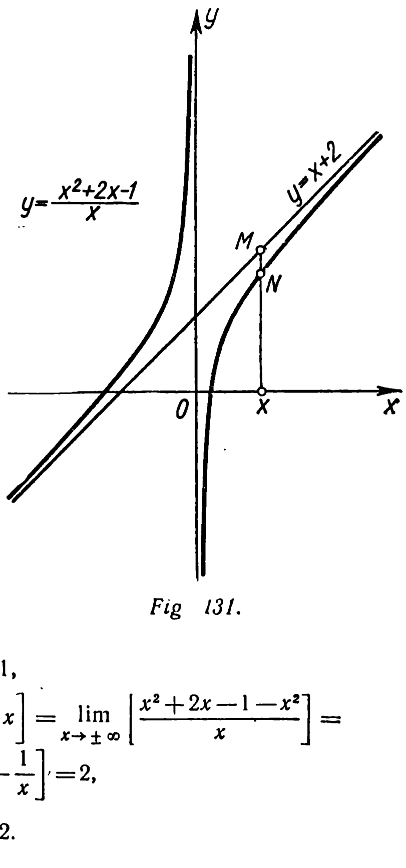


Fig 131.

that is,

$$k=1,$$

$$b=\lim_{x\to\pm\infty}[y-x]=\lim_{x\to\pm\infty}\left[\frac{x^2+2x-1}{x}-x\right]=\lim_{x\to\pm\infty}\left[\frac{x^2+2x-1-x^2}{x}\right]=$$
$$=\lim_{x\to\pm\infty}\left[2-\frac{1}{x}\right]=2,$$

or, finally,

$$b=2.$$

Therefore, the straight line

$$y=x+2$$

is an inclined asymptote to the given curve.

To investigate the mutual positions of a curve and an asymptote, let us consider the difference of the ordinates of the curve and the asymptote for one and the same value of x:

$$\frac{x^2+2x-1}{x}-(x+2)=-\frac{1}{x}.$$

This difference is negative for $x>0$, and positive for $x<0$; and so for $x>0$ the curve lies below the asymptote, and for $x<0$, it lies above the asymptote (Fig. 131).

Example 5. Find the asymptotes of the curve

$$y=e^{-x}\sin x+x.$$

Solution. 1) It is obvious that there are no vertical asymptotes.
2) Look for inclined asymptotes:

$$k=\lim_{x\to+\infty}\frac{y}{x}=\lim_{x\to+\infty}\frac{e^{-x}\sin x+x}{x}=\lim_{x\to+\infty}\left[\frac{e^{-x}\sin x}{x}+1\right]=1,$$

$$b=\lim_{x\to+\infty}[e^{-x}\sin x+x-x]=\lim_{x\to+\infty}e^{-x}\sin x=0.$$

Hence, the straight line

$$y=x$$

is an inclined asymptote as $x\to+\infty$.

The given curve has no asymptote as $x\to-\infty$. Indeed, the limit $\lim_{x\to-\infty}\frac{y}{x}$ does not exist, since $\frac{y}{x}=\frac{e^{-x}}{x}\times\sin x+1$. (Here, the first term increases without bound as $x\to-\infty$ and, therefore it has no limit.)

SEC. 11. GENERAL PLAN FOR INVESTIGATING FUNCTIONS AND CONSTRUCTING GRAPHS

The term "investigation of a function" usually implies the finding of:

1) the natural domain of the function;
2) the discontinuities of the function;
3) the intervals of increase and decrease of the function;
4) the maximum point and the minimum point, and also the maximal and minimal values of the functions;
5) the regions of convexity and concavity of the graph, and points of inflection;
6) the asymptotes of the graph of the function.

The graph of the function is constructed on the basis of such an investigation (it is sometimes wise to plot elements of the graph in the very process of investigation).

Note 1. If the function under investigation $y=f(x)$ is *even*, that is, such that upon change of sign of the argument the value of the function does not change, i. e., if

$$f(-x)=f(x),$$

then it is sufficient to investigate the function and construct its graph for positive values of the argument that lie within the domain of definition of the function. For negative values of the argument, the graph of the function is constructed on the grounds that the graph of an even function is symmetric about the ordinate axis.

Example 1. The function $y=x^2$ is even, since $(-x)^2=x^2$ (see Fig. 5).
Example 2. The function $y=\cos x$ is even, since $\cos(-x)=\cos x$ (see Fig. 17).

Note 2. If the function $y=f(x)$ is *odd*, that is, such that for any change in the argument the function changes sign, i. e., if

$$f(-x)=-f(x),$$

then it is sufficient to investigate this function in the case of positive values of the argument. The graph of an odd function is symmetric about the origin.

Example 3. The function $y=x^3$ is odd, since $(-x)^3=-x^3$ (see Fig. 7).
Example 4. The function $y=\sin x$ is odd, since $\sin(-x)=-\sin x$ (see Fig. 16).

Note 3. Since a knowledge of certain properties of a function allows us to judge of the other properties, it is sometimes advisable to choose the order of investigation on the basis of the specific peculiarities of the given function. For example, if we have found out that the given function is continuous and differentiable and if we have found the maximum point and the minimum point of this function, we have thus already determined also the range of increase and decrease of the function.

Example 5. Investigate the function

$$y=\frac{x}{1+x^2}$$

and construct its graph.

Solution. 1) The domain of the function is the interval $-\infty<x<\infty$. It will straightway be noted that for $x<0$ we have $y<0$, and for $x>0$ we have $y>0$.

2) The function is everywhere continuous.

3) Test the function for maximum and minimum, from the equation

$$y'=\frac{1-x^2}{(1+x^2)^2}=0.$$

Find the critical points:

$$x_1=-1,\ x_2=1.$$

Investigate the character of the critical points:

$$\text{for } x_1<-1 \text{ we have } y'<0;$$
$$\text{for } x_1>-1 \text{ we have } y'>0.$$

Hence, at $x=-1$ the function has a minimum:

$$y_{\min}=(y)_{x=-1}=-1.$$

And

$$\text{for } x<1 \text{ we have } y'>0;$$
$$\text{for } x>1 \text{ we have } y'<0.$$

Hence, at $x=1$ the function has a maximum:

$$y_{\max}=(y)_{x=1}=1.$$

4) Determine the domain of increase and decrease of the function:

for $-\infty<x<-1$ we have $y'<0$, the function decreases;
for $-1<x<1$ we have $y'>0$, the function increases;
for $1<x<\infty$ we have $y'<0$, the function decreases.

5) Determine the domains of convexity and concavity of the curve and the points of inflection: from the equality

$$y''=\frac{2x(x^2-3)}{(1+x^2)^3}=0$$

we get

$$x_1=-\sqrt{3},\quad x_2=0,\quad x_3=\sqrt{3}.$$

Investigating y'' as a function of x we find that

for $-\infty<x<-\sqrt{3}$ $y''<0$, the curve is convex;
for $-\sqrt{3}<x<0$ $y''>0$, the curve is concave;
for $0<x<\sqrt{3}$ $y''<0$, the curve is convex;
for $\sqrt{3}<x<\infty$ $y''>0$, the curve is concave.

Thus, the point with coordinates $x=-\sqrt{3}$, $y=-\frac{\sqrt{3}}{4}$ is a point of inflection; in exactly the same way, the points (0, 0) and $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$ are points of inflection.

6) Determine the asymptotes of the curve:

for $x\to+\infty$ $y\to 0$,
for $x\to-\infty$ $y\to 0$.

Consequently, the straight line $y=0$ is the only inclined asymptote. The curve has no vertical asymptotes because the function does not approach infinity for a single finite value of x.

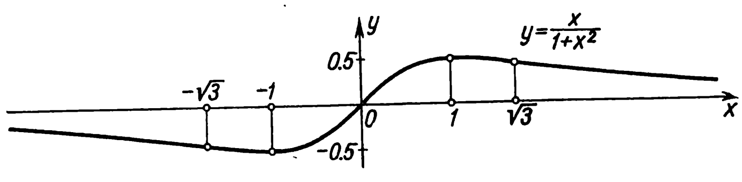


Fig. 132.

The graph of the curve under study is given in Fig. 132.

Example 6. Investigate the function

$$y=\sqrt[3]{2ax^2-x^3}$$

and construct its graph.

Solution. 1) The function is defined for all values of x.
2) The function is everywhere continuous.
3) Test the function for maximum and minimum:

$$y'=\frac{4ax-3x^2}{3\sqrt[3]{(2ax^2-x^3)^2}}=\frac{4a-3x}{3\sqrt[3]{x(2a-x)^2}}.$$

There is a derivative everywhere except for the points

$$x_1=0 \text{ and } x_2=2a.$$

Investigate the limiting values of the derivative as $x\to -0$ and as $x\to +0$:

$$\lim_{x\to -0}\frac{4a-3x}{3\sqrt[3]{x}\sqrt[3]{(2a+x)^2}}=-\infty,\quad \lim_{x\to +0}\frac{4a-3x}{3\sqrt[3]{x}\sqrt[3]{(2a+x)^2}}=+\infty$$

for $x<0$ $y'<0$, and for $x>0$ $y'>0$.

Hence, at $x=0$ the function has a minimum. The value of the function at this point is zero.

Now investigate the function at the other critical point $x_2=2a$. As $x\to 2a$ the derivative also approaches infinity. However, in this case, for all values of x close to $2a$ (both on the right and left of $2a$), the derivative is negative. Therefore, at this point the function has neither a maximum nor a minimum. At and about the point $x_2=2a$ the function decreases; the tangent to the curve at this point is vertical.

At $x=\frac{4a}{3}$ the derivative vanishes. Let us investigate the character of this critical point. Examining the expression of the first derivative, we note that

$$\text{for } x<\frac{4a}{3}\ y'>0, \text{ and for } x>\frac{4a}{3}\ y'<0.$$

Thus, at $x=\frac{4a}{3}$ the function has a maximum:

$$y_{\max}=\frac{2}{3}a\sqrt[3]{4}.$$

4) On the basis of this study we get the domains of increase and decrease of the function:

for $-\infty<x<0$ the function decreases;

for $0<x<\frac{4a}{3}$ the function increases;

for $\frac{4a}{3}<x<\infty$ the function decreases.

5) Determine the domains of convexity and concavity of the curve and the points of inflection: the second derivative

$$y''=-\frac{8a^2}{9x^{\frac{4}{3}}(2a-x)^{\frac{5}{3}}}$$

does not vanish at a single point. Yet there are two points at which the second derivative is discontinuous: $x_1=0$ and $x_2=2a$.

Let us investigate the sign of the second derivative near each of these points:

for $x<0$ we have $y''<0$ and the curve is convex up;

for $x>0$ we have $y''<0$ and the curve is convex up.

Hence, the point with abscissa $x=0$ is not a point of inflection.
For $x<2a$ we have $y''<0$ and the curve is convex upwards;
for $x>2a$ we have $y''>0$ and the curve is convex down.

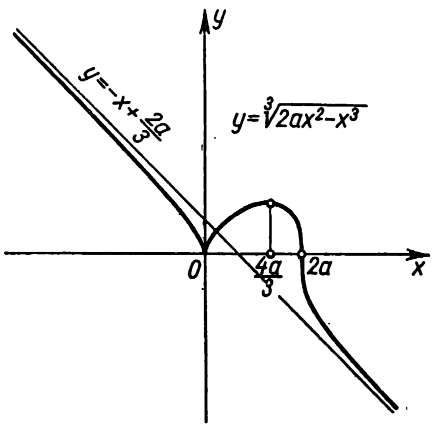


Fig. 133.

Hence, the point $(2a, 0)$ on the curve is a point of inflection.
6) Determine the asymptotes of the curve:

$$k=\lim_{x\to\pm\infty}\frac{y}{x}=\lim_{x\to\pm\infty}\frac{\sqrt[3]{2ax^2-x^3}}{x}=\lim_{x\to\pm\infty}\sqrt[3]{\frac{2a}{x}-1}=-1,$$

$$b=\lim_{x\to\pm\infty}\left[\sqrt[3]{2ax^2-x^3}+x\right]=$$

$$=\lim_{x\to\pm\infty}\frac{2ax^2-x^3+x^3}{\sqrt[3]{(2ax^2-x^3)^2}-x\sqrt[3]{2ax^2-x^3}+x^2}=\frac{2a}{3}.$$

Thus, the straight line

$$y=-x+\frac{2a}{3}$$

is an inclined asymptote to the curve $y=\sqrt[3]{2ax^2-x^3}$. The graph of this function is shown in Fig. 133.

SEC. 12. INVESTIGATING CURVES REPRESENTED PARAMETRICALLY

Let a curve be given by the parametric equations

$$\left.\begin{aligned} x&=\varphi(t),\\ y&=\psi(t). \end{aligned}\right\} \qquad (1)$$

In this case the investigation and construction of the curve is carried out just as for the curve given by the equation

$$y=f(x).$$

Evaluate the derivatives

$$\left.\begin{aligned} \frac{dx}{dt}&=\varphi'(t),\\ \frac{dy}{dt}&=\psi'(t). \end{aligned}\right\} \qquad (2)$$

For those points of the curve near which it is the graph of a certain function $y=f(x)$, evaluate the derivative

$$\frac{dy}{dx}=\frac{\psi'(t)}{\varphi'(t)}. \qquad (3)$$

We find the values of the parameter $t=t_1, t_2, \ldots, t_k$ for which at least one of the derivatives $\varphi'(t)$ or $\psi'(t)$ vanishes or becomes discontinuous. (We shall call these values of t critical values.) By formula (3), in each of the intervals (t_1, t_2); (t_2, t_3); ...; (t_{k-1}, t_k) and hence, in each of the intervals (x_1, x_2); (x_2, x_3); ...; (x_{k-1}, x_k) (where $x_i=\varphi(t_i)$), we determine the sign of $\frac{dy}{dx}$, in this way determining the domain of increase and decrease. This likewise enables us to determine the character of points that correspond to the values of the parameter $t_1, t_2, \ldots, t_k$. Next, evaluate

$$\frac{d^2y}{dx^2}=\frac{\psi''(t)\,\varphi'(t)-\varphi''(t)\,\psi'(t)}{[\varphi'(t)]^3}. \qquad (4)$$

From this formula, determine the direction of convexity of the curve at each point.

To find the asymptotes determine those values of t, upon approach to which either x or y approaches infinity, and those values of t upon approach to which both x and y approach infinity. Then carry out the investigation in the usual way.

The following examples will serve to illustrate some of the peculiarities that appear when investigating curves represented parametrically.

Example 1. Investigate the curve given by the equations

$$\left.\begin{aligned} x &= a\cos^3 t, \\ y &= a\sin^3 t. \end{aligned}\right\} \qquad (1')$$

Solution. The quantities x and y are defined for all values of t. But since the functions of $\cos^3 t$ and $\sin^3 t$ are periodic, with a period 2π, it is sufficient to consider the variation of the parameter t in the range from 0 to 2π; here the interval $[-a, a]$ is the range of x and the interval $[-a, a]$ is the range of y. Consequently, this curve has no asymptotes. Next, we find

$$\left.\begin{aligned} \frac{dx}{dt} &= -3a\cos^2 t \sin t, \\ \frac{dy}{dt} &= 3a\sin^2 t \cos t. \end{aligned}\right\} \qquad (2')$$

These derivatives vanish at $t=0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. Evaluate

$$\frac{dy}{dx} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t. \qquad (3')$$

On the basis of (2′) and (3′) we compile the following table:

Range of t	Corresponding range of x	Corresponding range of y	Sign of $\frac{dy}{dx}$	Type of variation of y as a function of x ($y=f(x)$)
$0<t<\frac{\pi}{2}$	$a>x>0$	$0<y<a$	$-$	Decreases
$\frac{\pi}{2}<t<\pi$	$0>x>-a$	$a>y>0$	$+$	Increases
$\pi<t<\frac{3\pi}{2}$	$-a<x<0$	$0>y>-a$	$-$	Decreases
$\frac{3\pi}{2}<t<2\pi$	$0<x<a$	$-a<y<0$	$+$	Increases

From the table it follows that equations (1′) define two continuous functions of the type $y=f(x)$, for $0\leqslant t\leqslant\pi$ $y\geqslant 0$ (see first two lines of the table), for $\pi<t\leqslant 2\pi$ $y<0$ (see two last lines of the table). From (3′) it follows that

$$\lim_{t\to\frac{\pi}{2}} \frac{dy}{dx} = \infty$$

and

$$\lim_{x\to\frac{3\pi}{2}} \frac{dy}{dx} = \infty.$$

At these points the tangent to the curve is vertical. We now find

$$\left.\frac{dy}{dt}\right|_{t=0} = 0, \quad \left.\frac{dy}{dt}\right|_{t=0} = 0, \quad \left.\frac{dy}{dt}\right|_{t=2\pi} = 0$$

At these points the tangent to the curve is horizontal. We then find

$$\frac{d^2y}{dx^2}=\frac{1}{3a\cos^4 t\sin t}.$$

Whence it follows that

for $0 < t < \pi$ $\frac{d^2y}{dx^2} > 0$ the curve is concave,

for $\pi < t < 2\pi$ $\frac{d^2y}{dx^2} < 0$ the curve is convex.

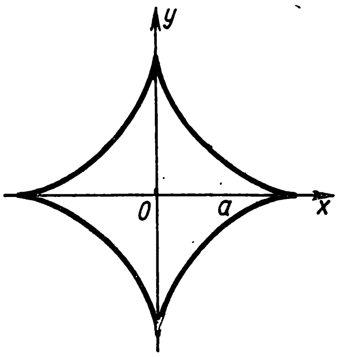


Fig. 134.

On the basis of this investigation we can construct a curve (Fig. 134), which is called an astroid.

Example 2. Construct a curve given by the following equations (folium of Descartes):

$$x=\frac{3at}{1+t^3},\quad y=\frac{3at^2}{1+t^3}. \qquad (1'')$$

Solution. Both functions are defined for all values of t except $t=-1$, and

$$\lim_{t\to-1-0} x=\lim_{t\to-1+0}\frac{3at}{1+t^3}=+\infty, \qquad \lim_{t\to-1-0} y=\lim_{t\to-1-0}\frac{3at^2}{1+t^3}=-\infty;$$

$$\lim_{t\to-1+0} x=-\infty, \qquad \lim_{t\to-1+0} y=+\infty.$$

Further note that

when $t=0$ $x=0$, $y=0$,
when $t\approx+\infty$ $x\to 0$, $y\to 0$,
when $t\approx-\infty$ $x\to 0$, $y\to 0$.

Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$:

$$\frac{dx}{dt}=\frac{6a\left(\frac{1}{2}-t^3\right)}{(1+t^3)^2},\quad \frac{dy}{dt}=\frac{3at(2-t^3)}{(1+t^3)^2}. \qquad (2'')$$

For t we get the following critical values:

$$t_1=-1,\quad t_2=0,\quad t_3=\frac{1}{\sqrt[3]{2}},\quad t_4=\sqrt[3]{2}.$$

Then we find

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{t(2-t^3)}{2\left(\frac{1}{2}-t^3\right)}. \qquad (3'')$$

On the basis of formulas (1″), (2″), and (3″) we compile the following table:

Range of t	Corresponding range of x	Corresponding range of y	Sign of $\frac{dy}{dx}$	Type of variation of y as a function of x $(y=f(x))$
$-\infty < t < -1$	$0 < x < +\infty$	$0 > y > -\infty$	$-$	Decreases
$-1 < t < 0$	$-\infty < x < 0$	$+\infty > y > 0$	$-$	Decreases
$0 < t < \frac{1}{\sqrt[3]{2}}$	$0 < x < a\sqrt[3]{4}$	$0 < y < a\sqrt[3]{2}$	$+$	Increases
$\frac{1}{\sqrt[3]{2}} < t < \sqrt[3]{2}$	$a\sqrt[3]{4} > x > a\sqrt[3]{2}$	$a\sqrt[3]{2} < y < a\sqrt[3]{4}$	$-$	Decreases
$\sqrt[3]{2} < t < \infty$	$a\sqrt[3]{2} > x > 0$	$a\sqrt[3]{4} > y > 0$	$+$	Increases

From (3″) we find

$$\left(\frac{dy}{dx}\right)_{\substack{t=0 \\ \binom{x=0}{y=0}}} = 0 \qquad \left(\frac{dy}{dx}\right)_{\substack{t=\infty \\ \binom{x=0}{y=0}}} = \infty.$$

Thus, the curve cuts the origin twice: with the tangent parallel to the x-axis and with the tangent parallel to the y-axis. Further,

$$\left(\frac{dy}{dx}\right)_{\substack{t=\frac{1}{\sqrt[3]{2}} \\ \left(\substack{x=a\sqrt[3]{4} \\ y=a\sqrt[3]{2}}\right)}} = \infty.$$

At this point the tangent to the curve is vertical.

$$\left(\frac{dy}{dx}\right)_{\substack{t=\sqrt[3]{2} \\ \left(\substack{x=a\sqrt[3]{2} \\ y=a\sqrt[3]{4}}\right)}} = 0.$$

At this point the tangent to the curve is horizontal. Let us investigate the question of the existence of an asymptote:

$$k = \lim_{x \to +\infty} \frac{y}{x} = \lim_{t \to -1-0} \frac{3at^2(1+t^3)}{3at(1+t^3)} = -1,$$

$$b = \lim_{x \to +\infty} (y - kx) = \lim_{x \to -1+0} \left[\frac{3at^2}{1+t^3} - (-1)\frac{3at}{1+t^3}\right] =$$

$$= \lim_{t \to -1-0} \left[\frac{3at(t+1)}{1+t^3}\right] = \lim_{t \to -1-0} \frac{3at}{1-t+t^2} = -a.$$

Hence, the straight line $y=-x-a$ is an asymptote to a branch of the curve as $x\to+\infty$.

Similarly we find

$$k=\lim_{x\to-\infty}\frac{y}{x}=-1,$$

$$b=\lim_{x\to-\infty}(y-kx)=-a.$$

Thus, the straight line is also an asymptote to a branch of the curve as $x\longrightarrow-\infty$.

On the basis of this investigation we construct the curve (Fig. 135).

Some problems involving investigation of curves will again be discussed in Chapter VIII "Singular Points of a Curve".

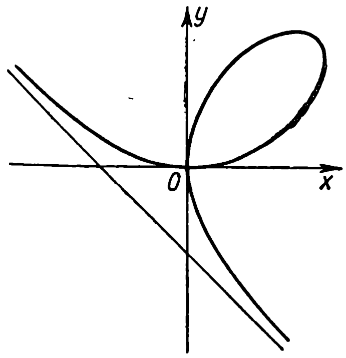


Fig. 135.

Exercises on Chapter V

Find the extremes of the functions: **1.** $y=x^2-2x+3$. *Ans.* $y_{min}=2$ at $x=1$. **2.** $y=\frac{x^3}{3}-2x^2+3x+1$. *Ans.* $y_{max}=\frac{7}{2}$ at $x=1$. **3.** $y=x^3-9x^2+15x+3$. *Ans.* $y_{max}=10$ at $x=1$, $y_{min}=-22$ at $x=5$. **4.** $y=-x^4+2x^2$. *Ans.* $y_{max}=1$ at $x=\pm1$, $y_{min}=0$ at $x=0$. **5.** $y=x^4-8x^2+2$. *Ans.* $y_{max}=2$ at $x=0$, $y_{min}=-14$ at $x=\pm2$. **6.** $y=3x^5-125x^3+2160x$. *Ans.* max at $x=-4$ and $x=3$, min at $x=-3$ and $x=4$. **7.** $y=2-(x-1)^{\frac{2}{3}}$. *Ans.* $y_{max}=2$ at $x=1$. **8.** $y=3-2(x+1)^{\frac{1}{3}}$. *Ans.* Neither max nor min. **9.** $y=\frac{x^2-3x+2}{x^2+3x+2}$. *Ans.* min at $x=\sqrt{2}$, max at $x=-\sqrt{2}$. **10.** $y=\frac{(x-2)(3-x)}{x^2}$. *Ans.* max at $x=\frac{12}{5}$. **11.** $y=2e^x+e^{-x}$. *Ans.* min at $x=-\frac{\ln 2}{2}$. **12.** $y=\frac{x}{\ln x}$. *Ans.* $y_{min}=e$ at $x=e$. **13.** $y=\cos x+\sin x\left(-\frac{\pi}{2}\leqslant x\leqslant\frac{\pi}{2}\right)$. *Ans.* $y_{max}=\sqrt{2}$ at $x=\frac{\pi}{4}$. **14.** $y=\sin 2x-x\left(-\frac{\pi}{2}\leqslant x\leqslant\frac{\pi}{2}\right)$. *Ans.* max at $x=\frac{\pi}{6}$, min at $x=-\frac{\pi}{6}$. **15.** $y=x+\tan x$. *Ans.* There is neither max nor min. **16.** $y=e^x\sin x$. *Ans.* min at $x=2k\pi-\frac{\pi}{4}$, max at $x=2k\pi+\frac{3}{4}\pi$. **17.** $y=x^4-2x^2+2$. *Ans.* max when $x=0$; two min when $x=-1$ and when $x=1$. **18.** $y=(x-2)^3(2x+1)$. *Ans.* $y_{min}\approx-8.24$ when $x=\frac{1}{8}$. **19.** $y=x+\frac{1}{x}$. *Ans.* min when $x=1$; max when $x=-1$. **20.** $y=x^2(a-x)^2$. *Ans.* $y_{max}=\frac{a^4}{16}$ when $x=\frac{a}{2}$; $y_{min}=0$ when $x=0$ and when $x=a$. **21.** $y=\frac{a^2}{x}+\frac{b^2}{a-x}$. *Ans.* max when $x=\frac{a^2}{a-b}$; min when $x=\frac{a^2}{a+b}$. **22.** $y=x+\sqrt{1-x}$. *Ans.* $y_{max}=1$ when $x=1$; $y_{min}=-1$

when $x=-1$. **23.** $y=x\sqrt{1-x}$ $(x\leqslant 1)$. *Ans.* $y_{\max}=\frac{2}{3}\sqrt{\frac{1}{3}}$ when $x=\frac{2}{3}$.
24. $y=\frac{x}{1+x^2}$. *Ans.* min when $x=-1$; max when $x=1$. **25.** $y=x\ln x$. *Ans.* min when $x=\frac{1}{e}$. **26.** $y=x\ln^2 x$. *Ans.* max when $x=e^{-\frac{1}{2}}$; min when $x=1$. 27. $y=\ln x-\text{arc tan } x$. *Ans.* The function increases. **28.** $y=\sin 3x-3\sin x$. *Ans.* min when $x=\frac{\pi}{2}$; max when $x=\frac{3\pi}{2}$. **29.** $y=2x+\text{arc tan } x$. *Ans.* No extrema. **30.** $y=\sin x\cos^2 x$. *Ans.* min when $x=\frac{\pi}{2}$; two max: when $x=\text{arc cos}\sqrt{\frac{2}{3}}$ and when $x=\text{arc cos}\left(-\sqrt{\frac{2}{3}}\right)$. **31.** $y=\text{arc sin }(\sin x)$. *Ans.* max when $x=\frac{(4m+1)\pi}{2}$; min when $x=\frac{(4m+3)\pi}{2}$.

Find the maximum and minimum values of the function on the indicated intervals: **32.** $y=-3x^4+6x^2-1$ $(-2\leqslant x\leqslant 2)$. *Ans.* Maximum $y=2$ at $x=\pm 1$, minimum $y=-25$ at $x=\pm 2$. **33.** $y=\frac{x^3}{3}-2x^2+3x+1$ $(-1\leqslant x\leqslant 5)$. *Ans* Maximum value $y=\frac{23}{3}$ at $x=5$, minimum value $y=-\frac{13}{3}$ at $x=-1$. 34. $y=\frac{x-1}{x+1}$ $(0\leqslant x\leqslant 4)$. *Ans.* Maximum value $y=\frac{3}{5}$ at $x=4$, minimum value $y=-1$ at $x=0$. **35.** $y=\sin 2x-x$ $\left(-\frac{\pi}{2}\leqslant x\leqslant\frac{\pi}{2}\right)$. *Ans.* Maximum value $y=\frac{\pi}{2}$ at $x=-\frac{\pi}{2}$, minimum value $y=-\frac{\pi}{2}$ at $x=\frac{\pi}{2}$.

36. Using square tin sheet with a side a, make a topless box of maximum volume by cutting equal squares at the corners and removing them and then bending the tin so as to form the sides of the box. What will the length of a side of the squares be? *Ans.* $\frac{a}{6}$.

37. Prove that of all rectangles that may be inscribed in a given circle, the square has the greatest area. Also show that the square will have the maximum perimeter as well.

38. Show that of all isosceles triangles inscribed in a given circle, an equilateral triangle has the largest perimeter.

39. Find a right triangle of maximum area with a hypotenuse h. *Ans.* Length of each side, $\frac{h}{2}$.

40. Find the height of a right cylinder with greatest volume that can be inscribed in a sphere of radius R. *Ans.* Height, $\frac{2R}{\sqrt{3}}$.

41. Find the height of a right cylinder with greatest lateral surface that may be inscribed in a given sphere of radius R. *Ans.* Height, $R\sqrt{2}$.

42. Find the height of a right cone with least volume circumscribed about a given sphere of radius R. *Ans.* $4R$ (the volume of the cone is equal to two volumes of the sphere).

43. A reservoir with a square bottom and open top is to be lined inside with lead. What are the dimensions of the reservoir (to hold 32 litres) that

will require the smallest amount of lead? *Ans.* Height, 0.2 metre, side of base, 0.4 metre (the side of the base must be twice the height).

44. A roofer wants to make an open channel of maximum capacity with bottom and sides 10 cm in width, and with the sides inclined at the same angle to the bottom. What is the width of the channel at the top? *Ans.* 20 cm.

45. Prove that a conic tent of given storage capacity requires the least material when its height is $\sqrt{2}$ times the radius of the base.

46. It is required to make a cylinder, open at the top, the walls and bottom of which have a given thickness. What should the dimensions of the cylinder be so that for a given storage capacity it will require the least material? *Ans.* If R is the inner radius of the base, v the inner volume of the cylinder, then $R=\sqrt[3]{\frac{v}{\pi}}$.

47. It is required to build a boiler out of a cylinder topped by two hemispheres and with walls of constant thickness so that for a given volume v it should have a minimum outer surface. *Ans.* It should have the shape of a sphere with inner radius $R=\sqrt[3]{\frac{3v}{4\pi}}$.

48. Construct an isosceles trapezoid, which for a given area S has a minimum perimeter; the angle at the base of the trapezoid is equal to α. *Ans.* The length of one of the nonparallel sides is $\sqrt{\frac{S}{\sin\alpha}}$.

49. Inscribe in a given sphere of radius R a regular triangular prism of maximum volume. *Ans.* The altitude of the prism is $\frac{2R}{\sqrt{3}}$.

50. It is required to circumscribe about a hemisphere of radius R a cone of minimum volume; the plane of the base of the cone coincides with that of the hemisphere; find the altitude of the cone. *Ans.* The altitude of the cone is $R\sqrt{3}$.

51. About a given cylinder of radius r circumscribe a right cone of minimum volume; we assume the planes and centres of the circular bases of the cylinder and the cone coincide. *Ans.* The radius of the base of the cone is equal to $\frac{3}{2}r$.

52. Out of sheet metal, having the shape of a circle of radius R, cut a sector such that it may be bent into a funnel of maximum storage capacity. *Ans.* The central angle of the sector is $2\pi\sqrt{\frac{2}{3}}$.

53. Of all circular cylinders inscribed in a given cube with side a so that their axes coincide with the diagonal of the cube and the circumferences of the bases touch its planes, find the cylinder with maximum volume. *Ans.* The altitude of the cylinder is equal to $\frac{a\sqrt{3}}{3}$; the radius of the base is $\frac{a}{\sqrt{6}}$.

54. Given, in a rectangular coordinate system, a point (x_0, y_0) lying in the first quadrant. Draw a straight line through this point so that it forms a triangle of least area with the positive directions of the axes. *Ans.* The straight line intercepts on the axes the segments $2x_0$ and $2y_0$; thus, it has the equation $\frac{x}{2x_0}+\frac{y}{2y_0}=1$.

55. Given a point on the axis of the parabola $y^2=2px$ at a distance a from the vertex, find the abscissa of the point of the curve closest to it. *Ans.* $x=a-p$.

56. Assuming that the strength of a beam of rectangular cross-section is directly proportional to the width and to the cube of the altitude, find the width of a beam of maximum strength that may be cut out of a log of diameter 16 cm. *Ans.* The width is 8 cm.

57. A torpedo boat is standing at anchor 9 km from the closest point of the shore; a messenger has to be sent to a camp 15 km (along the shore) from the point of the shore closest to the boat. Where should the messenger land so as to get to the camp in the shortest possible time, if he does 5 km/hr walking and 4 km/hr rowing. *Ans.* At a point 3 km from the camp.

58. A point moves over a plane in a medium situated outside the line MN with velocity v_1, and along the line MN with velocity v_2. What path between A and B, situated on MN, will it cover in the shortest time? The distance of A from MN is h, the distance of the projection α of A on the line MN from B is a. *Ans.* If ACB is the path of the point, then $\frac{\alpha C}{AC}=\frac{v_1}{v_2}$ for $\frac{\alpha B}{AB}\geqslant\frac{v_1}{v_2}$ and $\alpha C=\alpha B$ for $\frac{\alpha B}{AB}<\frac{v_1}{v_2}$.

59. A load w is hoisted by a lever; a force F is applied to one end, the point of support is at the other end of the lever. If the load is suspended from a point a centimetres from the fulcrum, and the lever rod weighs v grams per centimetre of length, what should the length of the rod be for the force (required to raise the load) to be a minimum? *Ans.* $x=\sqrt{\frac{2aw}{v}}$ cm.

60. For n measurements of an unknown quantity x the following readings have been obtained: $x_1, x_2, \ldots, x_n$. Show that the sum of the squares of the errors $(x-x_1)^2+(x-x_2)^2+\ldots+(x-x_n)^2$ will be least if for x we take the number $\frac{x_1+x_2+\ldots+x_n}{n}$.

61. To reduce the friction of a liquid against the walls of a channel, the area in contact with the liquid must be a minimum. Show that the best shape of an open rectangular channel with given cross-sectional area is that for which the width of the channel is twice its altitude.

Determine the points of inflection and the intervals of convexity and concavity of the curves.

62. $y=x^5$. *Ans.* For $x<0$ the curve is convex; for $x>0$ the curve is concave; at $x=0$ there is a point of inflection. 63. $y=1-x^2$. *Ans.* The curve is everywhere convex. 64. $y=x^3-3x^2-9x+9$. *Ans.* Point of inflection at $x=1$. 65. $y=(x-b)^3$. *Ans.* Point of inflection at $x=b$. 66. $y=x^4$. *Ans.* The curve is everywhere concave. 67. $y=\frac{1}{x^2+1}$. *Ans.* Point of inflection at $x=\pm\frac{1}{\sqrt{3}}$. 68. $y=\tan x$. *Ans.* Point of inflection at $x=n\pi$. 69. $y=xe^{-x}$. *Ans.* Point of inflection at $x=2$. 70. $y=a-\sqrt[3]{x-b}$. *Ans.* Point of inflection at $x=b$. 71. $y=a-\sqrt[5]{(x-b)^2}$. *Ans.* The curve has no point of inflection.

Find the asymptotes of the following curves: 72. $y=\frac{1}{x-1}$. *Ans.* $x=1$, $y=0$. 73. $y=\frac{1}{(x+2)^3}$. *Ans.* $x=-2$, $y=0$. 74. $y=c+\frac{a^3}{(x-b)^2}$. *Ans.* $x=b$,

$y=\frac{c}{4}$. 75. $y=e^{\frac{1}{x}}-1$. *Ans.* $x=0$, $y=0$. 76. $y=\ln x$. *Ans.* $x=0$.
77. $y^3=6x^2+x^3$. *Ans.* $y=x+2$. 78. $y^3=a^3-x^3$. *Ans.* $y+x=0$.
79. $y^2=\frac{x^3}{2a-x}$. *Ans.* $x=2a$. 80. $y^2(x-2a)=x^3-a^3$. *Ans.* $x=2a$, $y=\pm(x+a)$.

Investigate the following functions and construct their graphs:
81. $y=x^4-2x+10$. 82. $y=\frac{8a^3}{x^2+4a^2}$. 83. $y=e^{-\frac{1}{x}}$. 84. $y=\frac{6x}{1+x^2}$.
85. $y=\frac{4+x}{x^2}$. 86. $y=\frac{x}{x^2-1}$. 87. $y=\frac{x+2}{x^3}$. 88. $y=\frac{x^2}{1+x}$. 89. $y^2=x^3-x$.
90. $y=\frac{x^3}{3-x^2}$. 91. $y=\sqrt[3]{x^2}+2$. 92. $y=x-\sqrt[3]{x^3+1}$. 93. $y=\sqrt{\frac{x-1}{x+1}}$.
94. $y=xe^{-x}$. 95. $y=x^2e^{-x^2}$. 96. $y=x-\ln(x+1)$. 97. $y=\ln(x^2+1)$
98. $y=\sin 3x$. 99. $y=x+\sin x$. 100. $y=x\sin x$. 101. $y=e^{-x}\sin x$.
102. $y=\ln\sin x$. 103. $y=\frac{\ln x}{x}$. 104. $\begin{cases} x=t^2, \\ y=\frac{1}{2}t. \end{cases}$ 105. $\begin{cases} x=t^2, \\ y=t^3. \end{cases}$
106. $\begin{cases} x=a(t-\sin t), \\ y=a(1-\cos t). \end{cases}$ 107. $\begin{cases} x=ae^t\cos t, \\ y=ae^t\sin t. \end{cases}$

Additional Exercises

Find the asymptotes of the following lines: 108. $y=\frac{x^2+1}{1+x}$. *Ans.* $x=-1$; $y=x-1$. 109. $y=x+e^{-x}$. *Ans.* $y=x$. 110. $2y(x+1)^2=x^3$. *Ans.* $x=-1$; $y=\frac{1}{2}x-1$. 111. $y^3=a^3-x^3$. *Ans.* $x+y=0$. 112. $y=e^{-2x}\sin x$. *Ans.* $y=0$. 113. $y=e^{-x}\sin 2x+x$. *Ans.* $y=x$. 114. $y=x\ln\left(e+\frac{1}{x}\right)$. *Ans.* $x=-\frac{1}{e}$; $y=x+\frac{1}{e}$. 115. $y=xe^{\frac{1}{x^2}}$. *Ans.* $x=0$; $y=x$. 116. $x=\frac{2t}{1-t^2}$, $y=\frac{t^2}{1-t^2}$. *Ans.* $y=\pm\frac{1}{2}x-\frac{1}{2}$.

Investigate and graph the following functions: 117. $y=|x|$. 118. $y=\ln|x|$. 119. $y^2=x^3-x$. 120. $y=(x+1)^2(x-2)$. 121. $y=x+|x|$. 122. $y=\sqrt[3]{x^2}-x$. 123. $y=x^2\sqrt{x+1}$. 124. $y=\frac{x^2}{2}-\ln x$. 125. $y=\frac{x^2}{2}\ln x$. 126. $y=\frac{1}{e^x-1}$. 127. $y=\frac{x}{\ln x}$. 128. $y=x+\frac{\ln x}{x}$. 129. $y=x\ln x$. 130. $y=e^{\frac{1}{x}}-x$. 131. $y=|\sin 3x|$. 132. $y=\frac{\sin x}{x}$. 133. $y=x\arctan x$. 134. $y=x-2\arctan x$. 135. $y=e^{-2x}\sin 3x$. 136. $y=|\sin x|+x$. 137. $y=\sin x^2$. 138. $y=\cos^3 x+\sin^3 x$.

CHAPTER VI

THE CURVATURE OF A CURVE

SEC. 1. THE LENGTH OF AN ARC AND ITS DERIVATIVE

Let the arc of a curve M_0M (Fig. 136) be the graph of the function $y=f(x)$ defined on the interval (a, b). Let us determine the arc length of the curve. On the curve M_0M take the points $M_0, M_1, M_2, \ldots, M_{i-1}, M_i, \ldots, M_{n-1}, M$. Connecting the points we get a broken line $M_0M_1M_2\ldots M_{i-1}M_i\ldots M_{n-1}M$ inscribed in the arc M_0M. Denote the length of this broken line by P_n.

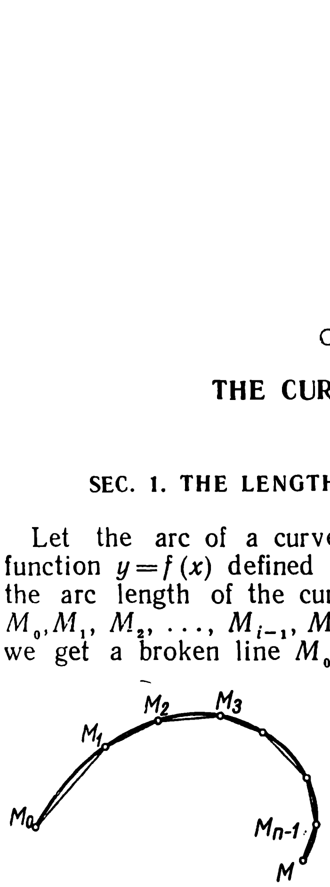


Fig 136.

The *length of the arc* M_0M is the limit (we denote it by s) approached by the length of the broken line as the largest of the lengths of the segments of the broken line $M_{i-1}M_i$ approaches zero, if this limit exists and is independent of any choice of points of the broken line $M_0M_1M_2\ldots M_{i-1}M_i\ldots M_{n-1}M$.

It will be noted that this definition of the arc length of an arbitrary curve is similar to the definition of the length of a circumference.

In Ch. XII it will be proved that if a function $f(x)$ and its derivative $f'(x)$ are continuous on an interval $[a, b]$, then the arc of the curve $y=f(x)$ lying between the points $[a, f(a)]$ and $[b, f(b)]$ has a definite length; a method will be shown for computing this length. There also, it will be established (as a corollary) that under the given conditions the ratio of the length of any arc of this curve to the length of its chord approaches unity when the length of the chord approaches zero, that is,

$$\lim_{M_0M \to 0} \frac{\text{length } \widehat{M_0M}}{\text{length } \overline{M_0M}} = 1.$$

This theorem may be readily proved for the circumference*) of

*) Consider the arc AB, the central angle of which is 2α. (Fig. 137). The length of this arc is $2R\alpha$ (R is the radius of the circle), and the length of its chord is $2R\sin\alpha$. Therefore, $\lim\limits_{\alpha \to 0} \frac{\text{length } \widehat{AB}}{\text{length } \overline{AB}} = \lim\limits_{\alpha \to 0} \frac{2R\alpha}{2R\sin\alpha} = 1$

a circle; however, in the general case we shall accept it without proof (Fig. 137).

Let us consider the following question.

On a plane we have a curve given by the equation

$$y = f(x).$$

Let $M_0(x_0, y_0)$ be some fixed point of the curve and $M(x, y)$, some variable point of the curve. Denote by s the arc length M_0M (Fig. 138).

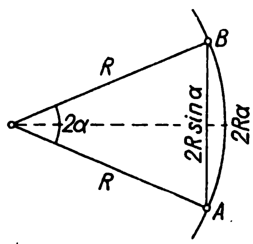


Fig. 137.

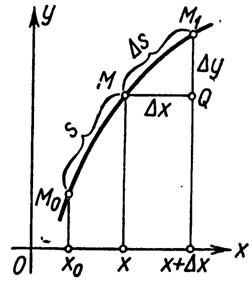


Fig. 138.

The arc length s will vary with changes in the abscissa x of the point M; in other words, s is a function of x. Find the derivative of s with respect to x.

Increase x by Δx. Then the arc s will change by $\Delta s =$ the length of $\overset{\frown}{MM_1}$. Let $\overline{MM_1}$ be the chord subtending this arc. In order to find $\lim\limits_{\Delta x \to 0} \frac{\Delta s}{\Delta x}$ do as follows: from ΔMM_1Q find

$$\overline{MM_1}^2 = (\Delta x)^2 + (\Delta y)^2.$$

Multiply and divide the left-hand side by Δs^2:

$$\left(\frac{\overline{MM_1}}{\Delta s}\right)^2 \Delta s^2 = (\Delta x)^2 + (\Delta y)^2.$$

Divide all terms of the equation by Δx^2:

$$\left(\frac{\overline{MM_1}}{\Delta s}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2.$$

Find the limit of the left and right sides as $\Delta x \to 0$. Taking into account that $\lim\limits_{\overline{MM_1} \to 0} \frac{\overline{MM_1}}{\Delta s} = 1$ and that $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ we get

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)$$

or

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}. \qquad (1)$$

For the *differential of the arc* we get the following expression:

$$ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad (2)$$

or *)

$$ds=\sqrt{dx^2+dy^2}. \qquad (2')$$

We have obtained an expression for the differential of arc length for the case when the curve is given by the equation $y=f(x)$. However, (2′) holds also for the case when the curve is represented by parametric equations.

If the curve is represented parametrically,

$$x=\varphi(t), \quad y=\psi(t),$$

then

$$dx=\varphi'(t)\,dt, \quad dy=\psi'(t)\,dt,$$

and expression (2′) takes the form

$$ds=\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt.$$

SEC. 2. CURVATURE

One of the elements that characterise the shape of a curve is the degree of its bentness, or curvature.

Let there be a curve that does not intersect itself and has a definite tangent at each point. Draw tangents to the curve at any two points A and B and denote the angle formed by these tangents by α [or, more precisely, the angle through which the tangent turns from A to B (Fig. 139)]. This angle is called the *angle of contingence* of the arc AB. Of two arcs of the same length, that arc is more curved which has a greater angle of contingence (Figs. 139 and 140).

On the other hand, when considering arcs of different length we cannot evaluate the degree of their curvature solely by the appro-

*) Strictly speaking, (2′) holds only for the case when $dx>0$. But if $dx<0$, then $ds=-\sqrt{dx^2+dy^2}$. For this reason, in the general case this formula is more correctly written as $|ds|=\sqrt{dx^2+dy^2}$.

priate angles of contingence. Whence it follows that a complete description of the curvature of a curve is given by the **ratio** of the angle of contingence to the length of the corresponding arc.

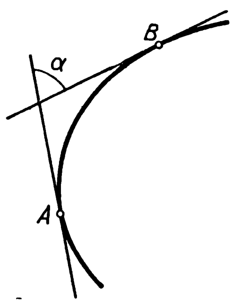


Fig. 139.

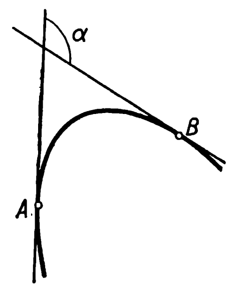


Fig. 140.

Definition 1. The *average curvature* K_{av} of an arc $\overset{\frown}{AB}$ is the ratio of the corresponding angle of contingence α to the length of the arc:

$$K_{av}=\frac{\alpha}{\overset{\frown}{AB}}.$$

For one and the same curve, the average curvature of its different parts (arcs) may be different; for example, for the curve shown in Fig. 141, the average curvature of the arc $\overset{\frown}{AB}$ is not equal to the average curvature of the arc $\overset{\frown}{A_1B_1}$, although the lengths of their arcs are the same. What is more, at different points the curvature of the curve differs. To characterise the degree of curvature of a given line in the immediate neighbourhood of a given point A, we introduce the concept of curvature of a curve at a given point.

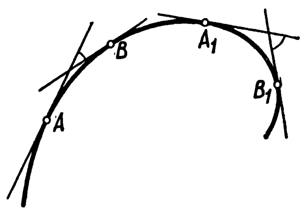


Fig. 141.

Definition 2. The *curvature* K_A of a line at a given point A is the limit of the average curvature of the arc $\overset{\frown}{AB}$ when the length of this arc approaches zero (that is, when the point B approaches the point A):

$$K_A=\lim_{B\to A} K_{av}=\lim_{\overset{\frown}{AB}\to 0}\frac{\alpha}{\overset{\frown}{AB}}\ ^{*)}.$$

*) We assume that the magnitude of the limit does not depend on which side of the point A we take the variable point B on the curve.

Example. For a circle of radius r: 1) determine the average curvature of the arc $\overset{\frown}{AB}$ subtending the central angle α (Fig. 142); 2) determine the curvature at the point A.

Solution. 1) Obviously the angle of contingence of the arc $\overset{\frown}{AB}$ is α, the length of the arc is αr. Hence,

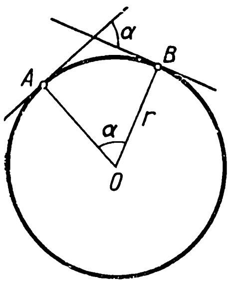


Fig. 142.

$$K_{av}=\frac{\alpha}{\alpha r}$$

or

$$K_{av}=\frac{1}{r}.$$

2) The curvature at the point A is

$$K=\lim_{\alpha\to 0}\frac{\alpha}{\alpha r}=\frac{1}{r}.$$

Thus, the average curvature of the arc of a circle of radius r is independent of the length and position of the arc, and for all arcs it is equal to $\frac{1}{r}$. Likewise, the curvature of a circle at any point is independent of the choice of this point and is equal to $\frac{1}{r}$.

Note. It should be noted that, generally speaking, for any curve the curvature at its various points differs (this will be seen later).

SEC. 3. CALCULATION OF CURVATURE

Let us develop a formula for finding the curvature of any line at any point $M(x, y)$. We shall assume that the curve is represented in the Cartesian coordinate system by an equation of the form

$$y=f(x) \qquad (1)$$

and that the function $f(x)$ has a continuous second derivative.

Draw tangents to the curve at the points M and M_1 with abscissas x and $x+\Delta x$ and denote by φ and $\varphi+\Delta\varphi$ the angles of inclination of these tangents (Fig. 143).

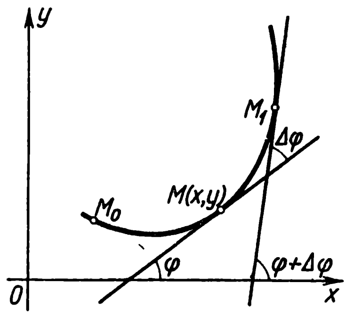


Fig. 143.

We reckon the length of the arc $\overset{\frown}{M_0M}$ from some fixed point M_0 and denote it by s; then $\Delta s=\overset{\frown}{M_0M_1}-\overset{\frown}{M_0M}$, and $|\Delta s|=\overset{\frown}{MM_1}$.

As will be seen from Fig. 143, the angle of contingence corres-

ponding to the arc $\widehat{MM_1}$ is equal to the absolute value*) of the difference of the angles φ and $\varphi+\Delta\varphi$, which means it is equal to $|\Delta\varphi|$.

According to the definition of average curvature of a curve, on the segment MM_1 we have

$$K_{av}=\frac{|\Delta\varphi|}{|\Delta s|}=\left|\frac{\Delta\varphi}{\Delta s}\right|.$$

To obtain the **curvature at the point** M, it is necessary to find the limit of the expression obtained on the condition that the arc length $\widehat{MM_1}$ approaches zero:

$$K=\lim_{\Delta s\to 0}\left|\frac{\Delta\varphi}{\Delta s}\right|.$$

Since the quantities φ and s both depend on x (are functions of x), φ may thus be considered as a function of s. We may consider that this function is represented parametrically by means of the parameter x. Then

$$\lim_{\Delta s\to 0}\frac{\Delta\varphi}{\Delta s}=\frac{d\varphi}{ds}$$

and, consequently,

$$K=\left|\frac{d\varphi}{ds}\right|. \tag{2}$$

To calculate $\frac{d\varphi}{ds}$, we make use of the formula for differentiating a function represented parametrically:

$$\frac{d\varphi}{ds}=\frac{\frac{d\varphi}{dx}}{\frac{ds}{dx}}.$$

To express the derivative $\frac{d\varphi}{dx}$ in terms of the function $y=f(x)$, we note that $\tan\varphi=\frac{dy}{dx}$ and, therefore,

$$\varphi=\operatorname{arc\,tan}\frac{dy}{dx}.$$

Differentiating the latter equality with respect to x, we get

$$\frac{d\varphi}{dx}=\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}.$$

*) It is obvious that for the curve given in Fig. 143, $|\Delta\varphi|=\Delta\varphi$ since $\Delta\varphi>0$.

As regards the derivative $\frac{ds}{dx}$, we found in Sec. 1, Ch. VI, that

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Therefore,

$$\frac{d\varphi}{ds}=\frac{\frac{d\varphi}{dx}}{\frac{ds}{dx}}=\frac{\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}=\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

or, since $K=\left|\frac{d\varphi}{ds}\right|$, we finally get

$$K=\frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}. \qquad (3)$$

It is thus possible to find the curvature at any point of a curve where there exists a second derivative $\frac{d^2y}{dx^2}$ and where it is continuous. Calculations are done with formula (3). It should be noted that when calculating the curvature of a curve only the arithmetical (positive) value of the root in the denominator should be taken, since the curvature of a line cannot (by definition) be negative.

Example 1. Determine the curvature of the parabola $y^2=2px$:
a) at an arbitrary point $M(x, y)$;
b) at the point $M_1(0, 0)$;
c) at the point $M_2\left(\frac{p}{2}, p\right)$.

Solution. Find the first and second derivatives of the function $y=\sqrt{2px}$:

$$\frac{dy}{dx}=\frac{p}{\sqrt{2px}}; \quad \frac{d^2y}{dx^2}=-\frac{p^2}{(2px)^{3/2}}.$$

Substituting the expressions obtained into (3), we get

a) $$K=\frac{p^2}{(2px+p^2)^{3/2}};$$

b) $$K_{\substack{x=0\\y=0}}=\frac{1}{p};$$

c) $$K_{\substack{x=\frac{p}{2}\\y=p}}=\frac{1}{2\sqrt{2p}}.$$

Example 2. Determine the curvature of the straight line $y=ax+b$ at an arbitrary point (x, y).

Solution.

$$y'=a, \ y''=0.$$

Referring to (3) we get

$$K=0.$$

Thus, a straight line is a "line of zero curvature". This very same result is readily obtainable directly from the definition of curvature.

SEC. 4. CALCULATION OF THE CURVATURE OF A LINE REPRESENTED PARAMETRICALLY

Let a curve be represented parametrically:

$$x=\varphi(t), \quad y=\psi(t).$$

Then (see Sec. 24, Ch. III).

$$\frac{dy}{dx}=\frac{\psi'(t)}{\varphi'(t)}, \quad \frac{d^2y}{dx^2}=\frac{\psi''\varphi'-\psi'\varphi''}{(\varphi')^3}.$$

Substituting the expressions obtained into formula (3) of the preceding section, we get

$$K=\frac{|\psi''\varphi'-\psi'\varphi''|}{[\varphi'^2+\psi'^2]^{3/2}}. \qquad (1)$$

Example. Determine the curvature of the cycloid

$$x=a(t-\sin t), \ y=a(1-\cos t)$$

at an arbitrary point (x, y).

Solution.

$$\frac{dx}{dt}=a(1-\cos t), \quad \frac{d^2x}{dt^2}=a\sin t, \quad \frac{dy}{dt}=a\sin t, \quad \frac{d^2y}{dt^2}=a\cos t.$$

Substituting the expressions obtained into (3), we get

$$K=\frac{|a(1-\cos t)a\cos t-a\sin t\cdot a\sin t|}{|a^2(1-\cos t)^2+a^2\sin^2 t|^{3/2}}=\frac{|\cos t-1|}{2^{3/2}a(1-\cos t)^{3/2}}=$$
$$=\frac{1}{2^{3/2}a(1-\cos t)^{1/2}}=\frac{1}{4a\left|\sin\frac{t}{2}\right|}.$$

SEC. 5. CALCULATION OF THE CURVATURE OF A LINE GIVEN BY AN EQUATION IN POLAR COORDINATES

Given a curve represented by an equation of the form

$$\rho=f(\theta). \qquad (1)$$

Write the transformation formulas from polar coordinates to Cartesian coordinates:

$$\left.\begin{aligned} x &= \varrho\cos\theta, \\ y &= \varrho\sin\theta. \end{aligned}\right\} \tag{2}$$

If in these formulas we replace ϱ by its expression in terms of θ, i.e., $f(\theta)$, we get

$$\left.\begin{aligned} x &= f(\theta)\cos\theta, \\ y &= f(\theta)\sin\theta. \end{aligned}\right\} \tag{3}$$

The latter equations may be regarded as parametric equations of curve (1), the parameter being θ.

Then

$$\frac{dx}{d\theta}=\frac{d\varrho}{d\theta}\cos\theta-\varrho\sin\theta,\quad \frac{dy}{d\theta}=\frac{d\varrho}{d\theta}\sin\theta+\varrho\cos\theta,$$

$$\frac{d^2x}{d\theta^2}=\frac{d^2\varrho}{d\theta^2}\cos\theta-2\frac{d\varrho}{d\theta}\sin\theta-\varrho\cos\theta,$$

$$\frac{d^2y}{d\theta^2}=\frac{d^2\varrho}{d\theta^2}\sin\theta+2\frac{d\varrho}{d\theta}\cos\theta-\varrho\sin\theta.$$

Substituting the latter expressions into (1) of the preceding section, we get a formula for calculating the curvature of a curve in polar coordinates:

$$K=\frac{|\varrho^2+2\varrho'^2-\varrho\varrho''|}{(\varrho^2+\varrho'^2)^{3/2}}. \tag{4}$$

Example. Determine the curvature of the spiral of Archimedes $\varrho=a\theta\,(a>0)$ at an arbitrary point (Fig. 144).

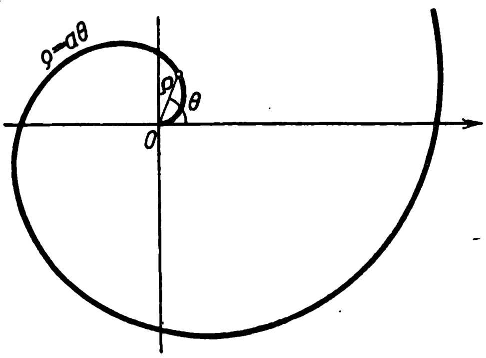


Fig. 144.

Solution.

$$\frac{d\varrho}{d\theta}=a;\qquad \frac{d^2\varrho}{d\theta^2}=0.$$

Hence

$$K=\frac{|a^2\theta^2+2a^2|}{(a^2\theta^2+a^2)^{3/2}}=\frac{1}{a}\,\frac{\theta^2+2}{(\theta^2+1)^{3/2}}.$$

It will be noted that for large values of θ we have the approximate equalities $\frac{\theta^2+2}{\theta^2}\approx 1$, $\frac{\theta^2+1}{\theta^2}\approx 1$; therefore, replacing θ^2+2 by θ^2 and θ^2+1 by θ^2 in the foregoing formula,

we get an approximate formula (for large values of θ)

$$K \approx \frac{1}{a} \frac{\theta^2}{(\theta^2)^{3/2}} = \frac{1}{a\theta}.$$

Thus, for large values of θ the spiral of Archimedes has, approximately, the same curvature as a circle of radius $a\theta$.

SEC. 6. THE RADIUS AND CIRCLE OF CURVATURE. CENTRE OF CURVATURE. EVOLUTE AND INVOLUTE

Definition. The quantity R, which is the reciprocal of the curvature K of a line at a given point M, is called the *radius of curvature* of the line at the point in question:

$$R = \frac{1}{K} \tag{1}$$

or

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}. \tag{2}$$

Draw a normal, at the point M, to a curve in the direction of the concavity of the curve, and lay off a segment MC equal to the radius R of the curvature of the curve at the point M. The

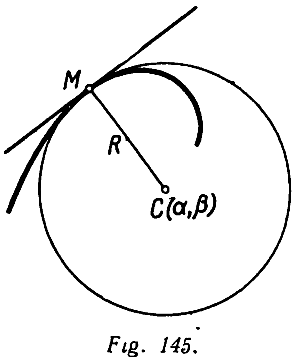


Fig. 145.

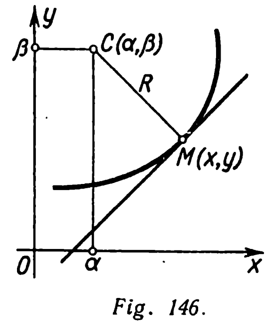


Fig. 146.

point C is called the *centre of curvature* of the given curve at M; the circle, of radius R, with centre at C (passing through M) is called the *circle of curvature* of the given curve at the point M (Fig. 145).

From the definition of circle of curvature it follows that at a given point the curvature of a curve and the curvature of a circle of curvature are the same.

Let us derive formulas defining the coordinates of the centre of curvature.

Let a curve be given by the equation

$$y=f(x). \tag{3}$$

Take a point $M(x, y)$ on this curve and determine the coordinates α and β of the centre of curvature corresponding to this point (Fig. 146). To do this, write the equation of the normal to the curve at M:

$$Y-y=-\frac{1}{y'}(X-x). \tag{4}$$

(Here, X and Y are the moving coordinates of the point of the normal.)

Since the point $C(\alpha, \beta)$ lies on the normal, its coordinates must satisfy equation (4):

$$\beta-y=-\frac{1}{y'}(\alpha-x). \tag{5}$$

Further, the point $C(\alpha, \beta)$ is separated from $M(x, y)$ by a distance equal to the radius of curvature R:

$$(\alpha-x)^2+(\beta-y)^2=R^2. \tag{6}$$

Solving equations (5) and (6) simultaneously, we find α and β:

$$(\alpha-x)^2+\frac{1}{y'^2}(\alpha-x)^2=R^2,$$

$$(\alpha-x)^2=\frac{y'^2}{1+y'^2}R^2.$$

Whence

$$\alpha=x\pm\frac{y'}{\sqrt{1+y'^2}}R, \quad \beta=y\mp\frac{1}{\sqrt{1+y'^2}}R,$$

and since $R=\frac{(1+y'^2)^{3/2}}{|y''|}$,

$$\alpha=x\pm\frac{y'(1+y'^2)}{|y''|}, \quad \beta=y\mp\frac{1+y'^2}{|y''|}.$$

In order to decide which signs (top or bottom) to take in the latter formulas, we must examine the case $y''>0$ and the case $y''<0$. If $y''>0$, then at this point the curve is concave, and, hence, $\beta>y$ (Fig. 146), and for this reason we take the bottom signs. Taking into account that in this case $|y''|=y''$, the formulas of the coordinates of the centre of curvature will be

$$\left.\begin{aligned}\alpha&=x-\frac{y'(1+y'^2)}{y''},\\ \beta&=y+\frac{1+y'^2}{y''}.\end{aligned}\right\} \tag{7}$$

Similarly, it may be shown that formulas (7) will hold for the case $y'' < 0$ as well.

If the curve is represented by the parametric equations

$$x = \varphi(t), \quad y = \psi(t),$$

then the coordinates of the centre of curvature are readily obtainable from (7) by substituting, in place of y' and y'', their expressions in terms of the parameter

$$y' = \frac{y'_t}{x'_t}; \quad y'' = \frac{x'_t y''_t - x''_t y'_t}{x'^3_t}.$$

Then

$$\left.\begin{aligned} \alpha &= x - \frac{y'(x'^2 + y'^2)}{x'y'' - x''y'}, \\ \beta &= y + \frac{x'(x'^2 + y'^2)}{x'y'' - x''y'}. \end{aligned}\right\} \qquad (7')$$

Example 1. To determine the coordinates of the centre of curvature of the parabola

$$y^2 = 2px:$$

a) at an arbitrary point $M(x, y)$; b) at the point $M_0(0, 0)$; c) at the point $M_1\left(\frac{p}{2}, p\right)$.

Solution. Substituting the values $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into (7) we get (Fig. 147):

a) $\alpha = 3x + p$, $\beta = \frac{(2x)^{3/2}}{\sqrt{p}}$;

b) at $x = 0$ we find $\alpha = p$, $\beta = 0$;

c) at $x = \frac{p}{2}$ we have $\alpha = \frac{5p}{2}$, $\beta = -p$.

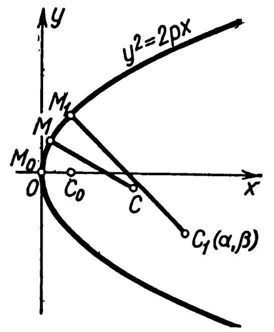


Fig. 147.

If at $M_1(x, y)$ of a given line the curvature differs from zero, then a very definite centre of curvature $C_1(\alpha, \beta)$ corresponds to this point. The totality of all centres of curvature of the given line forms a certain new line, called the **evolute**, with respect to the first.

Thus, the locus of centres of curvature of a given line is called the *evolute*. As related to its evolute, the given line is called the *evolvent* or *involute*.

If a given curve is defined by the equation $y = f(x)$, then equations (7) may be regarded as the parametric equations of the evo-

lute with parameter x. Eliminating from these equations the parameter x (if this is possible), we get an immediate relationship between the coordinates of the evolute α and β. But if the curve is given by parametric equations $x=\varphi(t)$, $y=\psi(t)$, then equations (7') yield the parametric equations of the evolute (since the quantities x, y, x', y', x'', y'' are functions of t).

Example 2. Find the equation of the evolute of the parabola

$$y^2=2px.$$

Solution. On the basis of Example 1 we have, for any point (x, y) of the parabola,

$$\alpha=3x+p,$$

$$\beta=-\frac{(2x)^{3/2}}{\sqrt{p}}.$$

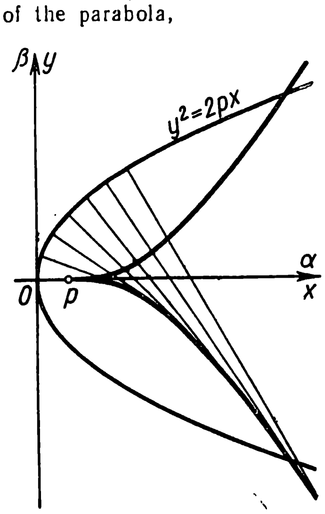


Fig. 148.

Eliminating the parameter x from these equations, we get

$$\beta^2=\frac{8}{27p}(\alpha-p)^3.$$

This is the equation of a semicubical parabola (Fig. 148).

Example 3. Find the equation of the evolute of an ellipse represented by the parametric equations

$$x=a\cos t,\quad y=b\sin t.$$

Solution. Evaluate the derivatives of x and y with respect to t:

$$x'=-a\sin t,\quad y'=b\cos t;$$
$$x''=-a\cos t,\quad y''=-b\sin t.$$

Substituting the expressions of the derivatives into (7'), we get

$$\alpha=a\cos t-\frac{b\cos t\,(a^2\cdot\sin^2 t+b^2\cos^2 t)}{ab\sin^2 t+ab\cos^2 t}=$$
$$=a\cos t-a\cos t\sin^2 t-\frac{b^2}{a}\cos^3 t=\left(a-\frac{b^2}{a}\right)\cos^3 t.$$

Thus,

$$\alpha=\left(a-\frac{b^2}{a}\right)\cos^3 t.$$

Similarly we get

$$\beta=\left(b-\frac{a^2}{b}\right)\sin^3 t.$$

Eliminating the parameter t, we get the equation of the evolute of the ellipse in the form

$$\left(\frac{\alpha}{b}\right)^{2/3}+\left(\frac{\beta}{a}\right)^{2/3}=\left(\frac{a^2-b^2}{ab}\right)^{2/3}.$$

Here, α and β are the coordinates of the evolute (Fig. 149).

Example 4. Find the parametric equations of the evolute of the cycloid

$$x=a(t-\sin t),$$
$$y=a(1-\cos t).$$

Solution.

$$x'=a(1-\cos t);\quad y'=a\sin t;$$
$$x''=a\sin t;\quad y''=-a\cos t.$$

Substituting the expressions obtained into (7′), we get

$$\alpha=a(t+\sin t),$$
$$\beta=-a(1-\cos t).$$

Rearrange the variables, putting

$$\alpha=\xi-\pi a,$$
$$\beta=\eta-2a,$$
$$t=\tau-\pi;$$

then the equations of the evolute will take the form

$$\xi=a(\tau-\sin\tau),$$
$$\eta=a(1-\cos\tau);$$

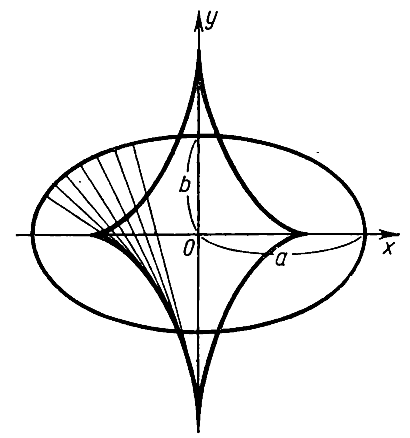


Fig. 149.

they define, in coordinates ξ, η, a cycloid with the same generating circle of radius a. Thus, the evolute of a cycloid is that same cycloid displaced along the x-axis by $-\pi a$ and along the y-axis by $-2a$ (Fig. 150).

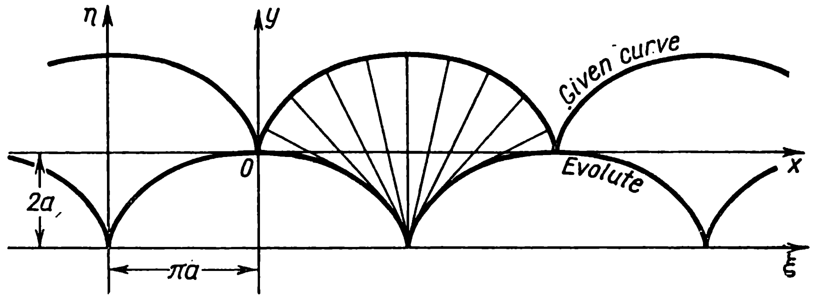


Fig. 150.

SEC. 7. THE PROPERTIES OF AN EVOLUTE

Theorem 1. *The normal to a given curve is a tangent to its evolute.*

Proof. The slope of the line tangent to an evolute defined by the parametric equations (7′) of the preceding section is

equal to

$$\frac{d\beta}{d\alpha}=\frac{\dfrac{d\beta}{dx}}{\dfrac{d\alpha}{dx}}.$$

Noting that [by virtue of the same equations (7′)]

$$\frac{d\alpha}{dx}=-\frac{3y''^2y'^2-y'y'''-y'^3y'''}{y''^2}=-y'\,\frac{3y'y''-y'''-y'y'''}{y''^2}, \qquad (1)$$

$$\frac{d\beta}{dx}=-\frac{3y''y'-y'''-y'^2y'''}{y''^2}, \qquad (2)$$

we get the relationship

$$\frac{d\beta}{d\alpha}=-\frac{1}{y'}.$$

But y' is the slope of the line tangent to the curve at the corresponding point; it therefore follows from the relationship obtained that the tangent to the curve and the tangent to its evolute at the corresponding point are mutually perpendicular; that is, the normal to a curve is the tangent to the evolute.

Theorem 2. *If, over a certain segment M_1M_2 of a curve, the radius of curvature varies monotonically (i.e., either only increases or only decreases), then the increment in the arc length of the evolute on this segment of the curve is equal (in absolute value) to the corresponding increment in the radius of curvature of the given curve.*

Proof. From formula (2′), Sec. 1, Ch. VI, we have

$$ds^2=d\alpha^2+d\beta^2$$

where ds is the differential of the arc length of the evolute; whence

$$\left(\frac{ds}{dx}\right)^2=\left(\frac{d\alpha}{dx}\right)^2+\left(\frac{d\beta}{dx}\right)^2.$$

Substituting, here, the expressions (1) and (2), we get

$$\left(\frac{ds}{dx}\right)^2=(1+y'^2)\left(\frac{3y'y''^2-y'''-y'^2y'''}{y''^2}\right)^2. \qquad (3)$$

Then find $\left(\frac{dR}{dx}\right)^2$. Since

$$R=\frac{(1+y'^2)^{3/2}}{y''},\quad R^2=\frac{(1+y'^2)^3}{y''^2}.$$

Differentiating both sides of this equation with respect to x, we get the following (after appropriate manipulations):

$$2R\,\frac{dR}{dx}=\frac{2(1+y'^2)^2(3y'y''^2-y'''-y'^2y''')}{(y'')^3}.$$

Dividing both sides of the equation by $2R=\frac{2(1+y'^2)^{3/2}}{y''}$, we have

$$\frac{dR}{dx}=\frac{(1+y'^2)^{1/2}(3y'y''^2-y'''-y'^2y''')}{y''^2}.$$

Squaring, we get

$$\left(\frac{dR}{dx}\right)^2=(1+y'^2)\left(\frac{3y'y''^2-y'''-y'^2y'''}{y''^2}\right)^2. \tag{4}$$

Comparing (3) and (4), we find

$$\left(\frac{dR}{dx}\right)^2=\left(\frac{ds}{dx}\right)^2,$$

whence

$$\frac{dR}{dx}=\mp\frac{ds}{dx}.$$

It is given that $\frac{dR}{dx}$ does not change sign (R only increases or only decreases); hence, $\frac{ds}{dx}$ does not change sign either. For the sake of definiteness, let $\frac{dR}{dx}\leqslant 0$, $\frac{ds}{dx}\geqslant 0$ (which corresponds to Fig. 151). Hence, $\frac{dR}{dx}=-\frac{ds}{dx}$.

Let the point M_1 have abscissa x_1 and M_2 have abscissa x_2. Apply the Cauchy theorem to the functions $s(x)$ and $R(x)$ on the interval $[x_1, x_2]$:

$$\frac{s(x_2)-s(x_1)}{R(x_2)-R(x_1)}=\frac{\left(\frac{ds}{dx}\right)_{x=\xi}}{\left(\frac{dR}{dx}\right)_{x=\xi}}=-1,$$

where ξ is a number lying between x_1 and x_2 $(x_1<\xi<x_2)$.

We introduce the designations (Fig. 151)

$$s(x_2)=s_2,\quad s(x_1)=s_1,\quad R(x_2)=R_2,\quad R(x_1)=R_1.$$

Then $\frac{s_2-s_1}{R_2-R_1}=-1$, or $s_2-s_1=-(R_2-R_1)$. But this means that

$$|s_2-s_1|=|R_2-R_1|.$$

This equality is proved in exactly the same manner if the radius of curvature increases.

We have proved Theorems 1 and 2 for the case when the curve is given by an explicit equation, $y=f(x)$.

If the curve is represented by parametric equations, these theorems also hold, and their proof is exactly the same.

Note. The following is a simple mechanical method for constructing a curve (involute) from its evolute.

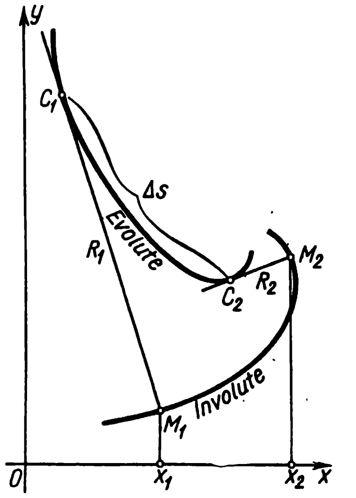


Fig. 151.

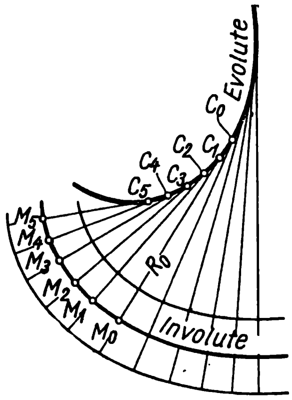


Fig. 152.

Let a flexible ruler be bent into the shape of an evolute C_0C_5 (Fig. 152). Suppose one end of an unstretchable string is attached to the point C_0 and bends round the ruler. If we hold the string taut and unwind it, the end of the string will describe a curve M_5M_0, which is the involute (or evolvent, the name coming from this process of "evolving"). Proof that this curve is indeed an involute may be carried out by means of the above-established properties of the evolute.

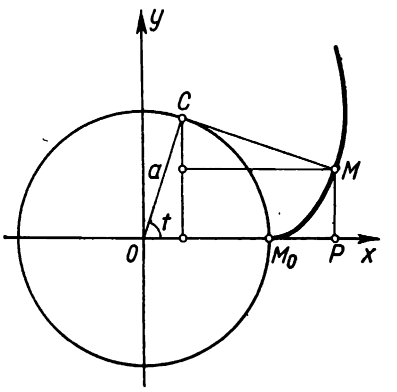


Fig. 153.

It should be noted that to a single evolute there correspond an infinitude of various involutes (Fig. 152).

Example. Let there be a circle of radius a (Fig. 153). Take the involute of this circle that passes through the point M_0 $(a, 0)$.

Taking into account that $CM = \widehat{CM_0} = at$, it is easy to obtain the equations of the involute of the circle:

$$OP = x = a(\cos t + t \sin t),$$
$$PM = y = a(\sin t - t \cos t).$$

It will be noted that the profile of a tooth of a gear wheel is most often in the shape of the involute of a circle.

SEC. 8. APPROXIMATING THE REAL ROOTS OF AN EQUATION

Methods of investigating the behaviour of functions enable us to approximate the roots of an equation:

$$f(x) = 0.$$

If the equation is an algebraic equation*) of the first, second, third, or fourth degree, there are formulas which permit expressing the roots of the equation in terms of its coefficients by means of a finite number of operations of addition, subtraction, multiplication, division and evolution. Generally speaking, there are no such formulas for equations above the fourth degree. If the coefficients of any equation algebraic or nonalgebraic (transcendental) are not literal but numerical, then the roots of the equation may be calculated approximately to any degree of accuracy. It should be noted that even when the roots of an algebraic equation are expressed in terms of radicals, it is sometimes better, practically speaking, to apply an approximation method of solving the equation. Below we give some methods of approximating the roots of an equation.

1. **Method of chords.** Let there be an equation

$$f(x) = 0 \tag{1}$$

where $f(x)$ is a continuous, doubly differentiable function on the interval $[a, b]$. Suppose that by investigating the function $y = f(x)$ within the interval $[a, b]$ we isolate a subinterval $[x_1, x_2]$ such that within this subinterval the function is monotonic (either increasing or decreasing), and at the end points the values of the function $f(x_1)$ and $f(x_2)$ are of different signs. For definiteness, we say that $f(x_1) < 0$, $f(x_2) > 0$ (Fig. 154). Since the function $y = f(x)$ is continuous on the interval $[x_1, x_2]$, its graph will cut the x-axis in some one point between x_1 and x_2.

Draw a chord AB connecting the end points of the curve $y = f(x)$, which correspond to abscissas x_1 and x_2. Then the

*) The equation $f(x) = 0$ is called algebraic if $f(x)$ is a polynomial (see Sec. 6, Ch. VII).

abscissa a_1 of the point of intersection of this chord with the x-axis will be the approximate value of the root (Fig. 155). In order to find this approximate value let us write the equation of the straight line AB that passes through two given points $A[x_1, f(x_1)]$ and $B[x_2, f(x_2)]$:

$$\frac{y-f(x_1)}{f(x_2)-f(x_1)}=\frac{x-x_1}{x_2-x_1}.$$

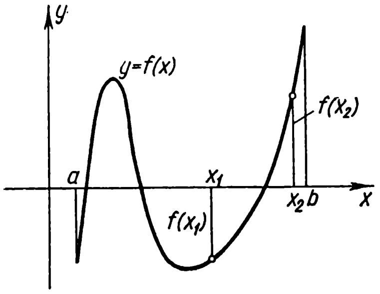


Fig. 154.

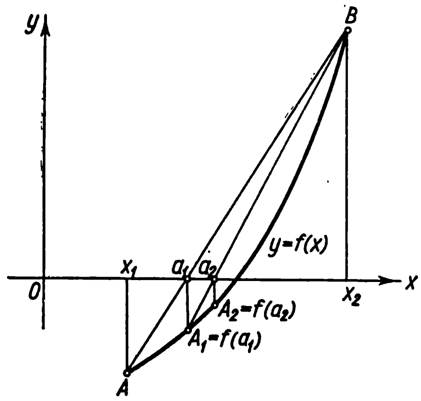


Fig. 155.

Since $y=0$ at $x=a_1$, it follows that

$$\frac{-f(x_1)}{f(x_2)-f(x_1)}=\frac{a_1-x_1}{x_2-x_1},$$

whence

$$a_1=x_1-\frac{(x_2-x_1)f(x_1)}{f(x_2)-f(x_1)}. \tag{2}$$

To obtain a more exact value of the root, we determine $f(a_1)$. If $f(a_1)<0$, then repeat the same procedure applying formula (2) to the interval $[a_1, x_2]$. If $f(a_1)>0$, then apply this formula to the interval $[x_1, a_1]$. By repeating this procedure several times we will obviously obtain more and more precise values of the root $a_2 a_3$, etc.

Example 1. Approximate the roots of the equation

$$f(x)=x^3-6x+2=0.$$

Solution. First find the segments where the function $f(x)$ is monotonic. Evaluating the derivative $f'(x)=3x^2-6$, we find that it is positive for $x<-\sqrt{2}$, negative for $-\sqrt{2}<x<+\sqrt{2}$ and again positive for $x>\sqrt{2}$ (Fig. 156). Thus, the function has three segments of monotonicity, on each of which there is one root.

To make the calculations more convenient, let us narrow these segments of monotonicity (but in such manner that there should be a corresponding

root on each segment). To do this, substitute into expression $f(x)$, at random, some values of x, then isolate (within each segment of monotonicity) such shorter intervals that the functions at the end points will have different signs:

$$\left.\begin{array}{ll} x_1=0, & f(0)=2, \\ x_2=1, & f(1)=-3, \end{array}\right\}$$
$$\left.\begin{array}{ll} x_3=-3, & f(-3)=-7, \\ x_4=-2, & f(-2)=6, \end{array}\right\}$$
$$\left.\begin{array}{ll} x_5=2, & f(2)=-2, \\ x_6=3, & f(3)=11. \end{array}\right\}$$

Thus, the roots lie within the intervals

$$(0,\ 1),\quad (-3,\ -2),\quad (2,\ 3).$$

Find the approximate value of the root in the interval (0, 1); from formula (2) we have

$$a_1=0-\frac{(1-0)\,2}{-3-2}=\frac{2}{5}=0.4.$$

Since

$$f(0.4)=0.4^3-6\cdot 0.4+2=-0.336,\quad f(0)=2,$$

it follows that the root lies between 0 and 0.4. Again applying (2) to this interval, we get the following approximation:

$$a_2=0-\frac{(0.4-0)\cdot 2}{-0.336-2}=\frac{0.8}{2.336}=0.342, \text{ etc.}$$

Similarly we approximate the roots in the other intervals.

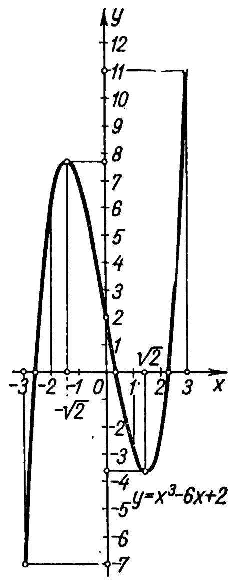


Fig. 156.

2. **Method of tangents (Newton's method).** Again, let $f(x_1)<0$, $f(x_2)>0$; and on the interval $[x_1, x_2]$ the first derivative does not change sign. Then there is one root of the equation $f(x)=0$ in the interval (x_1, x_2). Let us assume that the second derivative does not change sign in the interval $[x_1, x_2]$ either; this can be achieved by reducing the length of the interval within which the root lies.

Retention of the sign of the second derivative on the interval $[x_1, x_2]$ means that the curve is either only convex or only concave on $[x_1, x_2]$.

Draw a tangent to the curve at the point B (Fig. 157). The abscissa a_1 of the point of intersection of the tangent with the x-axis will be an approximate value of the root. To find this abscissa write the equation of the line tangent at the point B:

$$y-f(x_2)=f'(x_2)(x-x_2).$$

Noting that $x=a_1$ at $y=0$, we have

$$a_1=x_2-\frac{f(x_2)}{f'(x_2)}. \tag{3}$$

Then, drawing the line tangent at the point B_1, we analogously find a more exact value of the root a_2. By repeating this procedure

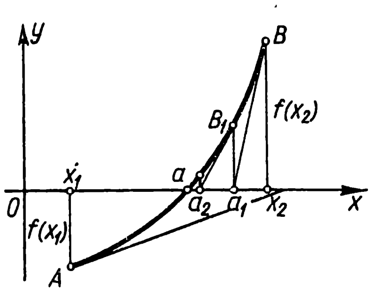


Fig. 157.

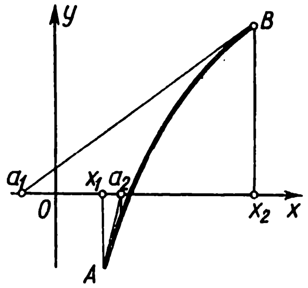


Fig. 158.

we can calculate the approximate value of the root to any desired degree of accuracy.

Note the following. If we drew the line tangent to the curve not at the point B but at A, it might appear that the point of intersection of the tangent with the x-axis lies outside the interval (x_1, x_2).

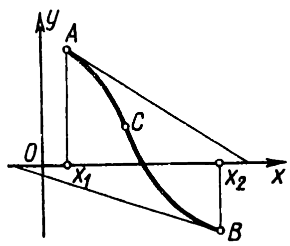


Fig. 159.

From Figs. 157 and 158 it follows that the tangent should be drawn at the end of the arc at which the signs of the function and its second derivative coincide. Since it is given that on the interval $[x_1, x_2]$ the second derivative retains its sign, the signs of the function and the second derivative must coincide at one of the end points. This rule also holds for the case when $f'(x) < 0$. If the line tangent is drawn at the left end point of the interval, then in formula (3) we must put x_1 in place of x_2:

$$a_1 = x_1 - \frac{f(x_1)}{f'(x_1)}. \qquad (3')$$

When there is a point of inflection C in the interval (x_1, x_2), the method of tangents can yield an approximate value of the root lying without the interval (x_1, x_2) (Fig. 159).

Example 2. Apply formula (3) to finding the root of the equation

$$f(x) = x^3 - 6x + 2 = 0$$

within the interval (0, 1). We have

$$f(0) = 2, \quad f'(0) = (3x^2 - 6)|_{x=0} = -6,$$

and so from (3) we get

$$a_1 = 0 - \frac{2}{-6} = \frac{1}{3} = 0.333.$$

3. **Combined method** (Fig. 160). Applying at the same time on the interval $[x_1, x_2]$ the method of chords and the method of tangents, we get two points a_1 and $\bar{a}_1$ lying on either side of the desired root a, since $f(a_1)$ and $f(\bar{a}_1)$ have different signs. Then, on the interval $[a_1, \bar{a}_1]$ again apply the method of chords and the method of tangents. This yields two numbers: a_2 and $\bar{a}_2$, which are still closer to the value of the root. We continue in this manner until the difference between the approximate values found is less than the required degree of accuracy.

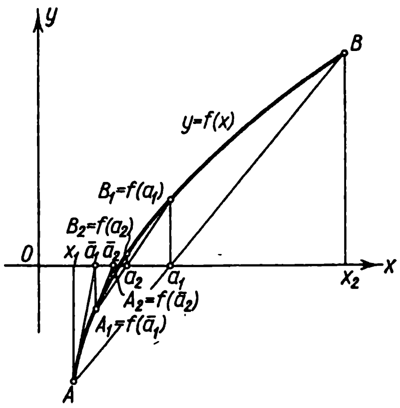


Fig. 160.

It will be noted that in the combined method we approach the sought-for root from two sides simultaneously (i. e., at the same time we approximate the root with an excess and with a deficit).

To illustrate, in the case we have examined it will be clear that by substitution we have

$$f(0.333) > 0, \quad f(0.342) < 0.$$

Hence, the root is between the approximate values obtained:

$$0.333 < x < 0.342.$$

Exercises on Chapter VI

Find the curvature of the curves at the indicated points:

1. $b^2x^2 + a^2y^2 = a^2b^2$ at the points $(0, b)$ and $(a, 0)$. *Ans.* $\frac{b}{a^2}$ at $(0, b)$; $\frac{a}{b^2}$ at $(a, 0)$.

2. $xy = 12$ at the point $(3, 4)$. *Ans.* $\frac{24}{124}$.

3. $y = x^3$ at the point (x_1, y_1). *Ans.* $\frac{6x_1}{(1+9x_1^4)^{3/2}}$.

4. $16y^2 = 4x^4 - x^6$ at the point $(2, 0)$. *Ans.* $\frac{1}{2}$.

5. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ at an arbitrary point. *Ans.* $\frac{1}{3(axy)^{\frac{1}{3}}}$.

Find the radius of curvature of the following curves at the indicated points; draw each curve and construct the appropriate circle of curvature.

6. $y^2=x^3$ at the point (4, 8). *Ans.* $R=\frac{80\sqrt{10}}{3}$.

7. $x^2=4ay$ at the point (0, 0). *Ans.* $R=2a$.

8. $b^2x^2-a^2y^2=a^2b^2$ at the point (x_1, y_1). *Ans.* $R=\frac{(b^4x_1+a^4y_1)^{3/2}}{a^4b^4}$.

9. $y=\ln x$ at the point (1, 0). *Ans.* $R=2\sqrt{2}$.

10. $y=\sin x$ at the point $\left(\frac{\pi}{2}, 1\right)$. *Ans.* $R=1$.

11. $\left.\begin{array}{l} x=a\cos^3 t \\ y=a\sin^3 t \end{array}\right\}$ for $t=t_1$. *Ans.* $R=3a\sin t\cos t$.

Find the radius of curvature of the indicated curves:

12. $\left.\begin{array}{l} x=3t^2 \\ y=3t-t^3 \end{array}\right\}$ for $t=1$. *Ans.* $R=6$.

13. Circle $\varrho=a\sin\theta$. *Ans.* $R=\frac{a}{2}$.

14. Spiral of Archimedes $\varrho=a\theta$. *Ans.* $R=\frac{(\varrho^2+a^2)^{3/2}}{\varrho^2+2a^2}$.

15. Cardioid $\varrho=a(1-\cos\theta)$. *Ans.* $R=\frac{2}{3}\sqrt{2a\varrho}$.

16. Lemniscate $\varrho^2=a^2\cos 2\theta$. *Ans.* $R=\frac{a^2}{3\varrho}$.

17. Parabola $\varrho=a\sec^2\frac{\theta}{2}$. *Ans.* $R=2a\sec^3\frac{\theta}{2}$.

18. $\varrho=a\sin^3\frac{\theta}{3}$. *Ans.* $R=\frac{3}{4}a\sin^2\frac{\theta}{3}$.

Find the points of curves at which the radius of curvature is a minimum:

19. $y=\ln x$. *Ans.* $\left(\frac{\sqrt{2}}{2}, -\frac{1}{2}\ln 2\right)$.

20. $y=e^x$. *Ans.* $\left(-\frac{1}{2}\ln 2, \frac{\sqrt{2}}{2}\right)$.

21. $\sqrt{x}+\sqrt{y}=\sqrt{a}$. *Ans.* $\left(\frac{a}{4}, \frac{a}{4}\right)$.

22. $y=a\ln\left(1-\frac{x^2}{a^2}\right)$. *Ans.* At the point (0, 0) $R=\frac{a}{2}$.

Find the coordinates of the centre of curvature (α, β) and the equation of the evolute for each of the following curves:

23. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. *Ans.* $\alpha=\frac{(a^2+b^2)x^3}{a^4}$; $\beta=-\frac{(a^2+b^2)y^3}{b^4}$.

24. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. *Ans.* $\alpha=x+3x^{\frac{1}{3}}y^{\frac{2}{3}}$; $\beta=y+3x^{\frac{2}{3}}y^{\frac{1}{3}}$.

25. $y^3=a^2x$. *Ans.* $\alpha=\frac{a^4+15y^4}{6a^2y}$; $\beta=\frac{a^4y-9y^5}{2a^4}$.

26. $\begin{cases} x=3t, \\ y=t^2-6. \end{cases}$ *Ans.* $\alpha=-\frac{4}{3}t^3$; $\beta=3t^2-\frac{3}{2}$.

27. $\begin{cases} x=k\ln\cot\frac{t}{2}-k\cos t, \\ y=k\sin t. \end{cases}$ *Ans.* $y=\frac{k}{2}\left(e^{\frac{x}{k}}+e^{-\frac{x}{k}}\right)$ (tractrix).

28. $\begin{cases} x=a(\cos t+t\sin t), \\ y=a(\sin t-t\cos t). \end{cases}$ *Ans.* $\alpha=a\cos t$; $\beta=a\sin t$.

29. $\begin{cases} x=a\cos^3 t, \\ y=a\sin^3 t. \end{cases}$ *Ans.* $\alpha=a\cos^3 t+3a\cos t\sin^2 t$;
$\beta=a\sin^3 t+3a\cos^2 t\sin t$.

30. Find the roots of the equation $x^3-4+2=0$ to three decimal places. *Ans.* $x_1=1.675$, $x_2=0.539$, $x_3=-2.214$.

31. For the equation $f(x)=x^5-x-0.2=0$, approximate the root in the interval (1, 1.1). *Ans.* 1.045.

32. Evaluate the roots of the equation $x^4+2x^2-6x+2=0$ to two decimal places. *Ans.* $0.38<x_1<0.39$; $1.24<x_2<1.25$.

33. Solve the equation $x^3-5=0$ approximately. *Ans.* $x_1\approx1.71$, $x_{2,3}=1.71\frac{-1\pm i\sqrt{3}}{2}$.

34. Approximate the root of the equation $x-\tan x=0$ lying between 0 and $\frac{3\pi}{2}$. *Ans.* 4.4935.

35. Evaluate the root of the equation $\sin x=1-x$ to three places of decimals. **Hint.** Reduce the equation to the form $f(x)=0$. *Ans.* $0.5110<x<$ <0.5111.

Miscellaneous Problems

36. Show that at each point of the lemniscate $\varrho^2=a^2\cos2\varphi$ the curvature is proportional to the radius vector of the point.

37. Find the greatest value of the radius of curvature of the curve $\varrho=a\sin^3\frac{\varphi}{3}$. *Ans.* $R=\frac{3}{4}a$.

38. Find the coordinates of the centre of curvature of the curve $y=x\ln x$ at the point where $y'=0$. *Ans.* $(e^{-1}, 0)$.

39. Prove that for points of the spiral of Archimedes $\varrho=a\varphi$ as $\varphi\to\infty$ the magnitude of the difference between the radius vector and the radius of curvature approaches zero.

40. Find the parabola $y=ax^2+bx+c$, which has common tangent and curvature with the sine curve $y=\sin x$ at the point $\left(\frac{\pi}{2}, 1\right)$. Make a drawing. *Ans.* $y=-\frac{x^2}{2}+\frac{\pi x}{2}+1-\frac{\pi^2}{8}$.

41. The function $y=f(x)$ is defined as follows:

$$f(x)=x^3 \text{ in the interval } -\infty<x\leqslant1,$$

$$f(x)=ax^2+bx+c \text{ in the interval } 1<x<+\infty.$$

What must a, b and c be for the line $y=f(x)$ to have continuous curvature everywhere? Make a drawing. *Ans.* $a=3$, $b=-3$, $c=1$.

42. Show that the radius of a curvature of a cycloid at any one of its points is twice the length of the normal at that point.

43. Write the equation of the circle of curvature of the parabola $y=x^2$ at the point (1, 1). *Ans.* $(x+4)^2+\left(y-\frac{7}{2}\right)^2=\frac{125}{4}$.

44. Write the equation of the circle of curvature of the curve $y=\tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$. *Ans.* $\left(x-\frac{\pi-10}{4}\right)^2+\left(y-\frac{9}{4}\right)^2=\frac{125}{16}$.

45. Find the length of the entire evolute of an ellipse whose semi-axes are a and b. *Ans.* $\frac{4(a^3-b^3)}{ab}$.

46. Find the approximate value of the roots of the equation $xe^x=2$ to within 0.01. *Ans.* The equation has only one real root, $x\approx 0.84$

47. Find the approximate value of the roots of the equation $x\ln x=0.8$ to within 0.01. *Ans.* The equation has only one real root, $x\approx 1.64$.

48. Find the approximate value of the roots of the equation $x^2 \operatorname{arc\,tan} x=1$ to within 0.001. *Ans.* The equation has only one real root, $x\approx 1.096$.

CHAPTER VII

COMPLEX NUMBERS. POLYNOMIALS

SEC. 1. COMPLEX NUMBERS. BASIC DEFINITIONS

A complex number is the expression

$$a+bi \tag{1}$$

where a and b are *real* numbers, i is the so-called *imaginary unit*, which is defined by the equalities

$$i=\sqrt{-1} \quad \text{or} \quad i^2=-1; \tag{2}$$

a is called the *real part*, and bi, the *imaginary part* of the complex number. Two complex numbers $a+bi$ and $a-bi$ that differ only in the sign of the imaginary part are called *conjugate*.

If $a=0$, the number $0+bi=bi$ is called a *pure imaginary;* if $b=0$, we get a real number: $a+0\cdot i=a$.

We agree upon the two following basic statements:

1) two complex numbers a_1+b_1i and a_2+b_2i are equal if

$$a_1=a_2, \quad b_1=b_2,$$

that is, if their real parts are equal and their imaginary parts are equal;

2) a complex number is equal to zero:

$$a+bi=0$$

if and only if $a=0$, $b=0$.

1. **Geometric representation of complex numbers.** Any complex number $a+bi$ may be represented in an xy-plane as a point $A(a, b)$ with coordinates a and b (Fig. 161); and conversely, any point $M(a, b)$ in an xy-plane may be regarded as the geometric image of a complex number $a+bi$.

But if to each point $A(a, b)$ there corresponds a complex number $a+bi$, then, to take a specific case, to points lying on the x-axis there correspond real numbers ($b=0$). But if a point lies on the y-axis, it represents a pure imaginary number, since $a=0$. For this reason, when complex numbers are represented in the plane, the y-axis is called the *imaginary axis* or *axis of imaginaries*, and the x-axis, the *real axis (axis of reals)*.

Joining the point $A(a, b)$ with the origin, we get a vector $\overline{OA}$. In certaini nstances, it is convenient to consider the *vector* $\overline{OA}$ as the geometric representation of the complex number $a+bi$.

2. **Trigonometric form of a complex number.** Denote by φ and $r\,(r \geqslant 0)$ the polar coordinates of the point $A(a, b)$ and consider the origin as the pole and the positive direction of the x-axis, the polar axis. Then (Fig. 161) we have the familiar relationships:

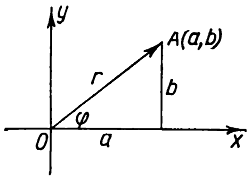


Fig. 161.

$$a = r\cos\varphi, \quad b = r\sin\varphi,$$

and, hence, the complex number may be given in the form

$$a + bi = r(\cos\varphi + i\sin\varphi). \qquad (3)$$

The expression on the right-hand side is called the trigonometric form of a complex number $a+bi$. The quantities r and φ are expressed in terms of a and b, by the formulas

$$r = \sqrt{a^2 + b^2}, \quad \varphi = \arctan\frac{b}{a}$$

and are called: r, the *modulus*, φ, the *argument* (**amplitude** or **phase**) of the complex number $a+bi$.

The amplitude of a complex number, the angle φ, is considered positive if it is reckoned from the positive x-axis counterclockwise, and negative, in the opposite sense. The amplitude φ is obviously not determined uniquely but to within the accuracy of the term $2\pi k$, where k is any integer.

The modulus r of the complex number $a+bi$ is sometimes denoted by the symbol $|a+bi|$:

$$r = |a + bi|.$$

It will be noted that the real number A can also be written in the form (3), namely:

$$A = |A|(\cos 0 + i\sin 0) \text{ for } A > 0,$$
$$A = |A|(\cos\pi + i\sin\pi) \text{ for } A < 0.$$

The modulus of the complex number 0 is zero: $|0| = 0$. Any angle φ may be taken for amplitude zero. Indeed, for any angle φ we have the equality

$$0 = 0\cdot(\cos\varphi + i\sin\varphi).$$

SEC. 2. BASIC OPERATIONS ON COMPLEX NUMBERS

1. **Addition of complex numbers.** The sum of two complex numbers $a_1 + b_1 i$ and $a_2 + b_2 i$ is a complex number defined by the equality

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)\,i. \qquad (1)$$

From (1) it follows that the addition of complex numbers given in vectors is performed by the rule of the addition of vectors.

2. **Subtraction of complex numbers.** The difference of two complex numbers a_2+b_2i and a_1+b_1i is a complex number such that when it is added to a_1+b_1i it yields a_2+b_2i.

It is easy to see that

$$(a_2+b_2i)-(a_1+b_1i)=(a_2-a_1)+(b_2-b_1)i. \quad (2)$$

It will be noted that the modulus of the difference of two complex numbers $\sqrt{(a_1-a_2)^2+(b_1-b_2)^2}$ is equal to the distance between the points representing these numbers in the plane of the complex variable (Fig. 162).

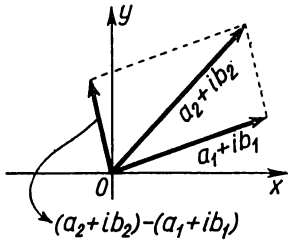


Fig. 162.

3. **Multiplication of complex numbers.** The product of two complex numbers a_1+b_1i and a_2+b_2i is a complex number obtained when these two numbers are multiplied as binomials by the rules of algebra, provided that

$$i^2=-1;\quad i^3=(-1)\,i=-i;\quad i^4=(-i)\,(i)=-i^2=1;\quad i^5=1\cdot i \text{ etc.,}$$

and, generally, for integral k,

$$i^{4k}=1;\quad i^{4k+1}=i;\quad i^{4k+2}=-1;\quad i^{4k+3}=-i.$$

From this rule we get

$$(a_1+b_1i)\,(a_2+b_2i)=a_1a_2+b_1a_2i+a_1b_2i+a_1b_2i^2,$$

or

$$(a_1+b_1i)\,(a_2+b_2i)=(a_1a_2-b_1b_2)+(b_1a_2+a_1b_2)\,i. \quad (3)$$

If the complex numbers are written in trigonometric form, we have

$$r_1(\cos\varphi_1+i\sin\varphi_1)\,r_2(\cos\varphi_2+i\sin\varphi_2)=$$
$$=r_1r_2[\cos\varphi_1\cos\varphi_2+i\sin\varphi_1\cos\varphi_2+i\cos\varphi_1\sin\varphi_2+i^2\sin\varphi_1\sin\varphi_2]=$$
$$=r_1r_2[(\cos\varphi_1\cos\varphi_2-\sin\varphi_1\sin\varphi_2)+i(\sin\varphi_1\cos\varphi_2+\cos\varphi_1\sin\varphi_2)]=$$
$$=r_1r_2[\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)].$$

Thus,

$$r_1(\cos\varphi_1+i\sin\varphi_1)\,r_2(\cos\varphi_2+i\sin\varphi_2)=$$
$$=r_1r_2[\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)], \quad (3')$$

the product of two complex numbers is a complex number, the modulus of which is equal to the product of the moduli of the

factors, and the amplitude is equal to the sum of the amplitudes of the factors.

Note 1. By virtue of (3), the conjugate numbers $a+bi$ and $a-bi$ satisfy the equality

$$(a+ib)(a-ib)=a^2+b^2;$$

the product of conjugate complex numbers is equal to the sum of the squares of the moduli of each of them.

4. Division of complex numbers. The division of complex numbers is defined as the inverse operation of multiplication: if

$$\frac{a_1+b_1i}{a_2+b_2i}=x+yi$$

(where $\sqrt{a_2^2+b_2^2}\neq 0$), then x and y must be such as to fulfil the equality

$$a_1+b_1i=(a_2+b_2i)(x+yi)$$

or

$$a_1+b_1i=(a_2x-b_2y)+(a_2y+b_2x)i.$$

Consequently,

$$a_1=a_2x-b_2y, \quad b_1=b_2x+a_2y,$$

whence we find

$$x=\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}, \quad y=\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2},$$

and finally we get

$$\frac{a_1+b_1i}{a_2+b_2i}=\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}+\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}i. \tag{4}$$

Actually, complex numbers are divided as follows: to divide a_1+ib_1 by a_2+ib_2, multiply the dividend and divisor by a number conjugate to the divisor (that is, by a_2-ib_2). Then the divisor will be a real number; dividing the real and imaginary parts of the dividend by it, we get the quotient

$$\frac{a_1+b_1i}{a_2+b_2i}=\frac{(a_1+b_1i)(a_2-b_2i)}{(a_2+b_2i)(a_2-b_2i)}=\frac{(a_1a_2+b_1b_2)+(a_2b_1-a_1b_2)i}{a^2+b^2}=$$

$$=\frac{a_1a_2+b_1b_2}{a^2+b^2}+\frac{a_2b_1-a_1b_2}{a^2+b^2}i.$$

For the trigonometric form of a complex number we have

$$\frac{r_1(\cos\varphi_1+i\sin\varphi_1)}{r_2(\cos\varphi_2+i\sin\varphi_2)}=\frac{r_1}{r_2}[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)].$$

To verify this equality, multiply the divisor by the quotient:

$$r_2(\cos\varphi_2 + i\sin\varphi_2)\frac{r_1}{r_2}[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)] =$$

$$= r_2\frac{r_1}{r_2}[\cos(\varphi_2 + \varphi_1 - \varphi_2) + i\sin(\varphi_2 + \varphi_1 - \varphi_2)] = r_1(\cos\varphi_1 + i\sin\varphi_1).$$

Thus, *the modulus of the quotient of two complex numbers is equal to the quotient of the moduli of the dividend and the divisor; the amplitude of the quotient is equal to the difference between the amplitudes of the dividend and divisor.*

Note 2. From the rules of operations involving complex numbers it follows that the operations of addition, subtraction, multiplication and division of complex numbers yield a complex number.

If the rules of operations on complex numbers are applied to real numbers, regarding the latter as a special case of complex numbers, these rules will coincide with the ordinary rules of arithmetic.

Note 3. Returning to the definitions of a sum, difference, product and quotient of complex numbers, it is easy to show that if each complex number in these expressions is replaced by its conjugate, then the results of the aforementioned operations will yield conjugate numbers. Whence follows (as a particular instance) the following theorem.

Theorem. *If in a polynomial with* ***real*** *coefficients*

$$A_0x^n + A_1x^{n-1} + \ldots + A_n$$

we put the number $a+bi$ in place of x, and then the conjugate number $a-bi$ in place of x, the results of these substitutions will be mutually conjugate.

SEC. 3. POWERS AND ROOTS OF COMPLEX NUMBERS

1. **Powers.** From formula (3) of the preceding section it follows that if n is a positive integer, then

$$[r(\cos\varphi + i\sin\varphi)]^n = r^n(\cos n\varphi + i\sin n\varphi). \qquad (1)$$

This formula is called *De Moivre's formula*. It shows that *when a complex number is raised to a positive integral power the modulus is raised to this power, while the amplitude is multiplied by the exponent.*

Now consider another application of De Moivre's formula.

Setting $r = 1$ in this formula, we get

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi.$$

Expanding the left-hand side in a binomial expansion and equating the real and imaginary parts, we can express $\sin n\varphi$ and

$\cos n\varphi$ in terms of the powers of $\sin\varphi$ and $\cos\varphi$. For instance, if $n=3$ we have

$\cos^3\varphi + i\,3\cos^2\varphi\sin\varphi - 3\cos\varphi\sin^2\varphi - i\sin^3\varphi = \cos 3\varphi + i\sin 3\varphi$;

making use of the condition of equality of two complex numbers, we get:

$$\cos 3\varphi = \cos^3\varphi - 3\cos\varphi\sin^2\varphi,$$
$$\sin 3\varphi = -\sin^3\varphi + 3\cos^2\varphi\sin\varphi.$$

2. **Roots.** The nth root of a complex number is another complex number whose nth power is equal to the radicand, or

$$\sqrt[n]{r(\cos\varphi + i\sin\varphi)} = \varrho(\cos\psi + i\sin\psi),$$

if

$$\varrho^n(\cos n\psi + i\sin n\psi) = r(\cos\varphi + i\sin\varphi).$$

Since the moduli of equal complex numbers must be equal, while their amplitudes may differ by a number that is a multiple of 2π, we have

$$\varrho^n = r, \quad n\psi = \varphi + 2k\pi.$$

Whence we find

$$\varrho = \sqrt[n]{r}, \quad \psi = \frac{\varphi + 2k\pi}{n},$$

where k is any integer, $\sqrt[n]{r}$ is the arithmetic (real positive) value of the root of the positive number r. Therefore,

$$\sqrt[n]{r(\cos\varphi + i\sin\varphi)} = \sqrt[n]{r}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right). \quad (2)$$

Giving k the values $0, 1, 2, \ldots, n-1$, we get n different values of the root. For the other values of k, the amplitudes will differ from those obtained by a number which is a multiple of 2π, and, for this reason, root values will be obtained that coincide with those considered.

Thus, the nth root of a complex number has n different values.

The nth root of a real nonzero number A also has n values, since a real number is a special case of a complex number and may be represented in trigonometric form:

$$\text{if } A > 0, \text{ then } A = |A|\,(\cos 0 + i\sin 0);$$
$$\text{if } A < 0, \text{ then } A = |A|\,(\cos\pi + i\sin\pi).$$

Example 1. Find all the values of the cube root of unity.

Solution. We represent unity in trigonometric form:

$$1 = \cos 0 + i\sin 0.$$

By formula (2) we have

$$\sqrt[3]{1}=\sqrt[3]{\cos 0+i\sin 0}=\cos\frac{0+2k\pi}{3}+i\sin\frac{0+2k\pi}{3}.$$

Setting k equal to 0, 1, 2, we find three values of the root:

$$x_1=\cos 0+r\sin 0=1;\ x_2=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3};$$

$$x_3=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}.$$

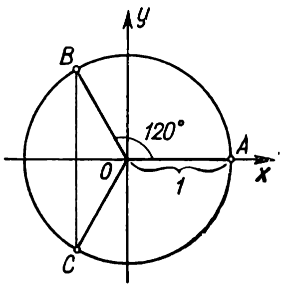


Fig. 163.

Noting that

$$\cos\frac{2\pi}{3}=-\frac{1}{2};\quad \sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2};\quad \cos\frac{4\pi}{3}=-\frac{1}{2};\quad \sin\frac{4\pi}{3}=-\frac{\sqrt{3}}{2},$$

we get

$$x_1=1;\ x_2=-\frac{1}{2}+i\frac{\sqrt{3}}{2};\quad x_3=-\frac{1}{2}-i\frac{\sqrt{3}}{2}.$$

In Fig. 163, the points A, B, C are geometric representations of the roots obtained.

3. Solution of a binomial equation. An equation of the form

$$x^n=A$$

is called a binomial equation. Let us find its roots.

If A is a real positive number, then

$$x=\sqrt[n]{A}\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right)$$
$$(k=0,\ 1,\ 2,\ \ldots,\ n-1).$$

The expression in the brackets gives all the values of the nth root of 1.

If A is a real negative number, then

$$x=\sqrt[n]{|A|}\left(\cos\frac{\pi+2k\pi}{n}+i\sin\frac{\pi+2k\pi}{n}\right).$$

The expression in the brackets gives all the values of the nth root of -1.

If A is a complex number, then the values of x are found from formula (2).

Example 2. Solve the equation

$$x^4=1.$$

Solution.

$$x=\sqrt[4]{\cos 2k\pi+i\sin 2k\pi}=\cos\frac{2k\pi}{4}+i\sin\frac{2k\pi}{4}.$$

Setting k equal to 0, 1, 2, 3, we get

$$x_1=\cos 0+i\sin 0=1,$$

$$x_2=\cos\frac{2\pi}{4}+i\sin\frac{2\pi}{4}=i,$$

$$x_3=\cos\frac{4\pi}{4}+i\sin\frac{4\pi}{4}=-1,$$

$$x_4=\cos\frac{6\pi}{4}+i\sin\frac{6\pi}{4}=-i.$$

SEC. 4. EXPONENTIAL FUNCTION WITH COMPLEX EXPONENT AND ITS PROPERTIES

Let $z=x+iy$. If x and y are real variables, then z is called a complex variable. To each value of the complex variable z in the xy-plane (**the complex plane**) there corresponds a definite point (see Fig. 161).

Definition. If to every value of the complex variable z, out of a certain range of complex values, there corresponds a definite value of another complex quantity w, then w is a *function of the complex variable z*. The functions of a complex argument are denoted by $w=f(z)$ or $w=w(z)$.

We introduce the concepts of the limit of a function of a complex variable, of the derivative, of the integral, and so forth.

Here, we consider one function of a complex variable, the exponential function:

$$w=e^z$$

or

$$w=e^{x+iy}.$$

The complex values of the function w are defined as follows:*)

$$e^{x+iy}=e^x(\cos y+i\sin y), \tag{1}$$

that is

$$w(z)=e^x(\cos y+i\sin y). \tag{2}$$

Examples:

1. $z=1+\frac{\pi}{4}i,\quad e^{1+\frac{\pi}{4}i}=e\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=e\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right).$

*) The advisability of this definition of the exponential function of a complex variable will also be shown later on, Sec. 21, Ch. XIII, and Sec. 18, Ch. XVI.

2. $z=0+\frac{\pi}{2}i,\quad e^{0+\frac{\pi}{2}i}=e^0\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=i,$

3. $z=1+i,\quad e^{1+i}=e^0(\cos 1+i\sin 1)=0.54+i\cdot 0.83,$

4. $z=x$ is a real number.
$e^{x+0y}=e^x(\cos 0+i\sin 0)=e^x$ is an ordinary exponential function.

Properties of an exponential function.

1. If z_1 and z_2 are two complex numbers, then

$$e^{z_1+z_2}=e^{z_1}e^{z_2}. \tag{3}$$

Proof. Let

$$z_1=x_1+iy_1,\quad z_2=x_2+iy_2;$$

then

$$e^{z_1+z_2}=e^{(x_1+iy_1)+(x_2+iy_2)}=e^{(x_1+x_2)+i\,(y_1+y_2)}=$$
$$=e^{x_1}e^{x_2}[\cos(y_1+y_2)+i\sin(y_1+y_2)]. \tag{4}$$

On the other hand, by the theorem of the product of two complex numbers in trigonometric form we will have

$$e^{z_1}e^{z_2}=e^{x_1+iy_1}e^{x_2+iy_2}=e^{x_1}(\cos y_1+i\sin y_1)e^{x_2}(\cos y_2+i\sin y_2)=$$
$$=e^{x_1}e^{x_2}[\cos(y_1+y_2)+i\sin(y_1+y_2)]. \tag{5}$$

In (4) and (5) the right sides are equal, hence the left sides are equal too:

$$e^{z_1+z_2}=e^{z_1}e^{z_2},\text{ etc.}$$

2. The following formula is similarly proved:

$$e^{z_1-z_2}=\frac{e^{z_1}}{e^{z_2}}. \tag{6}$$

3. If m is an integer, then

$$(e^z)^m=e^{mz}. \tag{7}$$

For $m>0$, this formula is readily obtained from (3); if $m<0$, then it is obtained from formulas (3) and (6).

4. The identity

$$e^{z+2\pi i}=e^z \tag{8}$$

holds.

Indeed, from (3) and (1) we get

$$e^{z+2\pi i}=e^ze^{2\pi i}=e^z(\cos 2\pi+i\sin 2\pi)=e^z.$$

From identity (8) it follows that the exponential function e^z is a periodic function with a period of $2\pi i$.

5. Let us now consider the complex quantity

$$w=u(x)+iv(x),$$

where $u(x)$ and $v(x)$ are real functions of the real variable x. This is the *complex function of a real variable*.

a) Let there exist the limits

$$\lim_{x \to x_0} u(x) = u(x_0), \quad \lim_{x \to x_0} v(x) = v(x_0).$$

Then $u(x_0) + iv(x_0) = w_0$ is called the limit of the complex variable w.

b) If the derivatives $u'(x)$ and $v'(x)$ exist, then we shall call the expression

$$w'_x = u'(x) + iv'(x) \tag{9}$$

the *derivative* of the complex function of a real variable with respect to a real argument.

Let us now consider the following exponential function:

$$w = e^{\alpha x + i\beta x} = e^{(\alpha + i\beta)x},$$

where α and β are constant real numbers, and x is a real variable. This is a complex function of a real variable, which function may be rewritten, according to (1), as follows:

$$w = e^{\alpha x}[\cos \beta x + i \sin \beta x]$$

or

$$w = e^{\alpha x} \cos \beta x + ie^{\alpha x} \sin \beta x.$$

Let us find the derivative w'_x. From (9) we have

$$\begin{aligned} w'_x &= (e^{\alpha x} \cos \beta x)' + i(e^{\alpha x} \sin \beta x)' = \\ &= e^{\alpha x}(\alpha \cos\beta x - \beta \sin \beta x) + ie^{\alpha x}(\alpha \sin \beta x + \beta \cos\beta x) = \\ &= \alpha [e^{\alpha x}(\cos \beta x + i \sin \beta x)] + i\beta [e^{\alpha x}(\cos\beta x + i\sin\beta x)] = \\ &= (\alpha + i\beta)[e^{\alpha x}(\cos\beta x + i \sin\beta x)] = (\alpha + i\beta) e^{(\alpha + i\beta)x}. \end{aligned}$$

To summarise then, if $w = e^{(\alpha + i\beta)x}$ then $w' = (\alpha + i\beta) e^{(\alpha + i\beta)x}$ or

$$[e^{(\alpha + i\beta)x}]' = (\alpha + i\beta) e^{(\alpha + i\beta)x}. \tag{10}$$

Thus, if k is a complex number (or, in the special case, a real number) and x is a real number, then

$$(e^{kx})' = ke^{kx}. \tag{9'}$$

We have thus obtained the ordinary formula for differentiation of an exponential function.

Further,

$$(e^{kx})'' = [(e^{kx})']' = k(e^{kx})' = k^2e^{kx}$$

and for arbitrary n

$$(e^{kx})^{(n)} = k^n e^{kx}.$$

We shall need these formulas later on.

SEC. 5. EULER'S FORMULA. THE EXPONENTIAL FORM OF A COMPLEX NUMBER

If we put $x=0$ in formula (1) of the preceding section, we get

$$e^{iy} = \cos y + i \sin y. \qquad (1)$$

This is *Euler's formula*, which expresses an exponential function with an imaginary exponent in terms of trigonometric functions.

Replacing y by $-y$ in (1) we get

$$e^{-iy} = \cos y - i \sin y. \qquad (2)$$

From (1) and (2) we find $\cos y$ and $\sin y$:

$$\left.\begin{aligned} \cos y &= \frac{e^{iy}+e^{-iy}}{2}, \\ \sin y &= \frac{e^{iy}-e^{-iy}}{2i}. \end{aligned}\right\} \qquad (3)$$

These formulas are used, among other things, to express the powers of $\cos\varphi$ and $\sin\varphi$ and their products in terms of the sine and cosine of multiple arcs.

Examples: 1. $\cos^2 y = \left(\frac{e^{iy}+e^{-iy}}{2}\right)^2 = \frac{1}{4}(e^{i2y}+2+e^{-i2y}) =$

$$= \frac{1}{4}[(\cos 2y + i\sin 2y) + 2 + (\cos 2y - i \sin 2y)] =$$

$$= \frac{1}{4}(2\cos 2y + 2) = \frac{1}{2}(1+\cos 2y).$$

2. $\cos^2\varphi \sin^2\varphi = \left(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right)^2 \left(\frac{e^{i\varphi}-e^{-i\varphi}}{2i}\right)^2 =$

$$= \frac{(e^{i2\varphi}-e^{-i2\varphi})^2}{4\cdot 4i^2} = -\frac{1}{8}\cos 4\varphi + \frac{1}{8}.$$

The exponential form of a complex number. Let us represent a complex number in trigonometric form:

$$z = r(\cos\varphi + i\sin\varphi),$$

where r is the modulus of the complex number and φ is the amplitude of the complex number. By Euler's formula,

$$\cos\varphi + i\sin\varphi = e^{i\varphi}.$$

Thus, any complex number may be represented in the so-called **exponential** form:

$$z = re^{i\varphi}.$$

Examples. Represent the numbers 1, i, -2, $-i$ in the exponential form.

Solution.

$$1=\cos 2k\pi+i\sin 2k\pi=e^{2k\pi i},$$

$$i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=e^{\frac{\pi}{2}i},$$

$$-2=2(\cos\pi+i\sin\pi)=2e^{\pi i},$$

$$-i=\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}=e^{-\frac{\pi}{2}i}.$$

SEC. 6. FACTORING A POLYNOMIAL

The function

$$f(x)=A_0x^n+A_1x^{n-1}+\ldots+A_n,$$

where n is an integer, is known as a *polynomial* or a *rational integral function* of x; the number n is called the *degree of the polynomial*. Here, the coefficients A_0, A_1, ..., A_n are real or complex numbers; the independent variable x can also take on both real and complex values. The *root* of a polynomial is that value of the variable x at which the polynomial becomes zero.

Theorem 1 (Remainder Theorem). *Division of a polynomial $f(x)$ by $x-a$ yields a remainder equal to $f(a)$.*

Proof. The quotient obtained by the division of $f(x)$ by $x-a$ will be a polynomial $f_1(x)$ of degree one less than that of $f(x)$, and the remainder will be a constant R. We can thus write

$$f(x)=(x-a)f_1(x)+R. \tag{1}$$

This equality holds for all values of x different from a (division by $x-a$ when $x=a$ is meaningless).

Now let x approach a. Then the limit of the left side of (1) will equal $f(a)$, while the limit of the right side will equal R. Since the functions $f(x)$ and $(x-a)f_1(x)+R$ are equal for all $x\neq a$, their limits are likewise equal as $x\to a$, that is, $f(a)=R$.

Corollary. *If a is a root of the polynomial, that is, if $f(a)=0$ then $x-a$ divides $f(x)$ without remainder and,* hence, $f(x)$ is represented in the form of a product

$$f(x)=(x-a)f_1(x)$$

where $f_1(x)$ is a polynomial.

Example 1. The polynomial $f(x)=x^3-6x^2+11x-6$ becomes zero for $x=1$; thus, $f(1)=0$, and so $x-1$ divides this polynomial without remainder:

$$x^3-6x^2+11x-6=(x-1)(x^2-5x+6).$$

Let us now consider equations in one unknown, x.

Any number (real or complex) which, when substituted into the equation in place of x, converts the equation into an identity is called a *root* of the equation.

Example 2. The numbers $x_1=\frac{\pi}{4}$; $x_2=\frac{5\pi}{4}$; $x_3=\frac{9\pi}{4}$; ..., are the roots of the equation $\cos x=\sin x$.

If the equation is of the form $P(x)=0$, where $P(x)$ is a polynomial of degree n, it is called an *algebraic* equation of degree n. From the definition it follows that the roots of an algebraic equation $P(x)=0$ are the same as are the roots of the polynomial $P(x)$.

Quite naturally the question arises: Does every equation have roots?

In the case of nonalgebraic equations, the answer is no: there are nonalgebraic equations which do not have a single root, either real or complex; for example, the equation $e^x=0$. *)

But in the case of an algebraic equation the answer is yes. This is given by the fundamental theorem of algebra.

Theorem 2 (Fundamental Theorem of Algebra). *Every rational integral function $f(x)$ has at least one root, real or complex.*

The proof of this theorem is given in higher algebra. Here we give it without proof.

With the aid of the fundamental theorem of algebra it is easy to prove the following theorem.

Theorem 3. *Every polynomial of degree n may be factored into n linear factors of the form $x-a$ and a factor equal to the coefficient of x^n.*

Proof. Let $f(x)$ be a polynomial of degree n:

$$f(x)=A_0x^n+A_1x^{n-1}+\ldots+A_n.$$

By virtue of the fundamental theorem, this polynomial has at least one root; we denote it by a_1. Then, by the corollary of the remainder theorem, we can write

$$f(x)=(x-a_1)f_1(x)$$

where $f_1(x)$ is a polynomial of degree $n-1$; $f_1(x)$ also has a root. We designate it by a_2. Then

$$f_1(x)=(x-a_2)f_2(x)$$

where $f_2(x)$ is a polynomial of degree $n-2$. Similarly,

$$f_2(x)=(x-a_3)f_3(x).$$

*) Indeed, if the number $x_1=a+bi$ were the root of this equation, we would have the identity $e^{a+bi}=0$ or (by Euler's formula) $e^a(\cos b+i\sin b)=0$. But e^a cannot equal zero for any real value of a; neither is $\cos b+i\sin b$ equal to zero (because the modulus of this number is $\sqrt{\cos^2 b+\sin^2 b}=1$ for any b). Hence, the product $e^a(\cos b+i\sin b)\neq 0$, i. e., $e^{a+bi}\neq 0$; but this means that the equation $e^x=0$ has no roots.

Continuing this process of factoring out linear factors, we arrive at the relation

$$f_{n-1}(x) = (x - a_n) f_n$$

where f_n is a polynomial of degree zero, i. e., some fixed number. This number is obviously equal to the coefficient of x^n; that is, $f_n = A_0$.

On the basis of the equalities obtained we can write

$$f(x) = A_0 (x - a_1)(x - a_2) \dots (x - a_n). \tag{2}$$

From the expansion (2) it follows that the numbers $a_1, a_2, \dots, a_n$ are roots of the polynomial $f(x)$, since upon the substitution $x = a_1$, $x = a_2, \dots, x = a_n$ the right side, and hence, the left, becomes zero.

Example 3. The polynomial $f(x) = x^3 - bx^2 + 11x - 6$ becomes zero when

$$x = 1, \; x = 2, \; x = 3.$$

Therefore,

$$x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3).$$

No value $x = a$ that is different from $a_1, a_2, \dots, a_n$ can be a root of the polynomial $f(x)$, since no factor on the right side of (2) vanishes when $x = a$. Whence the following proposition.

A polynomial of degree n cannot have more than n distinct roots.

But then the following theorem obtains.

Theorem 4. *If the values of two polynomials of degree n,* $\varphi_1(x)$ *and* $\varphi_2(x)$, *coincide for* $n+1$ *distinct values* $a_0, a_1, a_2, \dots, a_n$ *of the argument x, then these polynomials are identical.*

Proof. Denote the difference of the polynomials by $f(x)$:

$$f(x) = \varphi_1(x) - \varphi_2(x).$$

It is given that $f(x)$ is a polynomial of degree not higher than n that becomes zero at the points $a_1, \dots, a_n$. It can therefore be represented in the form

$$f(x) = A_0 (x - a_1)(x - a_2) \dots (x - a_n).$$

But it is given that $f(x)$ also vanishes at the point a_0. Then $f(a_0) = 0$ and not a single one of the linear factors equals zero. For this reason, $A_0 = 0$ and then from (2) it follows that the polynomial $f(x)$ is identically equal to zero. Consequently, $\varphi_1(x) - \varphi_2(x) \equiv 0$ or $\varphi_1(x) \equiv \varphi_2(x)$.

Theorem 5. *If a polynomial*

$$P(x) = A_0 x^n + A_1 x^{n-1} + \dots + A_{n-1} x + A_n$$

is identically equal to zero, all its coefficients equal zero.

Proof. Let us write its factorisation using formula (2):
$$P(x)=A_0x^n+A_1x^{n-1}+\ldots+A_{n-1}x+A_n=A_0(x-a_1)\ldots(x-a_n). \quad (1')$$
If this polynomial is identically equal to zero, it is also equal to zero for some value of x different from $a_1, \ldots, a_n$. But then none of the bracketed values $x-a_1, \ldots, x-a_n$ is equal to zero, and, hence, $A_0=0$.

Similarly it is proved that $A_1=0$, $A_2=0$, and so forth.

Theorem 6. *If two polynomials are identically equal, the coefficients of one polynomial are equal to the corresponding coefficients of the other.*

This follows from the fact that the difference between the polynomials is a polynomial identically equal to zero. Therefore, from the preceding theorem all its coefficients are zeros.

Example 4. If the polynomial ax^3+bx^2+cx+d is identically equal to the polynomial x^2-5x, then $a=0$, $b=1$, $c=-5$, and $d=0$.

SEC. 7. THE MULTIPLE ROOTS OF A POLYNOMIAL

If, in the factorisation of a polynomial of degree n into linear factors
$$f(x)=A_0(x-a_1)(x-a_2)\ldots(x-a_n) \quad (1)$$
certain linear factors turn out the same, they may be combined, and then factorisation of the polynomial will yield
$$f(x)=A_0(x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a_m)^{k_m}. \quad (1')$$
And
$$k_1+k_2+\ldots+k_m=n.$$
In this case, the root a_1 is called a root of multiplicity k_1, or a k_1-tuple root, a_2, a root of multiplicity k_2, etc.

Example. The polynomial $f(x)=x^3-5x^2+8x-4$ may be factored into the following linear factors:
$$f(x)=(x-2)(x-2)(x-1).$$
This factorisation may be written as follows:
$$f(x)=(x-2)^2(x-1).$$
The root $a_1=2$ is a double root, $a_2=1$ is a simple root.

If a polynomial has a root a of multiplicity k, then we will consider that the polynomial has k coincident roots. Then from the theorem of factorisation of a polynomial into linear factors we get the following theorem.

Every polynomial of degree n has exactly n roots (real or complex).

Note. All that has been said of the roots of the polynomial

$$f(x)=A_0x^n+A_1x^{n-1}+\dots+A_n,$$

may obviously be formulated in terms of the roots of the algebraic equation

$$A_0x^n+A_1x^{n-1}+\dots+\dots A_n=0.$$

Let us further prove the following theorem.

Theorem. *If, for the polynomial $f(x)$, a_1 is a root of multiplicity $k_1>1$, then for the derivative $f'(x)$ this number is a root of multiplicity k_1-1.*

Proof. If a_1 is a root of multiplicity $k_1>1$, then it follows from formula (1′) that

$$f(x)=(x-a_1)^{k_1}\varphi(x)$$

where $\varphi(x)=(x-a_2)^{k_2}\dots(x-a_m)^{k_m}$ does not become zero at $x=a_1$; that is, $\varphi(a_1)\neq 0$. Differentiating, we get

$$f'(x)=k_1(x-a_1)^{k_1-1}\varphi(x)+(x-a_1)^{k_1}\varphi'(x)=$$
$$=(x-a_1)^{k_1-1}[k_1\varphi(x)+(x-a_1)\varphi'(x)].$$

Put

$$\psi(x)=k_1\varphi(x)+(x-a_1)\varphi'(x).$$

Then

$$f'(x)=(x-a_1)^{k_1-1}\psi(x)$$

and here

$$\psi(a_1)=k_1\varphi(a_1)+(a_1-a_1)\varphi'(a_1)=k_1\varphi(a_1)\neq 0.$$

In other words, $x=a_1$ is a root of multiplicity k_1-1 of the polynomial $f'(x)$. From the foregoing proof it follows that if $k_1=1$, then a_1 is not a root of the derivative $f'(x)$.

From the proved theorem it follows that a_1 is a root of multiplicity k_1-2 for the derivative $f''(x)$, a root of multiplicity k_1-3 for the derivative $f'''(x)\dots$, and a root of multiplicity one (simple root) for the derivative $f^{(k_1-1)}(x)$ and is not a root for the derivative $f^{(k_1)}(x)$, or

$$f(a_1)=0,\ f'(a_1)=0,\ f''(a_1)=0,\ \dots,\ f^{(k_1-1)}(a_1)=0,$$

but

$$f^{(k)}(a_1)\neq 0.$$

SEC. 8. FACTORISATION OF A POLYNOMIAL IN THE CASE OF COMPLEX ROOTS

In formula (1), Sec. 7, Chapter VII, the roots $a_1, a_2, \dots, a_n$ may be either real or complex. We have the following theorem.

Theorem. *If a polynomial $f(x)$ with real coefficients has a complex root $a+bi$, it also has a conjugate root $a-bi$.*

Proof. Substitute, in the polynomial $f(x)$, $a+bi$ in place of x, raise to a power and collect separately terms containing i and those not containing i; we then get

$$f(a+bi)=M+Ni,$$

where M and N are expressions that do not contain i.

Since $a+bi$ is a root of the polynomial, we have

$$f(a+bi)=M+Ni=0$$

whence

$$M=0, \quad N=0.$$

Now substitute the expression $a-bi$ for x in the polynomial. Then (on the basis of Note 3 at the end of Sec. 2 of this chapter) we get a number that is a conjugate of the number $M+Ni$, or

$$f(a-bi)=M-Ni.$$

Since $M=0$ and $N=0$, we have $f(a-bi)=0$; $a-bi$ is a root of the polynomial.

Thus, in the factorisation

$$f(x)=A_0(x-a_1)(x-a_2)\dots(x-a_n)$$

the complex roots enter as **conjugate pairs.**

Multiplying together the linear factors that correspond to a pair of complex conjugate roots, we get a trinomial of degree two with real coefficients:

$$[x-(a+bi)][x-(a-bi)]=$$
$$=[(x-a)-bi][(x-a)+bi]=$$
$$=(x-a)^2+b^2=x^2-2ax+a^2+b^2=x^2+px+q,$$

where $p=-2a$, $q=a^2+b^2$ are real numbers.

If the number $a+bi$ is a root of multiplicity k, the conjugate number $a-bi$ must be a root of the same multiplicity k, so that factorisation of the polynomial will yield the same number of linear factors $x-(a+bi)$ as those of the form $x-(a-bi)$.

Thus, *a polynomial with real coefficients may be factored into real factors of the first and second degree of corresponding multiplicity;* that is,

$$f(x)=A_0(x-a_1)^{k_1}(x-a_2)^{k_2}\dots$$
$$\dots(x-a_r)^{k_r}(x^2+p_1x+q_1)^{l_1}\dots(x^2+p_sx+q_s)^{l_s}$$

where

$$k_1+k_2+\dots+k_r+2l_1+\dots+2l_s=n.$$

SEC. 9. INTERPOLATION. LAGRANGE'S INTERPOLATION FORMULA

Let it be established, in the study of some phenomenon, that there is a functional relationship between the quantities y and x which describes the quantitative aspect of the phenomenon; the function $y=\varphi(x)$ is unknown, but experiment has established the values of this function $y_0, y_1, y_2, \ldots, y_n$ for certain values of the argument $x_0, x_1, x_2, \ldots, x_n$, in the interval $[a, b]$.

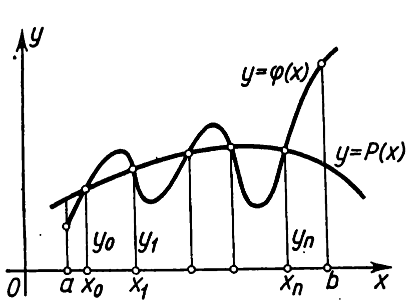


Fig. 164.

The problem is to find a function (as simple as possible from the computational standpoint; for example, a polynomial) which would represent the unknown function $y=\varphi(x)$ on the interval $[a, b]$ either exactly or approximately. In more abstract fashion the problem may be formulated as follows: given on the interval $[a, b]$ the values of an unknown function $y=\varphi(x)$ at $n+1$ distinct points $x_0, x_1, \ldots, x_n$:

$$y_0=\varphi(x_0), \quad y_1=\varphi(x_1), \quad \ldots, \quad y_n=\varphi(x_n);$$

it is required to find a **polynomial** $P(x)$ of degree $\leqslant n$ that approximately expresses the function $\varphi(x)$.

For such a polynomial, it is natural to take a polynomial whose values at the points $x_0, x_1, x_2, \ldots, x_n$ **coincide** with the corresponding values $y_0, y_1, y_2, \ldots, y_n$ of the function $\varphi(x)$ (Fig. 164). Then the problem, which is called the "problem of *interpolating* a function", is formulated thus: for a given function $\varphi(x)$ find a polynomial $P(x)$ of degree $\leqslant n$, which, for the given values of $x_0, x_1, \ldots, x_n$, will take on the values

$$y_0=\varphi(x_0), \; y_1=\varphi(x_1), \; \ldots, \; y_n=\varphi(x_n).$$

For the desired polynomial, take a polynomial of degree n of the form

$$\begin{aligned} P(x) = {} & C_0(x-x_1)(x-x_2)\ldots(x-x_n)+ \\ & +C_1(x-x_0)(x-x_2)\ldots(x-x_n)+ \\ & +C_2(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)+\ldots \\ & \ldots+C_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \end{aligned} \tag{1}$$

and define the coefficients $C_0, C_1, \ldots, C_n$ so that the following

conditions are fulfilled:

$$P(x_0)=y_0, \quad P(x_1)=y_1, \quad \ldots, \quad P(x_n)=y_n. \tag{2}$$

In (1) put $x=x_0$; then, taking into account equality (2), we get

$$y_0=C_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n),$$

whence

$$C_0=\frac{y_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}.$$

Then, setting $x=x_1$, we get

$$y_1=C_1(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n),$$

whence

$$C_1=\frac{y_1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}.$$

In the same way we find

$$C_2=\frac{y_2}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)};$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$C_n=\frac{y_n}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}.$$

Substituting these values of the coefficients into (1), we get

$$\begin{aligned}P(x)=&\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0+\\&+\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1+\\&+\frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}y_2+\ldots\\&\ldots+\frac{(x-x_0)(x-x_1)\ldots(x_n-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n.\end{aligned} \tag{3}$$

This formula is called the *Lagrange interpolation formula.*

Let it be noted, without proof, that if $\varphi(x)$ has a derivative of the $(n+1)$st order on the interval $[a, b]$, the error resulting from replacing the function $\varphi(x)$ by the polynomial $P(x)$, i. e., the quantity $R(x)=\varphi(x)-P(x)$, satisfies the inequality

$$|R(x)|<|(x-x_0)(x-x_1)\ldots(x-x_n)|\frac{1}{(n+1)!}\max|\varphi^{(n+1)}(x)|.$$

Note. From Theorem 4, Sec. 6, Ch. VII, it follows that the polynomial $P(x)$ which we found is the only one that satisfies the given conditions.

Example. From experiment we get the values of the function $y=\varphi(x)$: $y_0=3$ for $x_0=1$, $y_1=-5$ for $x_1=2$, $y_2=4$ for $x_2=-4$.

It is required to represent the function $y=\varphi(x)$ approximately by a polynomial of degree two.

Solution. From (3) we have (for $n=2$):

$$P(x)=\frac{(x-2)(x+4)}{(1-2)(1+4)}3+\frac{(x-1)(x+4)}{(2-1)(2+4)}(-5)+\frac{(x-1)(x-2)}{(-4-1)(-4-2)}4$$

or

$$P(x)=-\frac{39}{30}x^2-\frac{123}{30}x+\frac{252}{30}.$$

SEC. 10. ON THE BEST APPROXIMATION OF FUNCTIONS BY POLYNOMIALS. CHEBYSHEV'S THEORY

A natural question follows from what has been discussed in the previous section: If a continuous function $\varphi(x)$ is given on the closed interval $[a, b]$, can this function be represented approximately in the form of a polynomial $P(x)$ **to any preassigned degree of accuracy?** In other words, is it possible to choose a polynomial $P(x)$ such that the absolute difference between $\varphi(x)$ and $P(x)$ at all points of the interval $[a, b]$ should be less than any preassigned positive number ε? The following theorem, which we give without proof, answers this question in the affirmative. *)

Weierstrass' Approximation Theorem. *If a function $\varphi(x)$ is continuous on a closed interval $[a, b]$, then for every $\varepsilon>0$ there exists a polynomial $P(x)$ such that $|f(x)-P(x)|<\varepsilon$, for every x in the interval.*

The outstanding Soviet mathematician Academician S. N. Bernstein gave the following method of direct construction of such polynomials that are approximately equal to the continuous function $\varphi(x)$ on the given interval.

Let $\varphi(x)$ be continuous on the interval $[0, 1]$. We write the expression

$$B_n(x)=\sum_{m=0}^{n}\varphi\left(\frac{m}{n}\right)C_n^m x^m(1-x)^{n-m}.$$

Here, C_n^m are binomial coefficients, $\varphi\left(\frac{m}{n}\right)$ is the value of the given function at the point $x=\frac{m}{n}$. The expression $B_n(x)$ is an nth degree polynomial called the *Bernstein polynomial*.

*) It will be noted that the Lagrange interpolation formula [see (3) Sec. 9] cannot yet answer this question. Its values are equal to those of the function at the points $x_0, x_1, x_2, \ldots, x_n$, but they may be very far from the values of the function at other points of the interval $[a, b]$.

If an arbitrary $\varepsilon > 0$ is given, one can choose a Bernstein polynomial (that is, select its degree n) such that for all values of x on the interval $[0, 1]$, the following inequality will be fulfilled:

$$|B_n(x) - \varphi(x)| < \varepsilon.$$

It should be noted that consideration of the interval $[0, 1]$, and not an arbitrary interval $[a, b]$, is not an essential limitation of generality, since by changing the variable $x = a + t(b - a)$ it is possible to convert any interval $[a, b]$ into $[0, 1]$. In this case, the nth degree polynomial will be transformed into a polynomial of the same degree.

The creator of the theory of best approximation of functions by polynomials is the brilliant Russian mathematician P. L. Chebyshev (1821-1894). In this field, he obtained the most profound results, which exerted a great influence on the work of later mathematicians. Studies involving the theory of articulated mechanisms, which are widely used in machines, served as the starting point of Chebyshev's theory. While studying these mechanisms he arrived at the problem of finding, among all polynomials of a given degree with the leading coefficient equal to unity, a polynomial of least deviation from zero on the given interval. He found these polynomials, which subsequently became known as the *Chebyshev polynomials*. They possess many remarkable properties, and at present are a powerful tool of investigation in many problems of mathematics and engineering.

Exercises on Chapter VII

1. Find $(3+5i)(4-i)$. *Ans.* $17+17i$. **2.** Find $(6+11i)(7+3i)$. *Ans.* $9+95i$. **3.** Find $\frac{3-i}{4+5i}$. *Ans.* $\frac{7}{41}-\frac{19}{41}i$. **4.** Find $(4-7i)^3$. *Ans.* $-524+7i$. **5.** Find $\sqrt{i}$. *Ans.* $\pm\frac{1+i}{\sqrt{2}}$. **6.** Find $\sqrt{-5-12i}$. *Ans.* $\pm(2-3i)$. **7.** Reduce the following expressions to trigonometric form: a) $1+i$. *Ans.* $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$; b) $1-i$. *Ans.* $\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$. **8.** Find $\sqrt[3]{i}$. *Ans.* $\frac{i+\sqrt{3}}{2}$; $-i$; $\frac{i-\sqrt{3}}{2}$. **9.** Express the following expressions in terms of powers of $\sin x$ and $\cos x$: $\sin 2x$, $\cos 2x$, $\sin 4x$, $\cos 4x$, $\sin 5x$, $\cos 5x$. **10.** Express the following in terms of the sine and cosine of multiple arcs: $\cos^2 x$, $\cos^3 x$, $\cos^5 x$, $\cos^6 x$; $\sin^2 x$, $\sin^3 x$, $\sin^4 x$, $\sin^5 x$. **11.** Divide $f(x)=x^3-4x^2+8x-1$ by $x+4$. *Ans.* $f(x)=(x+4)(x^2-8x+40)-161$, that is, the quotient is equal to $x^2-8x+40$; and the remainder is $f(-4)=-161$. **12.** Divide $f(x)=x^4+12x^3+54x^2+108x+81$ by $x+3$. *Ans.* $f(x)=(x+3)(x^3+9x^2+27x+27)$. **13.** Divide $f(x)=x^7-1$ by $x-1$. *Ans.* $f(x)=(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$.

Factor the following polynomials: **14.** $f(x)=x^4-1$. *Ans.* $f(x)=(x-1)(x+1)(x^2+1)$. **15.** $f(x)=x^2-x-2$. *Ans.* $f(x)=(x-2)(x+1)$. **16.** $f(x)=x^3+1$. *Ans.* $f(x)=(x+1)(x^2-x+1)$.

17. Experiment yielded the following values of y as a function of x:

$$y_1=4 \text{ for } x_1=0,$$
$$y_2=6 \text{ for } x_2=1,$$
$$y_3=10 \text{ for } x_3=2.$$

Represent (approximately) the function by a second-degree polynomial. *Ans.* x^2+x+4.

18. Find a polynomial of degree four that takes on the values 2, 1, -1, 5, 0 for $x=1$, 2, 3, 4, 5, respectively. *Ans.* $\frac{3}{2}x^4-17x^3+\frac{129}{2}x^2-92x+35$.

19. Find a polynomial of the lowest possible degree that takes on the values 3, 7, 9, 19 for $x=2$, 4, 5, 10, respectively. *Ans.* $2x-1$.

20. Find the Bernstein polynomials of degree 1, 2, 3 and 4 for the function $y=\sin\pi x$ on the interval [0, 1]. *Ans.* $B_1(x)=0$; $B_2(x)=2x(1-x)$; $B_3(x)=\frac{3\sqrt{3}}{2}x(1-x)$; $B_4(x)=2x(1-x)[(2\sqrt{2}-3)x^2-(2\sqrt{2}-3)x+\sqrt{2}]$.

CHAPTER VIII

FUNCTIONS OF SEVERAL VARIABLES

SEC. 1. DEFINITION OF A FUNCTION OF SEVERAL VARIABLES

When considering a function of one variable we pointed out that in the study of many phenomena one encounters functions of two and more independent variables. Some examples follow.

Example 1. The area S of a rectangle with sides of length x and y is expressed by the formula

$$S = xy.$$

To each pair of values of x and y there corresponds a definite value of the area S. S is a function of two variables.

Example 2. The volume V of a rectangular parallelepiped with edges of length x, y, z is expressed by the formula

$$V = xyz.$$

Here, V is a function of three variables, x, y, z.

Example 3. The range R of a shell fired with initial velocity v_0 from a gun, whose barrel is inclined to the horizon at an angle φ, is expressed by the formula

$$R = \frac{v_0^2 \sin 2\varphi}{g}$$

(air resistance is disregarded). Here, g is the acceleration of gravity.

For every pair of values of v_0 and φ this formula yields a definite value of R; in other words, R is a function of two variables, v_0 and φ.

Example 4.

$$u = \frac{x^2 + y^2 + z^2 + t^2}{\sqrt{1 + x^2}}.$$

Here, u is a function of four variables x, y, z, t.

Definition 1. If to each pair (x, y) of values of two independent variable quantities x and y (from some range D) there corresponds a definite value of the quantity z, we say that z is a *function of the two independent variables x and y* defined in D.

A function of two variables is symbolically given as

$$z = f(x, y), \quad z = F(x, y) \text{ and so forth.}$$

A function of two variables may be represented, for example, by means of a table or analytically (by a formula) as in the four examples given above. The formula may be used to construct

a table of values of the function for certain number pairs of the independent variables. From Example 1 we can build the following table:

$$S = xy$$

y \ x	0	1	1.5	2	3
1	0	1	1.5	2	3
2	0	2	3	4	6
3	0	3	4.5	6	9
4	0	4	6	8	12

In this table, the intersections of the lines and columns, which correspond to definite values of x and y, yield the corresponding values of the function S.

If the functional relation $z = f(x, y)$ is obtained as a result of changes in the quantity z in some experimental study of a phenomenon, we straightway get a table defining z as a function of two variables. In this case, the function is specified by the table alone.

As in the case of a single independent variable, a function of two variables does not, generally speaking, exist for all values of x and y.

Definition 2. The collection of pairs (x, y) of values of x and y, for which the function

$$z = f(x, y)$$

is defined, is called the *domain of definition* of this function.

The domain of a function is apparent when illustrated geometrically. If each number pair x and y is given as a point $M(x, y)$ in the xy-plane, then the domain of definition of the function will be a certain collection of points in the plane. We shall also call this collection of points the domain of definition of the function. In particular, the entire plane may be the domain. In future we shall mainly have to do with such domains as are **parts of the plane bounded by lines.** The line bounding the given domain we shall call the *boundary* of the domain. The points of the domain not lying on the boundary we shall call *interior* points of the domain. A domain consisting solely of interior points is called an *open* domain; that which includes the points of the boundary is called a *closed* domain.

Example 5. Determine the natural domain of definition of the function

$$z=2x-y.$$

The analitic expression $2x-y$ is meaningful for all values of x and y. Therefore, the entire xy-plane is the natural domain of the function.

Example 6. $z=\sqrt{1-x^2-y^2}$.

For z to have a real value it is necessary that the radicand be a nonnegative number; in other words, x and y must satisfy the inequality

$$1-x^2-y^2\geqslant 0, \quad \text{or} \quad x^2+y^2\leqslant 1.$$

All the points $M(x, y)$ whose coordinates satisfy the given inequality lie in a circle of radius 1 with centre at the origin and on the boundary of this circle.

Example 7.

$$z=\ln(x+y).$$

Since logarithms are defined only for positive numbers, the following inequality must be satisfied:

$$x+y>0 \quad \text{or} \quad y>-x.$$

This means that the natural domain of definition of the function z is the half-plane above the straight line $y=-x$, the line itself not included (Fig. 165).

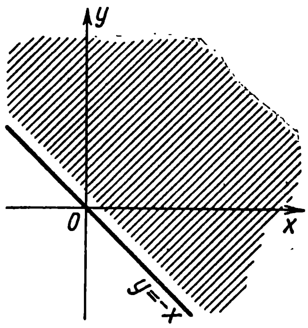


Fig. 165.

Example 8. The area of the triangle S is a function of the base x and the altitude y:

$$S=\frac{xy}{2}.$$

The domain of this function is $x>0$, $y>0$ (since the base of a triangle and its altitude cannot be negative or zero). We notice that the domain of this function does not coincide with the natural domain of definition of the analytic expression used to define the function, because the natural domain of the expression $\frac{xy}{2}$ is obviously the entire xy-plane.

It is easy to generalise the definition of a function of two variables to the case of three or more variables.

Definition 3. If to every collection of values of the variables $x, y, z, \ldots, u, t$ there corresponds a definite value of the variable w, we shall then call w *the function of the independent variables* $x, y, z, \ldots, u, t$ and write $w=F(x, y, z, \ldots, u, t)$ or $w=f(x, y, z, u, t)$, and so on.

Just as in the case of a function of two variables, we can speak of the domain of definition of a function of three, four and more variables.

To take an example, for a function of three variables, the domain of definition is a certain collection of number triples (x, y, z). Let it be noted that each number triple is associated with some point $M(x, y, z)$ in xyz-space. Consequently, the domain of

definition of a function of three variables is some collection of points in space.

Similarly, one can speak of the domain of definition of a function of four variables $u=f(x, y, z, t)$ as of a certain collection of number quadruples (x, y, z, t). However, the domain of definition of a function of four or a larger number of variables no longer permits of a simple geometric interpretation.

Example 2 gives a function of three variables defined for all values of x, y, z.

In Example 4 we have a function of four variables.

Example 9.

$$w=\sqrt{1-x^2-y^2-z^2-u^2}.$$

Here w is a function of the four variables x, y, z, u defined for values of the variables that satisfy the relationship

$$1-x^2-y^2-z^2-u^2 \geqslant 0.$$

SEC. 2. GEOMETRIC REPRESENTATION OF A FUNCTION OF TWO VARIABLES

We consider the function

$$z=f(x, y), \tag{1}$$

defined in the domain G in the xy-plane (as a particular case, this domain may be the entire plane), and a system of rec-

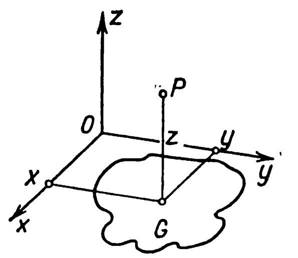


Fig. 166.

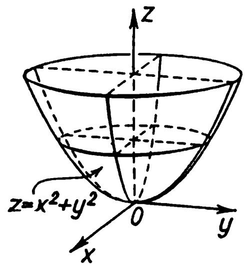


Fig. 167.

tangular Cartesian coordinates $Oxyz$ (Fig. 166). At each point (x, y) erect a perpendicular to the xy-plane and on it lay off a segment equal to $f(x, y)$.

This gives us a point P in space with coordinates

$$x, y, z=f(x, y).$$

The locus of points P whose coordinates satisfy equation (1) is the graph of a function of two variables. From the course of

analytic geometry we know that equation (1) defines a surface in space. Thus, the graph of a function of two variables is a surface projected onto the xy-plane in the domain of definition of the function. Each perpendicular to the xy-plane intersects the surface $z=f(x, y)$ at not more than one point.

Example. As we know from analytic geometry, the graph of the function $z=x^2+y^2$ is a paraboloid of revolution (Fig. 167).

Note. It is impossible to depict a function of three or more variables by means of a graph in space.

SEC. 3. PARTIAL AND TOTAL INCREMENT OF A FUNCTION

Consider the line of intersection PS of the surface

$$z=f(x, y)$$

with the plane y-const parallel to the xz-plane (Fig. 168).

Since in this plane y remains constant, z will vary along the curve PS depending only on the changes in x. Increase the independent variable x by Δx; then z will be increased; this increase is called the *partial increment of z with respect to x* and it is denoted by $\Delta_x z$ (the segment SS' in the figure), so that

$$\Delta_x z=f(x+\Delta x, y)-f(x, y). \quad (1)$$

Similarly, if x is held constant and y is increased by Δy, then z is increased, and this increase is called the *partial increment of z with respect to y* (symbolised by $\Delta_y z$, the segment TT' in the figure):

$$\Delta_y z=f(x, y+\Delta y)-f(x, y). \quad (2)$$

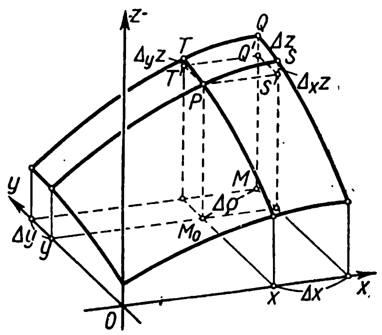


Fig. 168.

The function receives the increment $\Delta_y z$ "along the line" of intersection of the surface $z=f(x, y)$ with the plane x-const parallel to the yz-plane.

Finally, increasing the argument x by Δx, and the argument y by the increment Δy, we get for z a new increment Δz, which is called the *total increment* of the function z and is defined by the formula

$$\Delta z=f(x+\Delta x, y+\Delta y)-f(x, y). \quad (3)$$

In Fig. 168 Δz is shown as the segment QQ'.

It must be noted that, generally speaking, the total increment is not equal to the sum of the partial increments, $\Delta z \neq \Delta_x z + \Delta_y z$.

Example. $z = xy$.

$$\Delta_x z = (x + \Delta x) y - xy = y \Delta x,$$
$$\Delta_y z = x (y + \Delta y) - xy = x \Delta y,$$
$$\Delta z = (x + \Delta x)(y + \Delta y) - xy = y \Delta x + x \Delta y + \Delta x \Delta y.$$

For $x = 1$, $y = 2$, $\Delta x = 0.2$, $\Delta y = 0.3$ we have $\Delta_x z = 0.4$, $\Delta_y z = 0.3$, $\Delta z = 0.76$.

Similarly we define the partial and total increments of a function of any number of variables. Thus, for a function of three variables $u = f(x, y, t)$ we have

$$\Delta_x u = f(x + \Delta x, y, t) - f(x, y, t),$$
$$\Delta_y u = f(x, y + \Delta y, t) - f(x, y, t),$$
$$\Delta_t u = f(x, y, t + \Delta t) - f(x, y, t),$$
$$\Delta u = f(x + \Delta x, y + \Delta y, t + \Delta t) - f(x, y, t).$$

SEC. 4. CONTINUITY OF A FUNCTION OF SEVERAL VARIABLES

We introduce an important auxiliary concept, that of the neighbourhood of a given point.

The *neighbourhood*, of radius r, of a point $M_0(x_0, y_0)$ is the collection of all points (x, y) that satisfy the inequality $\sqrt{(x-x_0)^2 + (y-y_0)^2} < r$; that is, the set of all points that lie inside a circle of radius r with centre in the point $M_0(x_0, y_0)$.

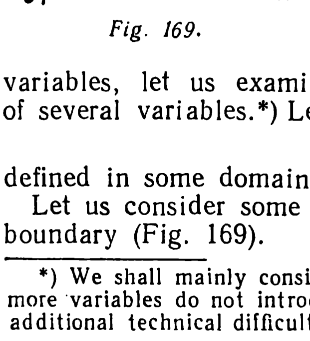


Fig. 169.

If we say that a function $f(x, y)$ possesses some property "near the point (x_0, y_0)" or "in the neighbourhood of the point (x_0, y_0)" we mean that there is a circle with centre at (x_0, y_0), at all points of which circle the given function possesses the given property.

Before considering the concept of continuity of a function of several variables, let us examine the notion of the limit of a function of several variables.*) Let there be a function

$$z = f(x, y)$$

defined in some domain G of an xy-plane.

Let us consider some definite point $M_0(x_0, y_0)$ in G or on its boundary (Fig. 169).

*) We shall mainly consider functions of two variables, since three and more variables do not introduce any fundamental changes, but do introduce additional technical difficulties.

Definition 1. The number A is called the *limit* of the function $f(x, y)$ as $M(x, y)$ approaches $M_0(x_0, y_0)$ if for every $\varepsilon > 0$ there is an $r > 0$ such that for all points $M(x, y)$ for which the inequality $\overline{MM_0} < r$ is fulfilled we have the inequality

$$|f(x, y) - A| < \varepsilon.$$

If A is the limit of $f(x, y)$ as $M(x, y) \to M_0(x_0, y_0)$, then we write

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = A.$$

Definition 2. Let the point $M_0(x_0, y_0)$ belong to the domain of definition of the function $f(x, y)$. The function $z = f(x, y)$ is called *continuous at the point* $M_0(x_0, y_0)$ if we have

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_0, y_0), \tag{1}$$

and $M(x, y)$ approaches $M_0(x_0, y_0)$ in arbitrary fashion all the while remaining in the domain of the function.

Designate $x = x_0 + \Delta x$, $y = y_0 + \Delta y$, then (1) may be rewritten as follows:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f(x_0 + \Delta x, \ y_0 + \Delta y) = f(x_0, y_0) \tag{1'}$$

or

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [f(x_0 + \Delta x, \ y_0 + \Delta y) - f(x_0, y_0)] = 0. \tag{1''}$$

We set $\Delta\varrho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ (see Fig. 168). As $\Delta x \to 0$ and $\Delta y \to 0$, $\Delta\varrho \to 0$; and conversely, if $\Delta\varrho \to 0$, then $\Delta x \to 0$ and $\Delta y \to 0$.

Noting further that the expression in the square brackets in (1″) is the total increment of the function Δz, (1″) may be rewritten in the form

$$\lim_{\Delta\varrho \to 0} \Delta z = 0. \tag{1'''}$$

A function continuous at each point of some domain is *continuous in the domain*.

If at some point $N(x_0, y_0)$ condition (1) is not fulfilled, then the point $N(x_0, y_0)$ is called a point of discontinuity of the function $z = f(x, y)$. For example, condition (1′) may not be fulfilled in the following cases:

1) $z = f(x, y)$ is defined at all points of a certain neighbourhood of the point $N(x_0, y_0)$ with the exception of the point $N(x_0, y_0)$ itself;

2) the function $z=f(x, y)$ is defined at all points of a neighbourhood of the point $N(x_0, y_0)$ but there is no limit $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x, y)$;

3) the function is defined at all points of the neighbourhood of $N(x_0, y_0)$ and the limit exists: $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x, y)$, but

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x, y)\neq f(x_0, y_0).$$

Example 1. The function

$$z=x^2+y^2$$

is continuous for all values of x and y; that is, it is continuous at every point in the xy-plane.

Indeed, no matter what the numbers x and y, Δx and Δy, we have

$$\Delta z=[(x+\Delta x)^2+(y+\Delta y)^2]-[x^2+y^2]=2x\,\Delta x+2y\,\Delta y+\Delta x^2+\Delta y^2.$$

Consequently,

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta z=0.$$

The following is an example of a discontinuous function.

Example 2. The function

$$z=\frac{2xy}{x^2+y^2}$$

is defined everywhere except at the point $x=0$, $y=0$ (Figs. 170, 171).

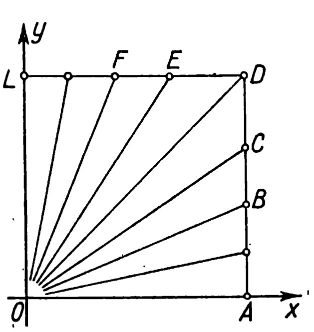


Fig. 170.

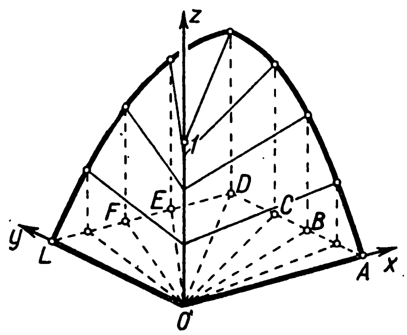


Fig. 171.

Let us examine the values of z along the straight line $y=kx$ ($k=$const). Obviously, along this line

$$z=\frac{2kx^2}{x^2+k^2x^2}=\frac{2k}{1+k^2}=\text{const}.$$

This means that a function z along any straight line passing through the origin retains a constant value that depends upon the slope k of the line. Thus, approaching the origin along different paths we will obtain different limiting values, and this means that the function $f(x, y)$ has no limit when the point (x, y) in the xy-plane approaches the origin. Thus, the function is discontinuous at this point. It is impossible to redefine this function at the coordinate origin so that it should become continuous. On the other hand, it is readily seen that the function is continuous at all other points.

SEC. 5. PARTIAL DERIVATIVES OF A FUNCTION OF SEVERAL VARIABLES

Definition. *The partial derivative, with respect to x,* of a function $z=f(x, y)$ is the limit of the ratio of the partial increment $\Delta_x z$, with respect to x, to the increment Δx as Δx approaches zero.

The partial derivative, with respect to x, of the function $z=f(x, y)$ is denoted by one of the symbols

$$z'_x;\quad f'_x(x, y);\quad \frac{\partial z}{\partial x};\quad \frac{\partial f}{\partial x}.$$

Thus, by definition,

$$\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x, y)-f(x, y)}{\Delta x}.$$

Similarly, the *partial derivative, with respect to y,* of a function $z=f(x, y)$ is defined as the limit of the ratio of the partial increment of the function $\Delta_y z$ with respect to y to the increment of Δy as Δy approaches zero. The partial derivative with respect to y is denoted by one of the following symbols:

$$z'_y;\quad f'_y;\quad \frac{\partial z}{\partial y};\quad \frac{\partial f}{\partial y}.$$

Thus,

$$\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{\Delta_y z}{\Delta y}=\lim_{\Delta y\to 0}\frac{f(x, y+\Delta y)-f(x, y)}{\Delta y}.$$

Noting that $\Delta_x z$ is calculated with y held constant, and $\Delta_y z$ with x held constant, we can formulate the definitions of partial derivatives as follows: *the partial derivative of the function $z=f(x, y)$ with respect to x* is the derivative with respect to x calculated on the assumption that y is constant. The *partial derivative of the function $z=f(x, y)$ with respect to y* is the derivative with respect to y calculated on the assumption that x is constant.

It is clear from this definition that the rules for computing partial derivatives coincide with the rules given for functions of one variable, and the only thing to remember is with respect to which variable the derivative is sought.

Example 1. Given the function $z=x^2 \sin y$; find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution.

$$\frac{\partial z}{\partial x}=2x \sin y; \quad \frac{\partial z}{\partial y}=x^2 \cos y.$$

Example 2. $z=x^y$.

Here

$$\frac{\partial z}{\partial x}=yx^{y-1},$$

$$\frac{\partial z}{\partial y}=x^y \ln x.$$

The partial derivatives of a function of any number of variables are determined similarly. Thus, if we have a function u of four variables x, y, z, t:

$$u=f(x, y, z, t)$$

then

$$\frac{\partial u}{\partial x}=\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z, t)-f(x, y, z, t)}{\Delta x},$$

$$\frac{\partial u}{\partial y}=\lim_{\Delta x \to 0} \frac{f(x, y+\Delta y, z, t)-f(x, y, z, t)}{\Delta y}, \text{ and so forth.}$$

Example 3.

$$u=x^2+y^2+xtz^3,$$

$$\frac{\partial u}{\partial x}=2x+tz^3; \quad \frac{\partial u}{\partial y}=2y; \quad \frac{\partial u}{\partial z}=3xtz^2; \quad \frac{\partial u}{\partial t}=xz^3.$$

SEC. 6. THE GEOMETRIC INTERPRETATION OF THE PARTIAL DERIVATIVES OF A FUNCTION OF TWO VARIABLES

Let the equation

$$z=f(x, y)$$

be the equation of a surface shown in Fig. 172.

Draw the plane $x=\text{const}$. The intersection of this plane with the surface yields the line PT. For a given x, let us consider a certain point $M(x, y)$ in the xy-plane. To the point M there corresponds a point $P(x, y, z)$ on the surface $z=f(x, y)$. Holding x constant, let us increase the variable y by $\Delta y=MN=PT'$. Then the function z will be increased by $\Delta_y z=TT'$ [to the point $N(x, y+\Delta y)$ there corresponds a point $T(x, y+\Delta_y, z+\Delta_y z)$ on the surface $z=f(x, y)$].

The ratio $\frac{\Delta_y z}{\Delta_y}$ is equal to the tangent of the angle formed by the *secant line* PT with the positive y-direction:

$$\frac{\Delta_y z}{\Delta_y} = \tan\ T\widehat{P}T'.$$

Consequently, the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \frac{\partial z}{\partial y}$$

is equal to the tangent of the angle β formed by the tangent line PB to the curve PT at the point P with the positive y-direction:

$$\frac{\partial z}{\partial y} = \tan\ \beta.$$

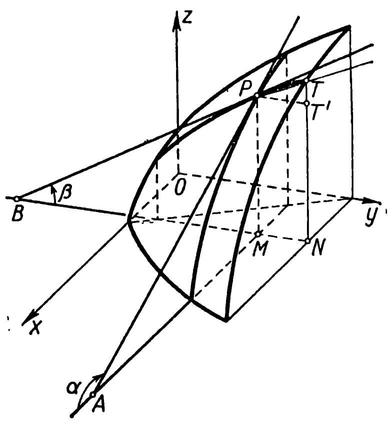


Fig. 172.

Thus, the partial derivative $\frac{\partial z}{\partial y}$ is numerically equal to the tangent of the angle of inclination of the tangent line to the curve resulting from the surface $z = f(x, y)$ being cut by the plane $x = \text{const}$.

Similarly, the partial derivative $\frac{\partial z}{\partial x}$ is numerically equal to the tangent of the angle of inclination α of the tangent line to the surface $z = f(x, y)$ cut by the plane $y = \text{const}$.

SEC. 7. TOTAL INCREMENT AND TOTAL DIFFERENTIAL

By the definition of the total increment of the function $z = f(x, y)$ we have (see Sec. 3, Ch. VIII)

$$\Delta z = f(x + \Delta x,\ y + \Delta y) - f(x, y). \tag{1}$$

Let us suppose that $f(x, y)$ has continuous partial derivatives at the point (x, y) under consideration.

Express Δz in terms of partial derivatives. To do this, add to and subtract from the right side of (1) $f(x, y + \Delta y)$:

$$\Delta z = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]. \tag{2}$$

The expression

$$f(x, y + \Delta y) - f(x, y)$$

in the second square brackets may be regarded as the difference between two values of the function of the variable y alone (the

value of x remaining constant). Applying to this difference the Lagrange theorem, we get

$$f(x,\ y+\Delta y)-f(x,\ y)=\Delta y\,\frac{\partial f(x,\ \bar{y})}{\partial y}, \tag{3}$$

where $\bar{y}$ lies between y and $y+\Delta y$.

In exactly the same way the expression in the first square brackets of (2) may be regarded as the difference between two values of the function of the variable x alone (the second argument retains the same value $y+\Delta y$). Applying the Lagrange theorem to this difference, we have

$$f(x+\Delta x,\ y+\Delta y)-f(x,\ y+\Delta y)=\Delta x\,\frac{\partial f(\bar{x},\ y+\Delta y)}{\partial x}, \tag{4}$$

where $\bar{x}$ lies between x and $x+\Delta x$.

Introducing expressions (3) and (4) into (2) we get

$$\Delta z=\Delta x\frac{\partial f(\bar{x},\ y+\Delta y)}{\partial x}+\Delta y\frac{\partial f(x,\ \bar{y})}{\partial y}. \tag{5}$$

Since it is assumed that the partial derivatives are continuous,

$$\left.\begin{aligned} \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\partial f(\bar{x},\ y+\Delta y)}{\partial x}&=\frac{\partial f(x,\ y)}{\partial x},\\ \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\partial f(x,\ \bar{y})}{\partial y}&=\frac{\partial f(x,\ y)}{\partial y}\end{aligned}\right\} \tag{6}$$

(because $\bar{x}$ and $\bar{y}$ respectively lie between x and $x+\Delta x$, and y and $y+\Delta y$, $\bar{x}$ and $\bar{y}$ approach x and y, respectively, as $\Delta x\to 0$ and $\Delta y\to 0$). Equalities (6) may be rewritten in the form

$$\left.\begin{aligned} \frac{\partial f(\bar{x},\ y+\Delta y)}{\partial x}&=\frac{\partial f(x,\ y)}{\partial x}+\gamma_1,\\ \frac{\partial f(x,\ \bar{y})}{\partial y}&=\frac{\partial f(x,\ y)}{\partial y}+\gamma_2,\end{aligned}\right\} \tag{6'}$$

where the quantities γ_1 and γ_2 approach zero as Δx and Δy approach zero (that is, as $\Delta\varrho=\sqrt{\Delta x^2+\Delta y^2}\to 0$).

By virtue of (6'), relation (5) becomes

$$\Delta z=\frac{\partial f(x,\ y)}{\partial x}\Delta x+\frac{\partial f(x,\ y)}{\partial y}\Delta y+\gamma_1\Delta x+\gamma_2\Delta y. \tag{5'}$$

The sum of the two latter terms of the right side is an infinitesimal of higher order relative to $\Delta\varrho=\sqrt{\Delta x^2+\Delta y^2}$. Indeed, the ratio $\frac{\gamma_1\Delta x}{\Delta\varrho}\to 0$ as $\Delta\varrho\to 0$, since γ_1 is an infinitesimal and $\frac{\Delta x}{\Delta\varrho}$

is bounded $\left(\left|\frac{\Delta x}{\Delta \varrho}\right| \leqslant 1\right)$. In similar fashion it is verified that $\frac{\gamma_2 \Delta y}{\Delta \varrho} \to 0$.

The sum of the first two terms is a linear expression in Δx and Δy. For $f'_x(x, y) \neq 0$ and $f'_y(x, y) \neq 0$, this expression is the **principal** part of the increment, differing from Δz by an infinitesimal of higher order relative to $\varrho = \sqrt{\Delta x^2 + \Delta y^2}$.

Definition. The function $z = f(x, y)$ [the total increment (Δz) of which at the given point (x, y) may be represented as a sum of two terms: a linear expression in Δx and Δy, and an infinitesimal of higher order relative to $\Delta \varrho$] is called *differentiable at the given point*, while the linear part of the increment is known as the *total differential* and is denoted by dz or df.

From (5') it follows that if the function $f(x, y)$ has continuous partial derivatives at a given point, it is differentiable at this point and has a total differential:

$$dz = f'_x(x, y) \Delta x + f'_y(x, y) \Delta y.$$

Equality (5') may be rewritten in the form

$$\Delta z = dz + \gamma_1 \Delta x + \gamma_2 \Delta y,$$

and, to within infinitesimals of higher order relative to $\Delta \varrho$, we may write the following **approximate** equality:

$$\Delta z \approx dz.$$

We shall call the increments of the independent variables Δx and Δy *differentials* of the independent variables x and y and we shall denote them by dx and dy respectively. Then the expression of the total differential will assume the form

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Thus, if the function $z = f(x, y)$ has continuous partial derivatives, it is differentiable at the point (x, y), and its total differential is equal to the sum of the products of the partial derivatives by the differentials of the corresponding independent variables.

Example 1. Find the total differential and the total increment of the function $z = xy$ at the point (2, 3) for $\Delta x = 0.1$, $\Delta y = 0.2$.

Solution.

$$\Delta z = (x + \Delta x)(y + \Delta y) - xy = y\,\Delta x + x\,\Delta y + \Delta x\,\Delta y,$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = y\,dx + x\,dy = y\,\Delta x + x\,\Delta y.$$

Consequently,

$$\Delta z = 3\cdot 0.1 + 2\cdot 0.2 + 0.1\cdot 0.2 = 0.72;$$
$$dz = 3\cdot 0.1 + 2\cdot 0.2 = 0.7.$$

Fig. 173 is an illustration of this example.

The foregoing reasoning and definitions are appropriately generalised to functions of any number of arguments.

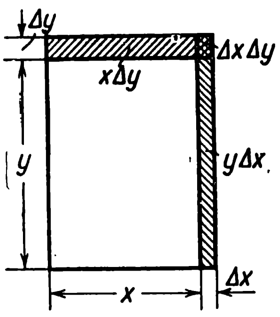


Fig. 173.

If we have a function of any number of variables

$$w = f(x, y, z, u, \ldots, t),$$

and all partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \ldots, \frac{\partial f}{\partial t}$ are continuous at the point $(x, y, z, u, \ldots, t)$, the expression

$$dw = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \ldots + \frac{\partial f}{\partial t}dt$$

is the principal part of the total increment of the function and is called the total differential. Proof of the fact that the difference $\Delta w - dw$ is an infinitesimal of higher order than $\sqrt{(\Delta x)^2 + (\Delta y)^2 + \ldots + (\Delta t)^2}$ is conducted in exactly the same way as for a function of two variables.

Example 2. Find the total differential of the function $u = e^{x^2+y^2}\sin^2 z$ of three variables x, y, z.

Solution. Noting that the partial derivatives

$$\frac{\partial u}{\partial x} = e^{x^2+y^2}2x\sin^2 z,$$

$$\frac{\partial u}{\partial y} = e^{x^2+y^2}2y\sin^2 z,$$

$$\frac{\partial u}{\partial z} = e^{x^2+y^2}2\sin z\cos z = e^{x^2+y^2}\sin 2z$$

are continuous for all values of x, y, z, we find that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = e^{x^2+y^2}(2x\sin^2 z\,dx + 2y\sin^2 z\,dy + \sin 2z\,dz).$$

SEC. 8. APPROXIMATION BY TOTAL DIFFERENTIALS

Let the function $z = f(x, y)$ be differentiable at the point (x, y). Find the total increment of this function:

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y),$$

whence

$$f(x+\Delta x,\ y+\Delta y)=f(x,\ y)+\Delta z. \tag{1}$$

We had the approximate formula

$$\Delta z\approx dz, \tag{2}$$

where

$$dz=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y. \tag{3}$$

Substituting, into formula (1), the expanded expression for dz in place of Δz, we get the approximate formula

$$f(x+\Delta x,\ y+\Delta y)\approx f(x,\ y)+\frac{\partial f(x,\ y)}{\partial x}\Delta x+\frac{\partial f(x,\ y)}{\partial y}\Delta y, \tag{4}$$

to within infinitesimals of higher order relative to Δx and Δy.

We shall now show how formulas (2) and (4) are used for approximate calculations.

Problem. Calculate the volume of material needed to make a cylindrical glass of the following dimensions (Fig. 174):

radius of interior cylinder R,
altitude of interior cylinder H,
thickness of walls and bottom of glass k.

Solution. We give two solutions of this problem: exact and approximate.

a) **Exact solution.** The desired volume v is equal to the difference between the volumes of the exterior cylinder and interior cylinder. Since the radius of the exterior cylinder is equal to $R+k$, and the altitude is $H+k$,

$$v=\pi(R+k)^2(H+k)-\pi R^2H$$

or

$$v=\pi(2RHk+R^2k+Hk^2+2Rk^2+k^3). \tag{5}$$

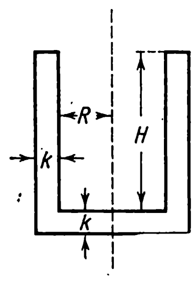


Fig. 174.

b) **Approximate solution.** Let us denote by f the volume of the interior cylinder, then $f=\pi R^2H$. This is a function of two variables R and H. If we increase R and H by k, then the function f will increase by Δf; but this will be the sought-for volume v, $v=\Delta f$.

On the basis of relation (1) we have the approximate equality

$$v\approx df$$

or

$$v\approx\frac{\partial f}{\partial R}\Delta R+\frac{\partial f}{\partial H}\Delta H.$$

But since

$$\frac{\partial f}{\partial R}=2\pi RH,\quad \frac{\partial f}{\partial H}=\pi R^2,\quad \Delta R=\Delta H=k,$$

we get

$$v \approx \pi (2RHk + R^2k). \tag{6}$$

Comparing the results of (5) and (6), we see that they differ by the quantity $\pi (Hk^2 + 2Rk^2 + k^3)$, which consists of terms of second and third order of smallness relative to k.

Let us apply these formulas to numerical examples.

Let $R = 4$ cm, $H = 20$ cm, $k = 0.1$ cm.

Applying (5), we get, exactly,

$$v = \pi (2\cdot 4\cdot 20\cdot 0.1 + 4^2\cdot 0.1 + 20\cdot 0.1^2 + 2\cdot 4\cdot 0.1^2 + 0.1^3) = 17.881\pi.$$

Applying formula (6), we have, approximately,

$$v \approx \pi (2\cdot 4\cdot 20\cdot 0.1 + 4^2\cdot 0.1) = 17.6\pi.$$

Hence, the approximate formula (6) gives an answer with an error less than 0.3π, which is $100\cdot\frac{0.3\pi}{17.881\pi}$ %, which is less than 2% of the measured quantity.

SEC. 9. ERROR APPROXIMATION BY DIFFERENTIALS

Let some quantity u be a function of the quantities $x, y, z, \ldots, t$

$$u = f(x, y, z, \ldots, t)$$

and let there be errors $\Delta x, \Delta y, \ldots, \Delta t$ made in determining the values of the quantities $x, y, z, \ldots, t$. Then the value of u computed from the inexact values of the arguments will be obtained with an error

$$\Delta u = f(x + \Delta x, y + \Delta y, \ldots, z + \Delta z, t + \Delta t) - f(x, y, z, \ldots, t).$$

Below we shall investigate the evaluation of the error Δu, provided the errors $\Delta x, \Delta y, \ldots, \Delta t$ are known.

For sufficiently small absolute values of the quantities $\Delta x, \Delta y, \ldots, \Delta t$ we can replace, approximately, the total increment by the total differential:

$$\Delta u \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \ldots + \frac{\partial f}{\partial t}\Delta t.$$

Here, the values of the partial derivatives and the errors of the arguments may be either positive or negative. Replacing them by the absolute values, we get the inequality

$$|\Delta u| \leqslant \left|\frac{\partial f}{\partial x}\right| |\Delta x| + \left|\frac{\partial f}{\partial y}\right| |\Delta y| + \ldots + \left|\frac{\partial f}{\partial t}\right| |\Delta t|. \tag{1}$$

If in terms of $|\Delta^* x|, |\Delta^* y|, \ldots, |\Delta^* u|$ we denote the *maximum absolute* errors of the corresponding quantities (the boundaries for the absolute values of the errors), it is obviously possible to take

$$|\Delta^* u| = \left|\frac{\partial f}{\partial x}\right| |\Delta^* x| \left|\frac{\partial f}{\partial y}\right| |\Delta^* y| + \ldots + \left|\frac{\partial f}{\partial t}\right| |\Delta^* t|. \tag{2}$$

Examples.

1. Let $u=x+y+z$, then

$$|\Delta^* u|=|\Delta^* x|+|\Delta^* y|+|\Delta^* z|.$$

2. Let $u=x-y$, then

$$|\Delta^* u|=|\Delta^* x|+|\Delta^* y|.$$

3. Let $u=xy$, then

$$|\Delta^* u|=|x||\Delta^* y|+|y||\Delta^* x|.$$

4. Let $u=\frac{x}{y}$, then

$$|\Delta^* u|=\left|\frac{1}{y}\right||\Delta^* x|+\left|\frac{x}{y^2}\right||\Delta^* y|=\frac{|y||\Delta^* x|+|x||\Delta^* y|}{y^2}.$$

5. The hypotenuse c and the leg a of a right triangle ABC, determined with maximum absolute errors $|\Delta^* c|=0.2$, $|\Delta^* a|=0.1$, are, respectively, $c=75$, $a=32$. Determine the angle A from the formula $\sin A=\frac{a}{c}$; and determine the maximum absolute error $|\overline{\Delta A}|$ when calculating the angle A.

Solution. $\sin A=\frac{a}{c}$, $A=\arcsin\frac{a}{c}$, hence,

$$\frac{\partial A}{\partial a}=\frac{1}{\sqrt{c^2-a^2}},\quad \frac{\partial A}{\partial c}=-\frac{a}{c\sqrt{c^2-a^2}}.$$

From formula (2) we get

$$|\overline{\Delta A}|=\frac{1}{\sqrt{(75)^2-(32)^2}}\cdot 0.1+\frac{32}{75\sqrt{(75)^2-(32)^2}}\cdot 0.2=0.00275 \text{ radian}=9'38''.$$

Thus,

$$A=\arcsin\frac{32}{75}\pm 9'38''.$$

6. In the right triangle ABC, let the leg $b=121.56$ and the angle $A=25°21'40''$, and the maximum absolute error in determining the leg b is $|\Delta^* b|=0.05$ metre, the maximum absolute error in determining the angle A is $|\Delta^* A|=12''$.

Determine the maximum absolute error in calculating the leg a from the formula $a=b\tan A$.

Solution. From formula (2) we find

$$|\Delta^* a|=|\tan A||\Delta^* b|+\frac{|b|}{\cos^2 a}|\Delta^* A|.$$

Substituting the appropriate values (and remembering that $|\Delta^* A|$ must be expressed in radians), we get

$$|\Delta^* a|=\tan 25°21'40''\cdot 0.05+\frac{121.56}{\cos^2 25°21'40''}\,\frac{12}{206\,265}=$$
$$=0.0237+0.0087=0.0324 \text{ metre}.$$

The ratio of the error Δx of some quantity to the approximate value of x of this quantity is called the *relative error* of the quantity. Let us designate it δx,

$$\delta x = \frac{\Delta x}{x}.$$

The *maximum relative error* of a quantity x is the ratio of the maximum absolute error to the absolute value of x and is denoted by $|\delta^* x|$,

$$|\delta^* x| = \frac{|\Delta^* x|}{|x|}. \tag{3}$$

To evaluate the maximum relative error of a function u, divide all numbers of (2) by $|u| = |f(x, y, z, \ldots, t)|$:

$$\frac{|\Delta^* u|}{|u|} = \left|\frac{\frac{\partial f}{\partial x}}{f}\right| |\Delta^* x| + \left|\frac{\frac{\partial f}{\partial y}}{f}\right| |\Delta^* y| + \ldots + \left|\frac{\frac{\partial f}{\partial t}}{f}\right| |\Delta^* t|, \tag{4}$$

but

$$\frac{\frac{\partial f}{\partial x}}{f} = \frac{\partial}{\partial x} \ln |f|; \qquad \frac{\frac{\partial f}{\partial y}}{f} = \frac{\partial}{\partial y} \ln |f|; \quad \ldots; \qquad \frac{\frac{\partial f}{\partial t}}{f} = \frac{\partial}{\partial t} \ln |f|.$$

For this reason, (3) may be rewritten as follows:

$$|\delta^* u| = \left|\frac{\partial}{\partial x} \ln |f|\right| |\Delta^* x| + \left|\frac{\partial}{\partial x} \ln |f|\right| |\Delta^* y| + \ldots + \left|\frac{\partial}{\partial t} \ln |f|\right| |\Delta^* t| \ldots, \tag{5}$$

or briefly,

$$|\delta^* u| = |\Delta^* \ln |f||. \tag{6}$$

From both (3) and (5) it follows that the maximum relative error of the function is equal to the maximum absolute error of the logarithm of this function.

From (6) follow the rules used in approximate calculations.

1. Let $u = xy$.

Using the results of Example 3, we get

$$|\delta^* u| = \frac{|x||\Delta^* x|}{|xy|} + \frac{|y||\Delta^* y|}{|xy|} = \frac{|\Delta^* x|}{|x|} + \frac{|\Delta^* y|}{|y|} = |\delta^* x| + |\delta^* y|;$$

that is, the maximum relative error of a product is equal to the sum of the maximum relative errors of the factors.

2. If $u = \frac{x}{y}$, then, using the results of Example 4, we have

$$|\delta^* u| = |\delta^* x| + |\delta^* y|.$$

Note. From Example 2 it follows that if $u=x-y$, then

$$|\delta^* u|=\frac{|\Delta^* x|+|\Delta^* y|}{|x-y|}.$$

If x and y are close, it may happen that $|\delta^* u|$ will be very great compared with the quantity $x-y$ being determined. This should be taken into account when performing the calculations.

Example 7. The oscillation period of a pendulum is

$$T=2\pi\sqrt{\frac{l}{g}},$$

where l is the length of the pendulum and g is the acceleration of gravity.

What relative error will be made in determining T when using this formula if we take $\pi\approx 3.14$ (accurate to 0.005), $l=1$ m (accurate to 0.01 m), $g=9.8$ m/sec² (accurate to 0.02 m/sec²).

Solution. From (6) the maximum relative error is

$$|\delta^* T|=|\Delta^* \ln T|.$$

But

$$\ln T=\ln 2+\ln \pi+\frac{1}{2}\ln l-\frac{1}{2}\ln g.$$

Calculate $|\Delta^* \ln T|$. Taking into account that $\pi\approx 3.14$, $\Delta^*\pi=0.005$, $l=1$ m, $\Delta^* l=0.01$ m, $g=9.8$ m/sec², $\Delta^* g=0.02$ m/sec², we get

$$\Delta^* \ln T=\frac{\Delta^*\pi}{\pi}+\frac{\Delta^* l}{2l}+\frac{\Delta^* g}{2g}=\frac{0.005}{3.14}+\frac{0.01}{2}+\frac{0.02}{2\cdot 9.8}=0.0076.$$

Thus, the maximum relative error is

$$\delta^* T=0.0076=0.76\%.$$

SEC. 10. THE DERIVATIVE OF A COMPOSITE FUNCTION. THE TOTAL DERIVATIVE

Let us assume that in the equation

$$z=F(u, v) \tag{1}$$

u and v are functions of the independent variables x and y:

$$u=\varphi(x, y);\quad v=\psi(x, y). \tag{2}$$

In this case, z is a composite function of the arguments x and y.

Of course, z can be expressed directly in terms of x, y; namely,

$$z=F[\varphi(x, y),\ \psi(x, y)]. \tag{3}$$

Example 1. Let

$$z=u^3v^3+u+1;\quad u=x^2+y^2;\quad v=e^{x+y}+1;$$

then

$$z=(x^2+y^2)^3(e^{x+y}+1)^3+(x^2+y^2)+1.$$

Now suppose that the functions $F(u, v)$, $\varphi(x, y)$, $\psi(x, y)$ have continuous partial derivatives with respect to all their arguments, and we pose the problem: evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ on the basis of equations (1) and (2) without having recourse to equation (3).

Increase the argument x by Δx, holding the value of y constant. Then, by virtue of equation (2), u and v will increase by $\Delta_x u$ and $\Delta_x v$.

But if u and v receive increments $\Delta_x u$ and $\Delta_x v$, then the function $z = F(u, v)$ will receive an increment Δz defined by formula (5), Sec. 7, Ch. VIII:

$$\Delta z = \frac{\partial F}{\partial u}\Delta_x u + \frac{\partial F}{\partial v}\Delta_x v + \gamma_1 \Delta_x u + \gamma_2 \Delta_x v.$$

Divide all terms of this equality by Δx:

$$\frac{\Delta z}{\Delta x} = \frac{\partial F}{\partial u}\frac{\Delta_x u}{\Delta x} + \frac{\partial F}{\partial v}\frac{\Delta_x v}{\Delta x} + \gamma_1 \frac{\Delta_x u}{\Delta x} + \gamma_2 \frac{\Delta_x v}{\Delta x}.$$

If $\Delta x \to 0$, then $\Delta_x u \to 0$ and $\Delta_x v \to 0$ (by virtue of the continuity of the functions u and v). But then γ_1 and γ_2 also approach zero. Passing to the limit as $\Delta x \to 0$, we get

$$\lim_{\Delta x \to 0}\frac{\Delta z}{\Delta x} = \frac{\partial z}{\partial x};\ \lim_{\Delta x \to 0}\frac{\Delta_x u}{\Delta x} = \frac{\partial u}{\partial x};\ \lim_{\Delta x \to 0}\frac{\Delta_x v}{\Delta x} = \frac{\partial v}{\partial x};$$

$$\lim_{\Delta x \to 0}\gamma_1 = 0;\quad \lim_{\Delta x \to 0}\gamma_2 = 0$$

and, consequently,

$$\frac{\partial z}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}. \tag{4}$$

If we increased the variable y by Δy and held x constant, then by similar reasoning we would find that

$$\frac{\partial z}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}. \tag{4'}$$

Example 2.

$$z = \ln(u^2 + v);\ u = e^{x+y^2},\ v = x^2 + y;$$

$$\frac{\partial z}{\partial u} = \frac{2u}{u^2+v};\ \frac{\partial z}{\partial v} = \frac{1}{u^2+v};$$

$$\frac{\partial u}{\partial x} = e^{x+y^2};\ \frac{\partial u}{\partial y} = 2ye^{x+y^2};\quad \frac{\partial v}{\partial x} = 2x;\ \frac{\partial v}{\partial y} = 1.$$

Using formulas (4) and (4′) we find

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2+v}e^{x+y^2} + \frac{1}{u^2+v}2x = \frac{2}{u^2+v}(ue^{x+y^2} + x),$$

$$\frac{\partial z}{\partial y} = \frac{2u}{u^2+v}2ye^{x+y^2} + \frac{1}{u^2+v} = \frac{1}{u^2+v}(2uye^{x+y^2} + 1).$$

Formulas (4) and (4′) are readily generalised to the case of a larger number of variables.

For example, if $w = F(z, u, v, s)$ is a function of four arguments z, u, v, s, and each of them depends on x and y, then formulas (4) and (4′) assume the form

$$\left.\begin{aligned}\frac{\partial w}{\partial x} = \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x},\\ \frac{\partial w}{\partial y} = \frac{\partial w}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y}.\end{aligned}\right\} \qquad (5)$$

If a function is given $z = F(x, y, u, v)$, where y, u, v in turn depend on a single independent variable (argument) x:

$$y = f(x); \quad u = \varphi(x); \quad v = \psi(x),$$

then z is actually a function only of the **one** variable x, and we may pose the question of finding the derivative $\frac{dz}{dx}$.

This derivative is calculated from the first of the formulas (5):

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}.$$

But since y, u, v are functions of x **alone**, the partial derivatives become ordinary derivatives; in addition $\frac{\partial x}{\partial x} = 1$. For this reason,

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} + \frac{\partial z}{\partial u}\frac{du}{dx} + \frac{\partial z}{\partial v}\frac{dv}{dx}. \qquad (6)$$

This formula is known as the formula for calculating the *total derivative* $\frac{dz}{dx}$ (in contrast to the **partial** derivative $\frac{\partial z}{\partial x}$).

Example 3.

$$z = x^2 + \sqrt{y}; \quad y = \sin x,$$

$$\frac{\partial z}{\partial x} = 2x; \ \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{y}}; \ \frac{dy}{dx} = \cos x.$$

Formula (6), here, yields the following result:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 2x + \frac{1}{2\sqrt{y}}\cos x = 2x + \frac{1}{2\sqrt{\sin x}}\cos x.$$

SEC. 11. THE DERIVATIVE OF A FUNCTION DEFINED IMPLICITLY

Let us begin this discussion with the implicit function of one variable.*) Let some function y of x be defined by the equation

$$F(x, y)=0.$$

We shall prove the following theorem.

Theorem. *Let a continuous function y of x be defined implicitly by the equation*

$$F(x, y)=0$$

where $F(x, y)$, $F'_x(x, y)$, $F'_y(x, y)$ are continuous functions in some domain D containing the point (x, y) whose coordinates satisfy equation (1); also, at this point $F'_y(x, y)\neq 0$. Then the function y of x has the derivative

$$y'_x=-\frac{F'_x(x, y)}{F'_y(x, y)}.$$

Proof. Let the value of the function y correspond to some value of x. Here,

$$F(x, y)=0.$$

Increase the independent variable x by Δx. Then the function y will receive an increment Δy; that is, to the value of the argument $x+\Delta x$ there corresponds the value of the function $y+\Delta y$. By virtue of equation $F(x, y)=0$ we shall have

$$F(x+\Delta x, y+\Delta y)=0.$$

Hence

$$F(x+\Delta x, y+\Delta y)-F(x, y)=0.$$

The left member of the latter equality, which is the total increment of the function of two variables by formula (5'), Sec. 7, may be rewritten as follows:

$$F(x+\Delta x, y+\Delta y)-F(x, y)=\frac{\partial F}{\partial x}\Delta x+\frac{\partial F}{\partial y}\Delta y+\gamma_1\Delta x+\gamma_2\Delta y,$$

where γ_1 and γ_2 approach zero as Δx and Δy approach zero. Since the left side of the latter expression is equal to zero, we can

*) In Sec. 11, Ch. III, we solved the problem of the differentiation of an implicit function of one variable. We considered individual cases and did not find a general formula that would yield the derivative of an implicit function; likewise we failed to clarify the conditions of the existence of this derivative.

write

$$\frac{\partial F}{\partial x}\Delta x+\frac{\partial F}{\partial y}\Delta y+\gamma_1\Delta x+\gamma_2\Delta y=0.$$

Divide the latter equality by Δx and calculate $\frac{\Delta y}{\Delta x}$:

$$\frac{\Delta y}{\Delta x}=-\frac{\frac{\partial F}{\partial x}+\gamma_1}{\frac{\partial F}{\partial y}+\gamma_2}.$$

Let Δx approach zero. Then, taking into account that γ_1 and γ_2 also approach zero and that $\frac{\partial F}{\partial y}\neq 0$, we have, in the limit,

$$y'_x=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}. \qquad (1)$$

We have proved the existence of the derivative y'_x of a function defined implicitly, and we have found the formula for calculating it.

Example 1. The equation

$$x^2+y^2-1=0$$

defines y as an implicit function of x. Here,

$$F(x, y)=x^2+y^2-1,\ \frac{\partial F}{\partial x}=2x;\ \frac{\partial F}{\partial y}=2y.$$

Consequently, from (1),

$$\frac{dy}{dx}=-\frac{2x}{2y}=-\frac{x}{y}.$$

It will be noted that the given equation defines two different functions [since to every value of x in the interval $(-1, 1)$ there correspond two values of y]; however, the value that we found of y'_x holds for both functions.

Example 2. An equation is given that connects x and y:

$$e^y-e^x+xy=0.$$

Here, $F(x, y)=e^y-e^x+xy$,

$$\frac{\partial F}{\partial x}=-e^x+y;\ \frac{\partial F}{\partial y}=e^y+x.$$

Consequently, from formula (1) we get

$$\frac{dy}{dx}=-\frac{-e^x+y}{e^y+x}=\frac{e^x-y}{e^y+x}.$$

Let us now consider an equation of the form

$$F(x, y, z)=0. \tag{2}$$

If to each number pair x and y in some domain there correspond one or several values of z that satisfy equation (2), then this equation implicitly defines one or several single-valued functions z of x and y.

For instance, the equation

$$x^2+y^2+z^2-R^2=0$$

implicitly defines two continuous functions z of x and y, which functions may be expressed explicitly by solving the equation for z; in this case we have

$$z=\sqrt{R^2-x^2-y^2} \text{ and } z=-\sqrt{R^2-x^2-y^2}.$$

Let us find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of the implicit function z of x and y defined by equation (2).

When we seek $\frac{\partial z}{\partial x}$, we consider y fixed. And so formula (1) is applicable, provided x is considered the independent variable and z the function. Thus,

$$z'_x=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}.$$

In the same way we find

$$z'_y=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

Similarly, we determine the implicit functions of any number of variables and find their partial derivatives.

Example 3.

$$x^2+y^2+z^2-R^2=0,$$

$$\frac{\partial z}{\partial x}=-\frac{2x}{2z}=-\frac{x}{z}\,;\ \frac{\partial z}{\partial y}=-\frac{y}{z}.$$

Differentiating this function as an explicit function (after solving the equation for z), we would obtain the very same result.

Example 4.

$$e^z+x^2y+z+5=0.$$

Here, $F(x, y, z)=e^z+x^2y+z+5$,

$$\frac{\partial F}{\partial x}=2xy;\ \frac{\partial F}{\partial y}=x^2;\ \frac{\partial F}{\partial z}=e^z+1;$$

$$\frac{\partial z}{\partial x}=-\frac{2xy}{e^z+1};\ \frac{\partial z}{\partial y}=-\frac{x^2}{e^z+1}.$$

SEC. 12. PARTIAL DERIVATIVES OF DIFFERENT ORDERS

Let there be given a function of two variables:

$$z=f(x, y).$$

The partial derivatives $\frac{\partial z}{\partial x}=f'_x(x, y)$ and $\frac{\partial z}{\partial y}=f'_y(x, y)$ are, generally speaking, functions of the variables x and y. And so from them we can again find partial derivatives. Thus, there are **four** partial derivatives of the second order of a function of two variables, since each of the functions $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ may be differentiated both with respect to x and with respect to y.

The second partial derivatives are denoted as follows:

$\frac{\partial^2 z}{\partial x^2}=f''_{xx}(x, y)$; here f is differentiated twice successively with respect to x;

$\frac{\partial^2 z}{\partial x \partial y}=f''_{xy}(x, y)$; here f is first differentiated with respect to x and then the result is differentiated with respect to y;

$\frac{\partial^2 z}{\partial y \partial x}=f''_{yx}(x, y)$; here f is differentiated first with respect to y and then the result is differentiated with respect to x;

$\frac{\partial^2 z}{\partial y^2}=f''_{yy}(x, y)$; here the function f is differentiated twice successively with respect to y.

Derivatives of the second order may again be differentiated both with respect to x and y. We then get partial derivatives of the third order. Obviously, there will be eight of them:

$$\frac{\partial^3 z}{\partial x^3};\ \frac{\partial^3 z}{\partial x^2 \partial y};\ \frac{\partial^3 z}{\partial x \partial y \partial x};\ \frac{\partial^3 z}{\partial x \partial y^2};\ \frac{\partial^3 z}{\partial y \partial x^2};\ \frac{\partial^3 z}{\partial y \partial x \partial y};\ \frac{\partial^3 z}{\partial y^2 \partial x};\ \frac{\partial^3 z}{\partial y^3}.$$

Generally speaking, *a partial derivative of the nth order* is the first derivative of the derivative of the $(n-1)$st order. For example, $\frac{\partial^n z}{\partial x^p \partial y^{n-p}}$ is a derivative of the nth order; here the function

z was first differentiated p times with respect to x, and then $n-p$ times with respect to y.

For a function of any number of variables, the higher-order partial derivatives are determined in similar fashion.

Example 1. Compute the second-order partial derivatives of the function

$$f(x, y) = x^2y + y^3.$$

Solution. We find successively

$$\frac{\partial f}{\partial x} = 2xy; \quad \frac{\partial f}{\partial y} = x^2 + 3y^2;$$

$$\frac{\partial^2 f}{\partial x^2} = 2y; \quad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial(2xy)}{\partial y} = 2x; \quad \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial(x^2+3y^2)}{\partial x} = 2x; \quad \frac{\partial^2 f}{\partial y^2} = 6y.$$

Example 2. Compute $\frac{\partial^3 z}{\partial x^2\,\partial y}$ and $\frac{\partial^3 z}{\partial y\,\partial x^2}$ if $z = y^2e^x + x^2y^3 + 1$.

Solution. We successively find

$$\frac{\partial z}{\partial x} = y^2e^x + 2xy^3; \quad \frac{\partial^2 z}{\partial x^2} = y^2e^x + 2y^3; \quad \frac{\partial^3 z}{\partial x^2\,\partial y} = 2ye^x + 6y^2,$$

$$\frac{\partial z}{\partial y} = 2ye^x + 3x^2y^2; \quad \frac{\partial^2 z}{\partial y\,\partial x} = 2ye^x + 6xy^2; \quad \frac{\partial^3 z}{\partial y\,\partial x^2} = 2ye^x + 6y^2.$$

Example 3. Compute $\frac{\partial^4 u}{\partial x^2\,\partial y\,\partial z}$ if $u = z^2e^{x+y^2}$.

Solution.

$$\frac{\partial u}{\partial x} = z^2e^{x+y^2}; \quad \frac{\partial^2 u}{\partial x^2} = z^2e^{x+y^2}, \quad \frac{\partial^3 u}{\partial x^2\,\partial y} = 2yz^2e^{x+y^2}, \quad \frac{\partial^4 u}{\partial x^2\,\partial y\,\partial z} = 4yze^{x+y^2}.$$

The natural question that arises is whether the result of differentiating a function of several variables depends on the order of differentiation with respect to the different variables; in other words, will, for instance, the following derivatives be identically equal:

$$\frac{\partial^2 f}{\partial x\,\partial y} \text{ and } \frac{\partial^2 f}{\partial y\,\partial x}$$

or

$$\frac{\partial^3 f(x, y, t)}{\partial x\,\partial y\,\partial t} \text{ and } \frac{\partial^3 f(x, y, t)}{\partial t\,\partial x\,\partial y},$$

and so forth. It turns out that the following theorem is true.

Theorem. *If the function $z = f(x, y)$ and its partial derivatives f'_x, f'_y, f''_{xy} and f''_{yx} are defined and continuous at a point $M(x, y)$ and in some neighbourhood of it, then at this point*

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} \quad (f''_{xy} = f''_{yx}).$$

Proof. Consider the expression

$$A=[f(x+\Delta x,\ y+\Delta y)-f(x+\Delta x,\ y)]-[f(x,\ y+\Delta y)-f(x,\ y)].$$

If we introduce an auxiliary function $\varphi(x)$ defined by the equality

$$\varphi(x)=f(x,\ y+\Delta y)-f(x,\ y),$$

then A may be written in the form

$$A=\varphi(x+\Delta x)-\varphi(x).$$

Since it is assumed that f'_x is defined in the neighbourhood of the point (x, y), it follows that $\varphi(x)$ is differentiable on the interval $[x,\ x+\Delta x]$; but then, applying the Lagrange theorem, we get

$$A=\Delta x\varphi'(\bar{x})$$

where $\bar{x}$ lies between x and $x+\Delta x$.

But

$$\varphi'(\bar{x})=f'_x(\bar{x},\ y+\Delta y)-f'_x(\bar{x},\ y).$$

Since f''_{xy} is defined in the neighbourhood of the point (x, y), f'_x is differentiable on the interval $[y, y+\Delta y]$; and so by applying once again the Lagrange theorem (with respect to the variable y) to the difference obtained we have

$$f'_x(\bar{x},\ y+\Delta y)-f'_x(\bar{x},\ y)=\Delta y f''_{xy}(\bar{x},\ \bar{y}),$$

where $\bar{y}$ lies between y and $y+\Delta y$.

Consequently, the original expression of A is

$$A=\Delta x\Delta y f''_{xy}(\bar{x},\ \bar{y}). \tag{1}$$

Changing the places of the middle terms we get

$$A=[f(x+\Delta x,\ y+\Delta y)-f(x,\ y+\Delta y)]-[f(x+\Delta x,\ y)-f(x,\ y)].$$

Introducing the auxiliary function

$$\psi(y)=f(x+\Delta x,\ y)-f(x,\ y),$$

we have

$$A=\psi(y+\Delta y)-\psi(y).$$

Again applying the Lagrange theorem we get

$$A=\Delta y\psi'(\bar{\bar{y}}),$$

where $\bar{\bar{y}}$ lies between y and $y+\Delta y$.

But

$$\psi'(\bar{\bar{y}})=f'_y(x+\Delta x,\ \bar{\bar{y}})-f'_y(x,\ \bar{\bar{y}}).$$

Again applying the Lagrange theorem we get

$$f'_y(x+\Delta x,\ \bar{\bar{y}})-f'_y(x,\ \bar{\bar{y}})=\Delta x f''_{yx}(\bar{\bar{x}},\ \bar{\bar{y}}),$$

where $\bar{\bar{x}}$ lies between x and $x+\Delta x$.

Thus, the original expression of A may be written in the form

$$A=\Delta y\,\Delta x f''_{yx}(\bar{\bar{x}},\ \bar{\bar{y}}). \tag{2}$$

The left members of (1) and (2) are equal to A, therefore the right ones are equal too; that is,

$$\Delta x\,\Delta y f''_{xy}(\bar{x},\ \bar{y})=\Delta y\,\Delta x f''_{yx}(\bar{\bar{x}},\ \bar{\bar{y}}),$$

whence

$$f''_{xy}(\bar{x},\ \bar{y})=f''_{yx}(\bar{\bar{x}},\ \bar{\bar{y}}).$$

Passing to the limit in this equality as $\Delta x \to 0$ and $\Delta y \to 0$, we get

$$\lim_{\substack{\Delta x \to 0\\ \Delta y \to 0}} f''_{xy}(\bar{x},\ \bar{y})=\lim_{\substack{\Delta x \to 0\\ \Delta y \to 0}} f''_{yx}(\bar{\bar{x}},\ \bar{\bar{y}}).$$

Since the derivatives f''_{xy} and f''_{yx} are continuous at the point $(x,\ y)$, we have $\lim\limits_{\substack{\Delta x \to 0\\ \Delta y \to 0}} f''_{xy}(\bar{x},\ \bar{y})=f''_{xy}(x,\ y)$ and $\lim\limits_{\substack{\Delta x \to 0\\ \Delta y \to 0}} f''_{yx}(\bar{\bar{x}},\ \bar{\bar{y}})=f''_{yx}(x,\ y)$. And finally we get

$$f''_{xy}(x,\ y)=f''_{yx}(x,\ y),$$

as required.

A corollary of this theorem is that if the partial derivatives $\frac{\partial^n f}{\partial x^k\,\partial y^{n-k}}$ and $\frac{\partial^n f}{\partial y^{n-k}\,\partial x^k}$ are continuous, then

$$\frac{\partial^n f}{\partial x^k\,\partial y^{n-k}}=\frac{\partial^n f}{\partial y^{n-k}\,\partial x^k}.$$

A similar theorem holds also for a function of any number of variables.

Example 4. Find $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}$ and $\frac{\partial^3 u}{\partial y\,\partial z\,\partial x}$ if $u=e^{xy}\sin z$.

Solution.

$$\frac{\partial u}{\partial x}=ye^{xy}\sin z;\quad \frac{\partial^2 u}{\partial x\,\partial y}=e^{xy}\sin z+xye^{xy}\sin z=e^{xy}(1+xy)\sin z;$$

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=e^{xy}(1+xy)\cos z;\quad \frac{\partial u}{\partial y}=xe^{xy}\sin z;\quad \frac{\partial^2 u}{\partial x\,\partial z}=xe^{xy}\cos z;$$

$$\frac{\partial^3 u}{\partial y\,\partial z\,\partial x}=e^{xy}\cos z+xye^{xy}\cos z=e^{xy}(1+xy)\cos z.$$

Hence,

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=\frac{\partial^3 u}{\partial y\,\partial z\,\partial x}$$

(also see Examples 1 and 2 of this section).

SEC. 13. LEVEL SURFACES

In a space (x, y, z) let there be a region D in which the function

$$u=u(x, y, z) \tag{1}$$

is defined. In this case we say that a *scalar field* is defined in the region D. If, for example, $u(x, y, z)$ denotes the temperature at the point $M(x, y, z)$, then we say that a scalar field of temperatures is defined; if D is filled with a liquid or gas and $u(x, y, z)$ denotes pressure, we have a scalar field of pressures, etc.

Consider the points of a region D in which the function $u(x, y, z)$ has a fixed value c:

$$u(x, y, z)=c. \tag{2}$$

The totality of these points forms a certain surface. If a different value of c is taken, we obtain a different surface. These surfaces are called *level surfaces*.

Example 1. Let there be given a scalar field

$$u(x, y, z)=\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}.$$

Here, the level surfaces are

$$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=c$$

or ellipsoids with semi-axes $2\sqrt{c}$, $3\sqrt{c}$, $4\sqrt{c}$.

If the function u is a function of two variables x and y,

$$u=u(x, y),$$

then the level "surfaces" are lines on the xy-plane:

$$u(x, y)=c \tag{2'}$$

which are called *level lines*.

If we plot values of u on the z-axis:

$$z = u(x, y),$$

the level lines in the xy-plane will be projections of lines obtained at the intersection of the surface $z = u(x, y)$ with the planes $z = c$ (Fig. 175). Knowing the level lines, it is easy to study the character of the surface $z = u(x, y)$.

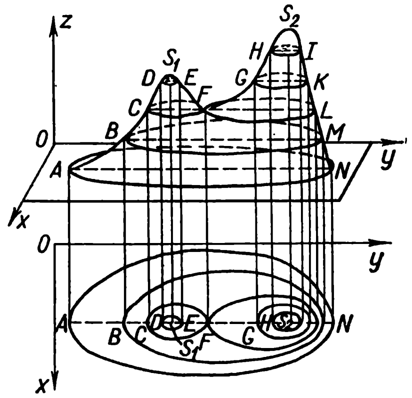


Fig. 175.

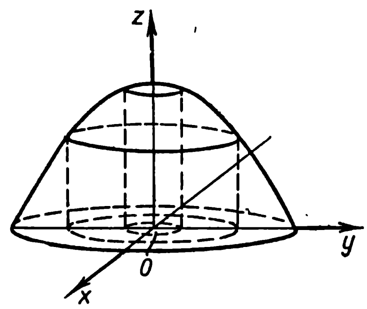


Fig. 176.

Example 2. Determine the level lines of the function $z = 1 - x^2 - y^2$. They are lines with equations $1 - x^2 - y^2 = c$, which are (Fig. 176) circles with radius $\sqrt{1-c}$. In particular, when $c = 0$ we get the circle $x^2 + y^2 = 1$.

SEC. 14. DIRECTIONAL DERIVATIVE

In a region D, consider the function $u = u(x, y, z)$ and the point $M(x, y, z)$. Draw from M a vector S whose direction cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$ (Fig. 177). On the vector S, at a distance

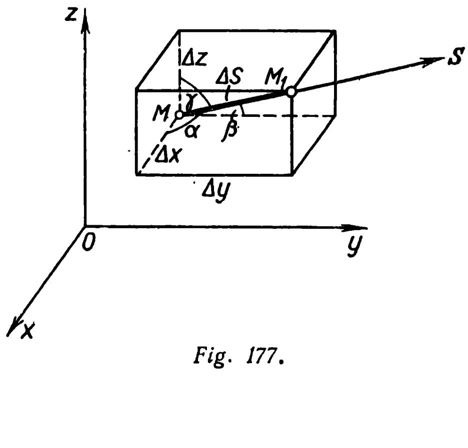


Fig. 177.

Δs from its origin, let us consider a point $M_1(x+\Delta x, y+\Delta y, z+\Delta z)$. Thus,

$$\Delta s=\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}.$$

We shall assume that the function $u(x, y, z)$ is continuous and has continuous derivatives with respect to their arguments in the region D.

As in Sec. 7, we will represent the total increment of the function as follows:

$$\Delta u=\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y+\frac{\partial u}{\partial z}\Delta z+\varepsilon_1\Delta x+\varepsilon_2\Delta y+\varepsilon_3\Delta z, \quad (1)$$

where ε_1, ε_2 and ε_3 approach zero as $\Delta s\to 0$. Divide all terms of (1) by Δs:

$$\frac{\Delta u}{\Delta s}=\frac{\partial u}{\partial x}\frac{\Delta x}{\Delta s}+\frac{\partial u}{\partial y}\frac{\Delta y}{\Delta s}+\frac{\partial u}{\partial z}\frac{\Delta z}{\Delta s}+\varepsilon_1\frac{\Delta x}{\Delta s}+\varepsilon_2\frac{\Delta y}{\Delta s}+\varepsilon_3\frac{\Delta z}{\Delta s}. \quad (2)$$

It is obvious that

$$\frac{\Delta x}{\Delta s}=\cos\alpha, \quad \frac{\Delta y}{\Delta s}=\cos\beta, \quad \frac{\Delta z}{\Delta s}=\cos\gamma.$$

Consequently, equation (2) may be rewritten as

$$\frac{\Delta u}{\Delta s}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma+\varepsilon_1\cos\alpha+\varepsilon_2\cos\beta+\varepsilon_3\cos\gamma. \quad (3)$$

The limit of the ratio $\frac{\Delta u}{\Delta s}$ as $\Delta s\to 0$ is called the *derivative of the function $u=u(x, y, z)$ at the point (x, y, z) along the direction of the vector S* and is denoted by $\frac{\partial u}{\partial s}$; thus

$$\lim_{\Delta s\to 0}\frac{\Delta u}{\Delta s}=\frac{\partial u}{\partial s}. \quad (4)$$

So, passing to the limit in (3), we get

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma. \quad (5)$$

From formula (5) it follows that if we know the partial derivatives it is easy to find the derivative along any direction S. The partial derivatives themselves are a particular case of a directional derivative.

For instance, when $\alpha=0$, $\beta=\frac{\pi}{2}$, $\gamma=\frac{\pi}{2}$, we get

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x}\cos 0+\frac{\partial u}{\partial y}\cos\frac{\pi}{2}+\frac{\partial u}{\partial z}\cos\frac{\pi}{2}=\frac{\partial u}{\partial x}.$$

Example. Given a function

$$u = x^2 + y^2 + z^2.$$

Find the derivative $\frac{\partial u}{\partial s}$ at the point $M(1, 1, 1)$: a) along the direction of the vector $S_1 = 2i + j + 3k$; b) along the direction of the vector $S_2 = i + j + k$.

Solution. a) Find the direction cosines of the vector S_1:

$$\cos\alpha = \frac{2}{\sqrt{4+1+9}} = \frac{2}{\sqrt{14}},$$

$$\cos\beta = \frac{1}{\sqrt{14}}, \quad \cos\gamma = \frac{3}{\sqrt{14}}.$$

Hence,

$$\frac{\partial u}{\partial s_1} = \frac{\partial u}{\partial x}\frac{2}{\sqrt{14}} + \frac{\partial u}{\partial y}\frac{1}{\sqrt{14}} + \frac{\partial u}{\partial z}\frac{3}{\sqrt{14}}.$$

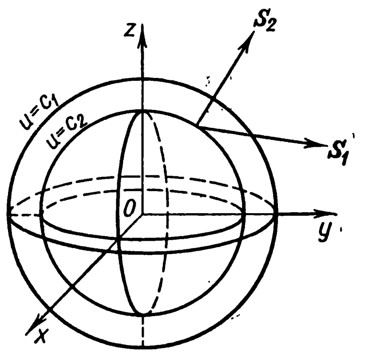


Fig. 178.

The partial derivatives at the point $M(1, 1, 1)$ are

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial u}{\partial z} = 2z;$$

$$\left(\frac{\partial u}{\partial x}\right)_M = 2, \quad \left(\frac{\partial u}{\partial y}\right)_M = 2, \quad \left(\frac{\partial u}{\partial z}\right)_M = 2.$$

Thus,

$$\frac{\partial u}{\partial s_1} = 2\cdot\frac{2}{\sqrt{14}} + 2\cdot\frac{1}{\sqrt{14}} + 2\cdot\frac{3}{\sqrt{14}} = \frac{12}{\sqrt{14}}.$$

b) Find the direction cosines of the vector S_2:

$$\cos\alpha = \frac{1}{\sqrt{3}}, \quad \cos\beta = \frac{1}{\sqrt{3}}, \quad \cos\gamma = \frac{1}{\sqrt{3}}.$$

Hence,

$$\frac{\partial u}{\partial s_2} = 2\cdot\frac{1}{\sqrt{3}} + 2\cdot\frac{1}{\sqrt{3}} + 2\cdot\frac{1}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}.$$

We note here (and it will be needed later on) that $2\sqrt{3} > \frac{12}{\sqrt{14}}$ (Fig. 178).

SEC. 15. GRADIENT

At every point of the region D, in which the function $u = u(x, y, z)$ is given, we determine the vector whose projections on the coordinate axes are the values of the partial derivatives

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ of this function at the appropriate point:

$$\operatorname{grad} u = \frac{\partial u}{\partial x} \boldsymbol{i} + \frac{\partial u}{\partial y} \boldsymbol{j} + \frac{\partial u}{\partial z} \boldsymbol{k}. \tag{1}$$

This vector is called the *gradient* of the function $u(x, y, z)$. We say that a *vector field of gradients* is defined in D. Let us now prove the following theorem which establishes a relationship between the gradient and the directional derivative.

Theorem. *Given a scalar field $u = u(x, y, z)$; in this field, let there be defined a field of gradients*

$$\operatorname{grad} u = \frac{\partial u}{\partial x} \boldsymbol{i} + \frac{\partial u}{\partial y} \boldsymbol{j} + \frac{\partial u}{\partial z} \boldsymbol{k}.$$

The derivative $\frac{\partial u}{\partial s}$ along the direction of some vector $\boldsymbol{S}$ is equal to the projection of the vector grad u *on the vector $\boldsymbol{S}$.*

Proof. Consider the unit vector $\boldsymbol{S}^0$, which corresponds to the vector $\boldsymbol{S}$:

$$\boldsymbol{S}^0 = \boldsymbol{i} \cos \alpha + \boldsymbol{j} \cos \beta + \boldsymbol{k} \cos \gamma.$$

Find the scalar product of the vectors grad u and $\boldsymbol{S}^0$:

$$\operatorname{grad} u \cdot \boldsymbol{S}^0 = \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma. \tag{2}$$

The expression on the right is a derivative of the function $u(x, y, z)$ along the vector $\boldsymbol{S}$. Hence, we can write

$$\operatorname{grad} u \cdot \boldsymbol{S}^0 = \frac{\partial u}{\partial s}.$$

If we designate the angle between the vectors grad u and $\boldsymbol{S}^0$ by φ (Fig. 179), we can write

$$|\operatorname{grad} u| \cos \varphi = \frac{\partial u}{\partial s} \tag{3}$$

or

$$\text{projection } \boldsymbol{S}^0 \operatorname{grad} u = \frac{\partial u}{\partial s} \tag{4}$$

and the theorem is proved.

This theorem gives us a clear picture of the relationship between the gradient and the derivative, at a given point, along any direction. Referring to Fig. 180, construct the vector grad u at some point $M(x, y, z)$. Construct a sphere for which grad u is the diameter. Draw the vector $\boldsymbol{S}$ from M. Denote by P the point of intersection of $\boldsymbol{S}$ with the surface of the sphere. It is

then obvious that $MP = |\operatorname{grad} u| \cos\varphi$, if φ is the angle between the directions of the gradient and the segment MP $\left(\text{here, } \varphi < \frac{\pi}{2}\right)$, or $MP = \frac{\partial u}{\partial s}$. Obviously, when the direction of the vector $\boldsymbol{S}$ is

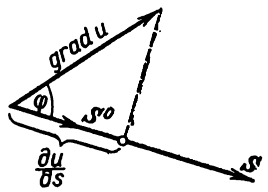


Fig. 179.

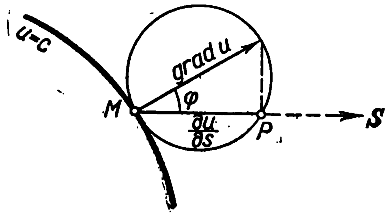


Fig. 180.

reversed the derivative changes sign, while its absolute value remains unchanged.

Let us establish certain properties of a gradient.

1) *The derivative at a given point along the direction of the vector* $\boldsymbol{S}$ *has a maximum if the direction of* $\boldsymbol{S}$ *coincides with that of the gradient; this maximal value of the derivative is equal to* $|\operatorname{grad} u|$.

The truth of this assertion follows directly from (3): $\frac{\partial u}{\partial s}$ will be a maximum when $\varphi = 0$, and in this case

$$\frac{\partial u}{\partial s} = |\operatorname{grad} u|.$$

2) *The derivative along a vector that is tangent to a level surface is zero.*

This assertion follows from formula (3). Indeed, in this case,

$$\varphi = \frac{\pi}{2}, \quad \cos\varphi = 0 \text{ and } \frac{\partial u}{\partial s} = |\operatorname{grad} u| \cos\varphi = 0.$$

Example 1. Given the function

$$u = x^2 + y^2 + z^2.$$

a) Determine the gradient at the point $M(1, 1, 1)$. The expression of the gradient of this function at an arbitrary point will be

$$\operatorname{grad} u = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}.$$

Hence,

$$(\operatorname{grad} u)_M = 2\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}, \quad |\operatorname{grad} u|_M = 2\sqrt{3}.$$

b) Determine the derivative of the function u at the point $M(1, 1, 1)$ along the direction of the gradient. The direction cosines of the gradient

will be

$$\cos\alpha=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{1}{\sqrt{3}},\ \cos\beta=\frac{1}{\sqrt{3}},\ \cos\gamma=\frac{1}{\sqrt{3}}.$$

And so

$$\frac{\partial u}{\partial s}=2\frac{1}{\sqrt{3}}+2\frac{1}{\sqrt{3}}+2\frac{1}{\sqrt{3}}=2\sqrt{3},$$

or

$$\frac{\partial u}{\partial s}=|\operatorname{grad} u|.$$

Note. If the function $u=u(x, y)$ is a function of two variables, then the vector

$$\operatorname{grad} u=\frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial u}{\partial y}\mathbf{j}$$

lies in the xy-plane. We shall prove that grad u is *perpendicular to the level line* $u(x, y)=c$ lying in the xy-plane and passing through the corresponding point. Indeed, the slope k_1 of the tangent to the level line $u(x, y)=c$ will equal $k_1=-\frac{u'_x}{u'_y}$ (see Sec. 11). The slope k_2 of the gradient is $k_2=\frac{u'_y}{u'_x}$. Obviously, $k_1k_2=-1$. This proves our assertion (Fig. 181). A similar property of the gradient of a function of three variables will be established in Sec. 6 of Chapter IX.

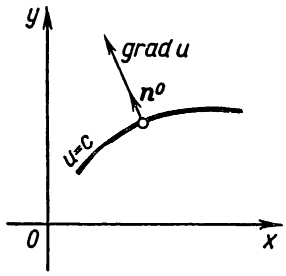


Fig. 181.

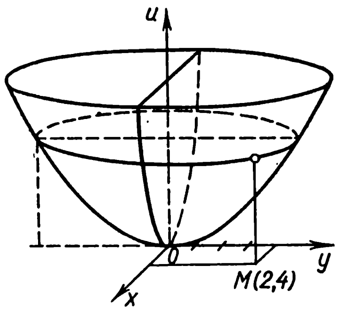


Fig. 182.

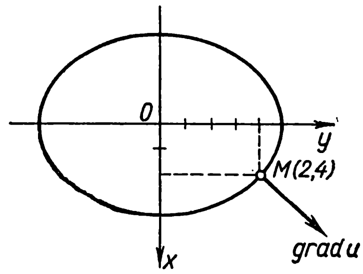


Fig. 183.

Example 2. Determine the gradient of the function $u=\frac{x^2}{2}+\frac{y^2}{3}$ (Fig. 182) at the point $M(2, 4)$.

Solution. Here

$$\frac{\partial u}{\partial x}=x\Big|_M=2,\quad \frac{\partial u}{\partial y}=\frac{2}{3}y\Big|_M=\frac{8}{3}.$$

Hence

$$\operatorname{grad} u=2i+\frac{8}{3}j.$$

The equation of the level line (Fig. 183) passing through the given point will be

$$\frac{x^2}{2}+\frac{y^2}{3}=\frac{22}{3}.$$

SEC. 16. TAYLOR'S FORMULA FOR A FUNCTION OF TWO VARIABLES.

Let there be a function of two variables

$$z=f(x,\ y)$$

which is continuous, together with all its partial derivatives up to the $(n+1)$st order inclusive, in some neighbourhood of the point $M(a,\ b)$. Then, like the case of one variable (see Sec. 6, Ch. IV), represent the function of two variables in the form of a sum of an nth degree polynomial in powers of $(x-a)$ and $(y-b)$ and some remainder. It will be shown below that for the case of $n=2$ this formula has the form

$$f(x,\ y)=A_0+D(x-a)+E(y-b)+$$
$$+\frac{1}{2!}[A(x-a)^2+2B(x-a)(y-b)+C(y-b)^2]+R_2, \qquad (1)$$

where the coefficients A_0, D, E, A, B, C are independent of x and y, while R_2 is the remainder, the structure of which is similar to the structure of the remainder in the Taylor formula for a function of one variable.

Let us apply the Taylor formula for a function $f(x,\ y)$ of one variable y considering x constant (we shall confine ourselves to second-order terms):

$$f(x,\ y)=f(x,\ b)+\frac{y-b}{1}f'_y(x,\ b)+$$
$$+\frac{(y-b)^2}{1\cdot 2}f''_{yy}(x,\ b)+\frac{(y-b)^3}{1\cdot 2\cdot 3}f'''_{yyy}(x,\ \eta_1), \qquad (2)$$

where $\eta_1=b+\theta_1(y-b)$, $0<\theta_1<1$. Expand the functions $f(x,\ b)$, $f'_y(x,\ b)$, $f''_{yy}(x,\ b)$ in a Taylor's series in powers of $(x-a)$, confining yourself to mixed derivatives up to the third order inclusive:

$$f(x,\ b)=$$

$$=f(a,\ b)+\frac{x-a}{1}f'_x(a,\ b)+\frac{(x-a)^2}{1\cdot 2}f''_{xx}(a,\ b)+\frac{x-a^3}{1\cdot 2\cdot 3}f'''_{xxx}(\xi_1,\ b), \quad (3)$$

where

$$\xi_1=x+\theta_2(x-a),\ 0<\theta_2<1;$$

$$f'_y(x,\ b)=f'_y(a,\ b)+\frac{x-a}{1}f''_{yx}(a,\ b)+\frac{(x-a)^2}{1\cdot 2}f'''_{yxx}(\xi_2,\ b), \quad (4)$$

where

$$\xi_2=x+\theta_3(x-a),\ 0<\theta_3<1;$$

$$f''_{yy}(x,b)=f''_{yy}(a,b)+\frac{x-a}{1}f'''_{yyx}(\xi_3,\ b), \quad (5)$$

where

$$\xi_3=x+\theta_4(x-a),\ 0<\theta_4<1.$$

Substituting expressions (3), (4) and (5) into formula (2), we get

$$f(x,y)=f(a,\ b)+\frac{x-a}{1}f'_x(a,\ b)+\frac{(x-a)^2}{1\cdot 2}f''_{xx}(a,\ b)+$$
$$+\frac{(x-a)^3}{1\cdot 2\cdot 3}f'''_{xxx}(\xi_1,\ b)+\frac{y-b}{1}\left[f'_y(a,\ b)+\frac{x-a}{1}f''_{yx}(a,\ b)+\right.$$
$$\left.+\frac{(x-a)^2}{1\cdot 2}f'''_{yxx}(\xi_2,\ b)\right]+\frac{(y-b)^2}{1\cdot 2}\left[f''_{yy}(a,\ b)+\frac{x-a}{1}f'''_{yyx}(\xi_3,\ b)\right]+$$
$$+\frac{(y-b)^3}{1\cdot 2\cdot 3}f'''_{yyy}(x,\ \eta).$$

Arranging the numbers as indicated in formula (1), we get

$$f(x,y)=f(a,\ b)+(x-a)f'_x(a,\ b)+(y-b)f'_y(a,\ b)+$$
$$+\frac{1}{2!}[(x-a)^2f''_{xx}(a,\ b)+2(x-a)(x-b)f''_{xy}(a,\ b)+$$
$$+(y-b)^2f''_{yy}(a,\ b)]+\frac{1}{3!}[(x-a)^3f'''_{xxx}(\xi_1,\ b)+$$
$$+3(x-a)^2(x-b)f'''_{xxy}(\xi_2,\ b)+3(x-a)(y-b)^2f'''_{xyy}(\xi_3 b)+$$
$$+(y-b)^3f'''_{yyy}(a,\ \eta)]. \quad (6)$$

This is Taylor's formula for $n=2$. The expression

$$R_2=\frac{1}{3!}[(x-a)^3f'''_{xxx}(\xi_1,\ b)+3(x-a)^2(y-b)f'''_{xxy}(\xi_2,\ b)+$$
$$+3(x-a)(y-b)^2f'''_{xyy}(\xi_3,\ b)+(y-b)^3f'''_{yyy}(a,\ \eta)]$$

is called the remainder. Further, let us denote $x-a=\Delta x$, $y-b=\Delta y$, $\Delta\varrho=\sqrt{(\Delta x)^2+(\Delta y)^2}$. Transform R_2:

$$R_2=\frac{1}{3!}\left[\frac{\Delta x^3}{\Delta\varrho^3}f'''_{xxx}(\xi_1, b)+3\frac{\Delta x^2\Delta y}{\Delta\varrho^3}f'''_{xxy}(\xi_2, b)+\right.$$

$$\left.+3\frac{\Delta x\Delta y^2}{\Delta\varrho^3}f'''_{xyy}(\xi_3, b)+\frac{\Delta y^3}{\Delta\varrho^3}f'''_{yyy}(a, \eta)\right]\Delta\varrho^3.$$

Since $|\Delta x|<\Delta\varrho$, $|\Delta y|<\Delta\varrho$ and the third derivatives are bounded (this is given), the coefficient of $\Delta\varrho^3$ is bounded in the domain under consideration; let us denote it by α_0.

Then we can write

$$R_2=\alpha_0\Delta\varrho^3.$$

In this notation, Taylor's formula (6) will then, for the case $n=2$, take the form

$$f(x, y)=f(a, b)+\Delta x f'_x(a, b)+\Delta y f'_y(a, b)+$$

$$+\frac{1}{2!}[\Delta x^2 f''_{xx}(a, b)+2\Delta x\Delta y f''_{xy}(a, b)+\Delta y^2 f''_{yy}(a, b)]+\alpha_0\Delta\varrho^3. \quad (6')$$

Taylor's formula is of a similar form for arbitrary n.

SEC. 17. MAXIMUM AND MINIMUM OF A FUNCTION OF SEVERAL VARIABLES

Definition 1. We say that the function $z=f(x, y)$ has a *maximum* at the point $M_0(x_0, y_0)$ (that is, when $x=x_0$ and $y=y_0$) if

$$f(x_0, y_0)>f(x, y)$$

for all points (x, y) sufficiently close to the point (x_0, y_0) and different from it.

Definition 2. Quite analogously we say that a function $z=f(x, y)$ has a *minimum* at the point $M_0(x_0, y_0)$ if

$$f(x_0, y_0)<f(x, y)$$

for all points (x, y) sufficiently close to the point (x_0, y_0) and different from it.

The maximum and minimum of a function are called *extrema* of the function; we say that a function has an extremum at a given point if this function has a maximum or minimum at the given point.

Example 1. The function

$$z=(x-1)^2+(y-2)^2-1$$

attains a minimum at $x=1$, $y=2$; i. e., at the point (1, 2). Indeed, $f(1, 2)=-1$,

and since $(x-1)^2$ and $(y-2)^2$ are always positive for $x \neq 1$, $y \neq 2$, so

$$(x-1)^2+(y-2)^2-1>-1,$$

that is,

$$f(x, y) > f(1, 2).$$

The geometric analogy of this case is shown in Fig. 184.

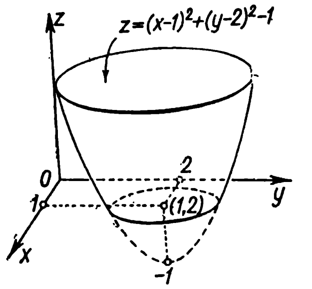


Fig. 184.

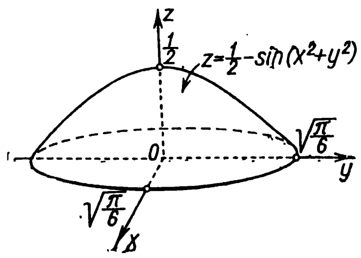


Fig. 185.

Example 2. The function

$$z=\frac{1}{2}-\sin(x^2+y^2)$$

for $x=0$, $y=0$ (coordinate origin) attains a maximum (Fig. 185).

Indeed,

$$f(0, 0)=\frac{1}{2}.$$

Inside the circle $x^2+y^2=\frac{\pi}{6}$ let us take the point (x, y) different from the point $(0, 0)$. Then for $0<x^2+y^2<\frac{\pi}{6}$,

$$\sin(x^2+y^2)>0$$

and therefore

$$f(x, y)=\frac{1}{2}-\sin(x^2+y^2)<\frac{1}{2}$$

or

$$f(x, y)<f(0, 0).$$

The definition, given above, of the maximum and minimum of a function may be rephrased as follows.

Let $x=x_0+\Delta x$; $y=y_0+\Delta y$; then

$$f(x, y)-f(x_0, y_0)=f(x_0+\Delta x, y_0+\Delta y)-f(x_0, y_0)=\Delta f.$$

1) If $\Delta f<0$ for all sufficiently small increments in the independent variables, then the function $f(x, y)$ reaches a *maximum* at the point $M(x_0, y_0)$.

2) If $\Delta f > 0$ for all sufficiently small increments in the independent variables, then the function $f(x, y)$ reaches a *minimum* at the point $M(x_0, y_0)$.

These formulations may be extended, without any change, to functions of any number of variables.

Theorem 1. (Necessary conditions of an extremum). *If a function $z = f(x, y)$ attains an extremum at $x = x_0$, $y = y_0$ then each first-order partial derivative with respect to z either vanishes for these values of the arguments or does not exist.*

Indeed, give the variable y a definite value $y = y_0$. Then the function $f(x, y_0)$ will be a function of one variable, x. Since at $x = x_0$ it has an extremum (maximum or minimum), it follows that $\left(\frac{\partial z}{\partial x}\right)_{\substack{x=x_0\\y=y_0}}$ is either equal to zero or does not exist. In exactly the same fashion it is possible to prove that $\left(\frac{\partial z}{\partial x}\right)_{\substack{x=x_0\\y=y_0}}$ is either equal to zero or does not exist.

This theorem is not sufficient for investigating the extremal values of a function, but permits finding these values for cases in which we are sure of the existence of a maximum or minimum. Otherwise, more investigation is required.

For instance, the function $z = y^2 - x^2$ has derivatives $\frac{\partial z}{\partial x} = -2x$; $\frac{\partial z}{\partial y} = +2y$, which vanish at $x = 0$ and $y = 0$. But for the given values, this function has neither maximum nor minimum. Indeed, this function is equal to zero at the origin and takes on both positive and negative values at points arbitrarily close to the origin. Hence, the value zero is neither a maximum nor a minimum (Fig. 186).

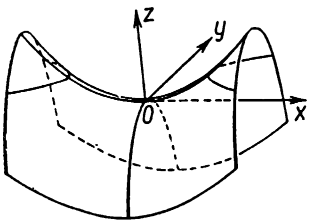


Fig. 186.

Points at which $\frac{\partial z}{\partial x} = 0$ (or does not exist) and $\frac{\partial z}{\partial y} = 0$ (or does not exist) are called *critical* points of the function $z = f(x, y)$. If a function reaches an extremum at some point, then (by virtue of Theorem 1) this can occur only at a critical point.

For investigating a function at critical points, let us establish sufficient conditions for the extremum of a function of two variables:

Theorem 2. *Let a function $f(x, y)$ have continuous partial derivatives up to order three inclusive in a certain domain containing the point $M_0(x_0, y_0)$; in addition, let the point $M_0(x_0, y_0)$ be a*

critical point of the function $f(x, y)$; that is,

$$\frac{\partial f(x_0, y_0)}{\partial x}=0, \quad \frac{\partial f(x_0, y_0)}{\partial y}=0.$$

Then for $x=x_0$, $y=y_0$:

1) *$f(x, y)$ has a maximum if*

$$\frac{\partial^2 f(x_0, y_0)}{\partial x^2}\cdot\frac{\partial^2 f(x_0, y_0)}{\partial y^2}-\left(\frac{\partial^2 f(x_0, y_0)}{\partial x\,\partial y}\right)^2>0 \text{ and } \frac{\partial^2 f(x_0, y_0)}{\partial x^2}<0;$$

2) *$f(x, y)$ has a minimum if*

$$\frac{\partial^2 f(x_0, y_0)}{\partial x^2}\cdot\frac{\partial^2 f(x_0, y_0)}{\partial y^2}-\left(\frac{\partial^2 f(x_0, y_0)}{\partial x\,\partial y}\right)^2>0 \text{ and } \frac{\partial^2 f(x_0, y_0)}{\partial x^2}>0;$$

3) *$f(x, y)$ has neither maximum nor minimum if*

$$\frac{\partial^2 f(x_0, y_0)}{\partial x^2}\cdot\frac{\partial^2 f(x_0, y_0)}{\partial y^2}-\left(\frac{\partial^2 f(x_0, y_0)}{\partial x\,\partial y}\right)^2<0;$$

4) *if* $\frac{\partial^2 f(x_0, y_0)}{\partial x^2}\cdot\frac{\partial^2 f(x_0, y_0)}{\partial y^2}-\left(\frac{\partial^2 f(x_0, y_0)}{\partial x\,\partial y}\right)^2=0$, *then there may or may not be an extremum* (in this case, an additional investigation is required).

Proof. Let us write the second-order Taylor formula for the function $f(x, y)$ [Formula (6), Sec. 16]. Assuming

$$a=x_0, \quad b=y_0, \quad x=x_0+\Delta x, \quad y=y_0+\Delta y,$$

we will have

$$f(x_0+\Delta x, y_0+\Delta y)=f(x_0, y_0)+\frac{\partial f(x_0, y_0)}{\partial x}\Delta x+\frac{\partial f(x_0, y_0)}{\partial y}\Delta y+$$
$$+\frac{1}{2}\left[\frac{\partial^2 f(x_0, y_0)}{\partial x^2}\Delta x^2+2\frac{\partial^2 f(x_0, y_0)}{\partial x\,\partial y}\Delta x\Delta y+\frac{\partial^2 f(x_0, y_0)}{\partial y^2}\Delta y^2\right]+\alpha_0(\Delta\varrho)^2,$$

where $\Delta\varrho=\sqrt{\Delta x^2+\Delta y^2}$ and α_0 approaches zero as $\Delta\varrho\to 0$.

It is given that

$$\frac{\partial f(x_0, y_0)}{\partial x}=0, \quad \frac{\partial f(x_0, y_0)}{\partial y}=0.$$

Hence

$$\Delta f=f(x_0+\Delta x, y_0+\Delta y)-f(x_0, y_0)=$$
$$=\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}\Delta x^2+2\frac{\partial^2 f}{\partial x\,\partial y}\Delta x\Delta y+\frac{\partial^2 f}{\partial y^2}\Delta y^2\right]+\alpha_0(\Delta\varrho)^2. \qquad (1)$$

Let us now denote the values of the second partial derivatives at the point $M_0(x_0, y_0)$ in terms of A, B, C:

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{M_0}=A; \quad \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{M_0}=B; \quad \left(\frac{\partial^2 f}{\partial y^2}\right)_{M_0}=C.$$

Denote by φ the angle between the direction of the segment M_0M, where M is the point $M(x_0+\Delta x,\ y_0+\Delta y)$, and the x-axis; then

$$\Delta x = \Delta\varrho \cos\varphi; \quad \Delta y = \Delta\varrho \sin\varphi.$$

Substituting these expressions into the formula for Δf, we find

$$\Delta f = \frac{1}{2}(\Delta\varrho)^2 [A\cos^2\varphi + 2B\cos\varphi\sin\varphi + C\sin^2\varphi + 2\alpha_0\Delta\varrho]. \quad (2)$$

Suppose that $A \neq 0$.

Dividing and multiplying by A the expression in the parentheses, we have

$$\Delta f = \frac{1}{2}(\Delta\varrho)^2 \left[\frac{(A\cos\varphi + B\sin\varphi)^2 + (AC - B^2)\sin^2\varphi}{A} + 2\alpha_0\Delta\varrho\right]. \quad (3)$$

Let us now consider four possible cases.

1) Let $AC - B^2 > 0$, $A < 0$. Then in the numerator of the fraction we have a sum of two nonnegative quantities. They do not vanish simultaneously because the first term vanishes for $\tan\varphi = -\frac{A}{B}$, while the second vanishes for $\sin\varphi = 0$.

If $A < 0$, then the fraction is a negative quantity that does not vanish. Denote it by $-m^2$; then

$$\Delta f = \frac{1}{2}(\Delta\varrho)^2 [-m^2 + 2\alpha_0\Delta\varrho],$$

where m is independent of $\Delta\varrho$, $\alpha_0\Delta\varrho \to 0$ as $\Delta\varrho \to 0$. Hence, for sufficiently small $\Delta\varrho$ we have

$$\Delta f < 0$$

or

$$f(x_0+\Delta x,\ y_0+\Delta y) - f(x_0,\ y_0) < 0.$$

But then for all points $(x_0+\Delta x,\ y_0+\Delta y)$ sufficiently close to the point (x_0, y_0) we have the inequality

$$f(x_0+\Delta x,\ y_0+\Delta y) < f(x_0,\ y_0),$$

which means that at the point $(x_0,\ y_0)$ the function attains a **maximum**.

2) Let $AC - B^2 > 0$, $A > 0$. Then, reasoning in the same way, we get

$$\Delta f = \frac{1}{2}(\Delta\varrho)^2 [m^2 + 2\alpha_0\Delta\varrho]$$

or

$$f(x_0+\Delta x,\ y_0+\Delta y) > f(x_0,\ y_0),$$

that is, $f(x, y)$ has a minimum at the point (x_0, y_0).

3′) Let $AC-B^2<0$, $A>0$. In this case the function **has neither a maximum nor a minimum.** The function increases when we move from the point (x_0, y_0) in certain directions and decreases when we move in other directions. Indeed, when moving along the ray $\varphi=0$, we have

$$\Delta f=\frac{1}{2}(\Delta\varrho)^2[A+2\alpha_0\Delta\varrho]>0;$$

when moving along this ray the function increases. But if we move along a ray $\varphi=\varphi_0$ such that $\tan\varphi_0=-\frac{A}{B}$, then for $A>0$ we have

$$\Delta f=\frac{1}{2}(\Delta\varrho)^2\left[\frac{AC-B^2}{A}\sin^2\varphi_0+2\alpha_0\Delta\varrho\right]<0;$$

when moving along this ray the function decreases.

3″) Let $AC-B^2<0$, $A<0$. Here the function again **has neither a maximum nor a minimum.** The investigation is conducted in the same way as for 3′.

3‴) Let $AC-B^2<0$, $A=0$. Then $B\neq 0$, and equality (2) may be rewritten as follows:

$$\Delta f=\frac{1}{2}(\Delta\varrho)^2[\sin\varphi(2B\cos\varphi+C\sin\varphi)+2\alpha_0\Delta\varrho].$$

For sufficiently small values of φ the expression in the parentheses retains its sign, since it is close to $2B$, while the factor $\sin\varphi$ changes sign depending on whether φ is greater or less than zero (after the choice of $\varphi>0$ and $\varphi<0$ we can take ϱ so small that $2\alpha_0$ will not change the sign of the whole square bracket). Consequently, in this case, too, Δf changes sign for different φ, that is, for different Δx and Δy; hence, in this case too there is neither a maximum nor a minimum.

Thus, no matter what the sign of A we always have the following situation:

If $AC-B^2<0$ at the point (x_0, y_0), then the function has neither a maximum nor a minimum at this point. In this case, the surface, which serves as a graph of the function, can, near this point, have, say, the shape of a saddle (see Fig. 186). The function at this point is said to have a *minimax*.

4) Let $AC-B^2=0$. In this case, by formulas (2) and (3), it is impossible to decide about the sign of Δf. For instance, when $A\neq 0$ we will have

$$\Delta f=\frac{1}{2}(\Delta\varrho)^2\left[\left(\frac{A\cos\varphi+B\sin\varphi}{A}\right)^2+2\alpha_0\Delta\varrho\right],$$

when $\varphi = \text{arc tan}\left(-\frac{A}{B}\right)$, the sign of Δf is determined by the sign of $2\alpha_0$; here, **a special additional investigation is required** (for example, with the aid of a higher-order Taylor formula or in some other way). Thus, Theorem 2 is fully proved.

Example 3. Test the following function for maximum and minimum:

$$z = x^2 - xy + y^2 + 3x - 2y + 1.$$

Solution. 1) Find the critical points

$$\frac{\partial z}{\partial x} = 2x - y + 3; \quad \frac{\partial z}{\partial y} = -x + 2y - 2.$$

Solving the system of equations

$$\left.\begin{aligned} 2x - y + 3 = 0, \\ -x + 2y - 2 = 0, \end{aligned}\right\}$$

we get

$$x = -\frac{4}{3}; \; y = \frac{1}{3}.$$

2) Find the second-order derivatives at the critical point $\left(-\frac{4}{3}, \frac{1}{3}\right)$ and determine the character of the critical point:

$$A = \frac{\partial^2 z}{\partial x^2} = 2; \; B = \frac{\partial^2 z}{\partial x\, \partial y} = -1; \; C = \frac{\partial^2 z}{\partial y^2} = 2;$$

$$AC - B^2 = 2 \cdot 2 - (-1)^2 = 3 > 0.$$

Thus, at the point $\left(-\frac{4}{3}, \frac{1}{3}\right)$ the given function has a minimum, namely

$$z_{\min} = -\frac{4}{3}.$$

Example 4. Test for a maximum and minimum the function $z = x^3 + y^3 - 3xy$.
Solution. 1) Find the critical points using the necessary conditions of an extremum:

$$\left.\begin{aligned} \frac{\partial z}{\partial x} = 3x^2 - 3y = 0, \\ \frac{\partial z}{\partial y} = 3y^2 - 3x = 0. \end{aligned}\right\}$$

Whence we get two critical points:

$$x_1 = 1, \quad y_1 = 1 \text{ and } x_2 = 0, \; y_2 = 0.$$

2) Find the second-order derivatives:

$$\frac{\partial^2 z}{\partial x^2} = 6x, \quad \frac{\partial^2 z}{\partial x\, \partial y} = -3, \quad \frac{\partial^2 z}{\partial y^2} = 6y.$$

3) Investigate the character of the first critical point:

$$A=\left(\frac{\partial^2 z}{\partial x^2}\right)_{\substack{x=1\\y=1}}=6;\quad B=\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)_{\substack{x=1\\y=1}}=-3;\quad C=\left(\frac{\partial^2 z}{\partial y^2}\right)_{\substack{x=1\\y=1}}=6;$$

$$AC-B^2=36-9=27>0;\quad A>0.$$

Hence, at the point (1, 1) the given function has a minimum, namely:

$$z_{\min}=-1.$$

4) Investigate the character of the second critical point $M_2(0, 0)$:

$$A=0;\quad B=-3;\quad C=0;$$

$$AC-B^2=-9<0.$$

Hence, at the second critical point the function has neither a maximum nor a minimum (minimax).

Example 5. Decompose a given positive number a into three positive terms so that their product is a maximum.

Solution. Denote the first term by x, the second by y; then the third will be $a-x-y$. The product of these terms is

$$u=x\cdot y\,(a-x-y).$$

It is given that $x>0$, $y>0$, $a-x-y>0$, that is, $x+y<a$, $u>0$. Hence, x and y can assume values in the domain bounded by the straight lines $x=0$, $y=0$, $x+y=a$.

Find the partial derivatives of the function u:

$$\frac{\partial u}{\partial x}=y\,(a-2x-y),$$

$$\frac{\partial u}{\partial y}=x\,(a-2y-x).$$

Equating the derivatives to zero, we get a system of equations:

$$y\,(a-2x-y)=0;\quad x\,(a-2y-x)=0.$$

Solving this system we get the critical points:

$$x_1=0,\quad y_1=0,\quad M_1(0,\ 0);$$
$$x_2=0,\quad y_2=a,\quad M_2(0,\ a);$$
$$x_3=a,\quad y_3=0,\quad M_3(a,\ 0);$$
$$x_4=\frac{a}{3},\quad y_4=\frac{a}{3},\quad M_4\left(\frac{a}{3},\ \frac{a}{3}\right).$$

The first three points lie on the boundary of the region, the last one, inside. On the boundary of the region, the function u is equal to zero, while inside it is positive; consequently, at the point $\left(\frac{a}{3},\ \frac{a}{3}\right)$, the function u has a maximum (since it is the only extremal point inside the triangle). The maximum value of the product

$$u_{\max}=\frac{a}{3}\,\frac{a}{3}\left(a-\frac{a}{3}-\frac{a}{3}\right)=\frac{a^3}{27}.$$

Investigate the character of the critical points using the sufficiency conditions. Find the second-order partial derivatives of the function u:

$$\frac{\partial^2 u}{\partial x^2}=-2y;\quad \frac{\partial^2 u}{\partial x\,\partial y}=a-2x-2y;\quad \frac{\partial^2 u}{\partial y^2}=-2x.$$

At the point $M_1(0, 0)$ we have $A=\frac{\partial^2 u}{\partial x^2}=0$; $B=\frac{\partial^2 u}{\partial x\partial y}=a$, $C=\frac{\partial^2 u}{\partial y^2}=0$, $AC-B^2=-a^2<0$. Hence, at the point M_1 there is neither a maximum nor a minimum. At the point $M_2(0, a)$ we have $A=\frac{\partial^2 u}{\partial x^2}=-2a$; $B=\frac{\partial^2 u}{\partial x\partial y}=-a$; $C=\frac{\partial^2 u}{\partial y^2}=0$;

$$AC-B^2=-a^2<0.$$

Which means that at the point M_2 there is neither a maximum nor a minimum. At the point $M_3(a, 0)$ we have $A=0$, $B=-a$, $C=-2a$:

$$AC-B^2=-a^2<0.$$

At M_3 too there is neither a maximum nor a minimum. At the point $M_4\left(\frac{a}{3}, \frac{a}{3}\right)$ we have $A=-\frac{2a}{3}$; $B=-\frac{a}{3}$; $C=-\frac{2a}{3}$;

$$AC-B^2=\frac{4a^2}{9}-\frac{a^2}{9}>0;\ A<0.$$

Hence, at M_4 we have a maximum.

SEC. 18. MAXIMUM AND MINIMUM OF A FUNCTION OF SEVERAL VARIABLES RELATED BY GIVEN EQUATIONS (CONDITIONAL MAXIMA AND MINIMA)

In many maximum and minimum problems, one has to find the extrema of a function of several variables that are not independent, but are related to one another by side conditions (for example, they must satisfy given equations).

By way of illustration let us consider the following problem. Using a piece of tin $2a$ in area it is required to build a closed box in the form of a parallelepiped of maximum volume.

Denote the length, width and height of the box by x, y, and z. The problem reduces to finding the maximum of the function

$$v=xyz$$

provided that $2xy+2xz+2yz=2a$. The problem here deals with a **conditional extremum:** the variables x, y, z **are restricted by the condition** that $2xy+2xz+2yz=2a$. In this section we shall consider methods of solving such problems.

Let us first consider the question of the conditional extremum of a function of two variables if these variables are restricted by a **single** condition.

Let it be required to find the maxima and minima of the function

$$u = f(x, y) \tag{1}$$

with the proviso that x and y are connected by the equation

$$\varphi(x, y) = 0. \tag{2}$$

Given condition (2), of the two variables x and y there will be only **one** which **is independent** (for instance, x) since y is determined from (2) as a function of x. If we solved equation (2) for y and put into (1) the expression found in place of y, we would obtain a function of **one** variable, x, and would reduce the problem to one that would involve finding the maximum and minimum of a function of one independent variable, x.

But the problem may be solved without solving equation (2) for x or y. For those values of x at which the function u can have a maximum or minimum, the derivative of u with respect to x should vanish.

From (1) we find $\frac{du}{dx}$, remembering that y is a function of x:

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$

Hence, at the points of the extremum

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

From equation (2) we find

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

This equality is satisfied for all x and y that satisfy equation (2) (see Sec. 11, Ch. VIII).

Multiplying the terms of (4) by an (as yet) undetermined coefficient λ and adding them to the corresponding terms of (3), we have

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right) + \lambda\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx}\right) = 0$$

or

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x}\right) + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y}\right)\frac{dy}{dx} = 0. \tag{5}$$

The latter equality is fulfilled at all extremum points. Choose λ such that for the values of x and y which correspond to the extre-

mum of the function u, the second parentheses in (5) should vanish: *)

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}=0.$$

But then, for these values of x and y, from (5) we have

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}=0.$$

It thus turns out that at the extremum points three equations (with three unknowns x, y, λ) are satisfied:

$$\left.\begin{aligned}\frac{\partial f}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}=0,\\ \frac{\partial f}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}=0,\\ \varphi(x, y)=0.\end{aligned}\right\} \qquad (6)$$

From these equations determine x, y, and λ; the latter only played an auxiliary role and will not be needed any more.

From this conclusion it follows that equations (6) are necessary conditions of a conditional extremum; or equations (6) are satisfied at the extremum points. But there will not be a conditional extremum for every x and y (and λ) that satisfy equations (6). A supplementary investigation of the nature of the critical point is required. In the solution of concrete problems it is sometimes possible to establish the character of the critical point from the statement of the problem. It will be noted that the left-hand sides of equations (6) are partial derivatives of the function

$$F(x, y, \lambda)=f(x, y)+\lambda\varphi(x, y) \qquad (7)$$

with respect to the variables x, y and λ.

Thus, in order to find the values of x and y which satisfy condition (2), for which the function $u=f(x, y)$ can have a conditional maximum or a conditional minimum, one has to construct an auxiliary function (7), equate to zero its derivatives with respect to x, y, and λ, and from the three equations (6) thus obtained determine the sought-for x, y (and the auxiliary factor λ). The foregoing method can be extended to a study of the conditional extremum of a function of any number of variables.

Let it be required to find the maxima and minima of a function of n variables, $u=f(x_1, x_2, \ldots, x_n)$ provided that the variables

*) For the sake of definiteness, we shall assume that at the critical points

$$\frac{\partial \varphi}{\partial y}\neq 0.$$

$x_1, x_2, \ldots, x_n$ are connected by $m\,(m<n)$ equations:

$$\left.\begin{array}{l}\varphi_1(x_1, x_2, \ldots, x_n)=0,\\ \varphi_2(x_1, x_2, \ldots, x_n)=0,\\ \cdots\cdots\cdots\cdots\\ \varphi_m(x_1, x_2, \ldots, x_n)=0.\end{array}\right\} \quad (8)$$

In order to find the values of $x_1, x_2, \ldots, x_n$, for which there may be conditional maxima and minima, one has to form the function

$$F(x_1, x_2, \ldots, x_n, \lambda_1, \ldots, \lambda_m)=f(x_1, \ldots, x_n)+\lambda_1\varphi_1(x_1, \ldots, x_n)+ \\ +\lambda_2\varphi_2(x_1, \ldots, x_n)+\ldots+\lambda_m\varphi_m(x_1, \ldots, x_n),$$

equate to zero its partial derivatives with respect to $x_1, x_2, \ldots, x_n$:

$$\left.\begin{array}{l}\dfrac{\partial f}{\partial x_1}+\lambda_1\dfrac{\partial\varphi_1}{\partial x_1}+\ldots+\lambda_m\dfrac{\partial\varphi_m}{\partial x_1}=0,\\ \dfrac{\partial f}{\partial x_2}+\lambda_1\dfrac{\partial\varphi_1}{\partial x_2}+\ldots+\lambda_m\dfrac{\partial\varphi_m}{\partial x_2}=0,\\ \cdots\cdots\cdots\cdots\cdots\\ \dfrac{\partial f}{\partial x_n}+\lambda_1\dfrac{\partial\varphi_1}{\partial x_n}+\ldots+\lambda_m\dfrac{\partial\varphi_m}{\partial x_n}=0\end{array}\right\} \quad (9)$$

and from the $m+n$ equations (8) and (9) determine $x_1, x_2, \ldots, x_n$ and the auxiliary unknowns $\lambda_1, \ldots, \lambda_m$. Just as in the case of a function of two variables, we shall, in the general case, leave undecided the question of whether the function, for the values found, will have a maximum or minimum or will have neither. We will decide this matter on the basis of additional reasoning.

Example 1. Let us return to the problem formulated at the beginning of this section: to find the maximum of the function

$$v=xyz$$

provided that

$$xy+xz+yz-a=0 \quad (x>0,\ y>0,\ z>0). \quad (10)$$

We form the auxiliary function

$$F(x, y, \lambda)=xyz+\lambda(xy+xz+yz-a).$$

Find its partial derivatives and equate them to zero:

$$\left.\begin{array}{l}yz+\lambda(y+z)=0,\\ xz+\lambda(x+z)=0,\\ xy+\lambda(x+y)=0.\end{array}\right\} \quad (11)$$

The problem reduces to solving a system of four equations (10) and (11) in four unknowns (x, y, z and λ). To solve this system, multiply the first of

equations (11) by x, the second by y, the third by z, and add; taking (10) into account we find that $\lambda = -\frac{3xyz}{2a}$. Putting this value of λ into equations (11) we get

$$yz\left[1-\frac{3x}{2a}(y+z)\right]=0,$$
$$xz\left[1-\frac{3y}{2a}(x+z)\right]=0,$$
$$xy\left[1-\frac{3z}{2a}(x+y)\right]=0.$$

Since it is evident from the statement of the problem that x, y, z are different from zero, we get from the latter equations

$$\frac{3x}{2a}(y+z)=1, \quad \frac{3y}{2a}(x+z)=1, \quad \frac{3z}{2a}(x+y)=1.$$

From the first two equations we find $x=y$, from the second and third equations, $y=z$. But then from equation (10) we get $x=y=z=\sqrt{\frac{a}{3}}$. This is the only system of values of x, y, and z, for which there **can be** a maximum or minimum.

It can be proved that the solution obtained yields a maximum. Incidentally, this is also evident from geometrical reasoning (the statement of the problem indicates that the volume of the box cannot be big without bound; it is therefore natural to expect that for some definite values of the sides the volume will be a maximum).

Thus, for the volume of the box to be a maximum, the box must be a cube, an edge of which is equal to $\sqrt{\frac{a}{3}}$.

Example 2. Determine the maximum value of the nth root of a product of numbers $x_1, x_2, \ldots, x_n$ provided that their sum is equal to a given number a. Thus, the problem is stated as follows: it is required to find the maximum of the function $u=\sqrt[n]{x_1 \ldots x_n}$ on the condition that

$$\begin{gathered} x_1+x_2+\ldots+x_n-a=0 \\ (x_1>0,\ x_2>0,\ \ldots,\ x_n>0). \end{gathered} \tag{12}$$

Form an auxiliary function:

$$F(x_1, \ldots, x_n, \lambda)=\sqrt[n]{x_1 \ldots x_n}+\lambda(x_1+x_2+\ldots+x_n-a).$$

Find its partial derivatives:

$$F'_{x_1}=\frac{1}{n}\frac{x_2x_3\ldots x_n}{(x_1\ldots x_n)^{\frac{n-1}{n}}}+\lambda=\frac{1}{n}\frac{u}{x_1}+\lambda=0 \text{ or } u=-n\lambda x_1,$$

$$F'_{x_2}=\frac{1}{n}\frac{u}{x_2}+\lambda=0 \qquad \text{or } u=-n\lambda x_2,$$

. . . .

$$F'_{x_n}=\frac{1}{n}\frac{u}{x_n}+\lambda=0 \qquad \text{or } u=-n\lambda x_n.$$

From the foregoing equations we find

$$x_1=x_2=\ldots=x_n,$$

and from equation (12) we have

$$x_1=x_2=\ldots=x_n=\frac{a}{n}.$$

By the meaning of the problem these values yield a maximum of the function $\sqrt[n]{x_1 \ldots x_n}$ equal to $\frac{a}{n}$.

Thus, for any positive numbers $x_1, x_2, \ldots, x_n$ connected by the relationship $x_1+x_2+\ldots+x_n=a$, the inequality

$$\sqrt[n]{x_1 \ldots x_n}\leqslant\frac{a}{n} \tag{13}$$

is fulfilled (since it has already been proved that $\frac{a}{n}$ is the maximum of this function). Now substituting into (13) the value of a obtained from (12), we get

$$\sqrt[n]{x_1x_2 \ldots x_n}\leqslant\frac{x_1+\ldots+x_n}{n}. \tag{14}$$

This inequality holds for all positive numbers $x_1, x_2, \ldots, x_n$. The expression on the left-hand side of (14) is called the *geometric mean* of these numbers. Thus, the geometric mean of several positive numbers is not greater than their arithmetic mean.

SEC. 19. SINGULAR POINTS OF A CURVE

The concept of a partial derivative is used in investigating curves.

Let a curve be given by the equation

$$F(x, y)=0.$$

The slope of the tangent to the curve is determined from the formula

$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

(see Sec. 11, Ch. VIII).

If at a given point $M(x, y)$ of the curve under consideration, at least one of the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ does not vanish, then at this point either $\frac{dy}{dx}$ or $\frac{dx}{dy}$ is completely determined. The curve $F(x, y)=0$ has a very definite line tangent at this point. In this case, the point $M(x, y)$ is called an *ordinary* point.

But if at some point $M_0(x_0, y_0)$ we have

$$\left(\frac{\partial F}{\partial x}\right)_{\substack{x=x_0\\y=y_0}}=0 \text{ and } \left(\frac{\partial F}{\partial y}\right)_{\substack{x=x_0\\y=y_0}}=0,$$

then the slope of the tangent becomes indeterminate.

Definition. If at the point $M_0(x_0, y_0)$ of the curve $F(x, y)=0$, both partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ vanish, then such a point is called a *singular point* of the curve. Thus, a singular point of a curve is defined by the system of equations

$$F=0; \quad \frac{\partial F}{\partial x}=0; \quad \frac{\partial F}{\partial y}=0.$$

Naturally, not every curve has singular points. For example, for the ellipse

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0,$$

obviously,

$$F(x, y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1; \quad \frac{\partial F}{\partial x}=\frac{2x}{a^2}; \quad \frac{\partial F}{\partial y}=\frac{2y}{b^2};$$

the derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ vanish only when $x=0$, $y=0$, but these values of x and y do not satisfy the equation of the ellipse. Consequently, the ellipse does not have any singular points.

Without undertaking a detailed investigation of the behaviour of a curve near a singular point, let us examine some examples of curves that have singular points.

Example 1. Investigate the singular points of the curve

$$y^2-x(x-a)^2=0 \quad (a>0).$$

Solution. Here, $F(x, y)=y^2-x(x-a)^2$ and therefore

$$\frac{\partial F}{\partial x}=(x-a)(a-3x); \quad \frac{\partial F}{\partial y}=2y.$$

Solving the three equations simultaneously,

$$F(x, y)=0, \quad \frac{\partial F}{\partial x}=0, \quad \frac{\partial F}{\partial y}=0,$$

we find the only system of values of x and y that satisfy them:

$$x_0=a, \quad y_0=0.$$

Consequently, the point $M_0(a, 0)$ is a singular point of the curve.

Let us investigate the behaviour of the curve near a singular point and then construct the curve.

Rewrite the equation in the form

$$y = \pm (x-a)\sqrt{x}.$$

From this formula it follows that the curve: 1) is defined only for $x \geqslant 0$; 2) is symmetrical about the x-axis; 3) cuts the x-axis at the points (0, 0) and (a, 0). The latter point is singular, as we have pointed out.

Let us first examine that part of the curve which corresponds to the plus sign:

$$y = (x-a)\sqrt{x}.$$

Find the first and second derivatives of y with respect to x:

$$y' = \frac{3x-a}{2\sqrt{x}}, \quad y'' = \frac{3x+a}{4x\sqrt{x}}.$$

For $x=0$ we have $y=\infty$. Thus, the curve touches the y-axis at the origin. For $x=\frac{a}{3}$ we have $y'=0$, $y''>0$, which means that for $x=\frac{a}{3}$ the function y has a minimum:

$$y = -\frac{2a}{3}\sqrt{\frac{a}{3}}.$$

On the interval $0<x<a$ we have $y<0$; for $x>\frac{a}{3}$ $y'>0$; as $x\to\infty$ $y\to\infty$.

For $x=a$ we have $y'=\sqrt{a}$, which means that at the singular point $M_0(a, 0)$ the branch of the curve $y=+(x-a)\sqrt{x}$ has a tangent

$$y = \sqrt{a}\,(x-a).$$

Since the second branch of the curve $y=-(x-a)\sqrt{x}$ is symmetrical with the first about the x-axis, the curve has also a second tangent (to the second branch) at the singular point

$$y = -\sqrt{a}\,(x-a).$$

The curve passes through the singular point twice. Such a point is called a *nodal* point.

The foregoing curve is shown in Fig. 187.

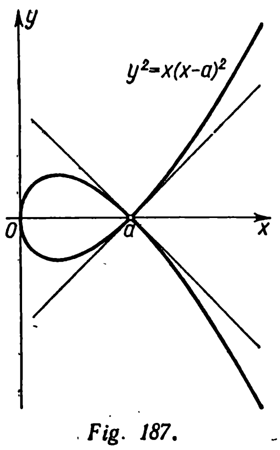


Fig. 187.

Example 2. Test for singular points the curve (semicubical parabola)

$$y^2 - x^3 = 0.$$

Solution. The coordinates of the singular points are determined from the following set of equations:

$$y^2 - x^3 = 0; \quad 3x^2 = 0; \quad 2y = 0.$$

Consequently, $M_0(0, 0)$ is a singular point.

Let us rewrite the given equation as

$$y = \pm\sqrt{x^3}.$$

To construct the curve let us first investigate the branch to which the plus sign in the equation

corresponds, since the branch of the curve corresponding to the minus sign is symmetric with the first about the x-axis.

The function y is defined only for $x \geqslant 0$, it is nonnegative and increases as x increases.

Let us find the first and second derivatives of the function $y=\sqrt{x^3}$:

$$y'=\frac{3}{2}\sqrt{x};\quad y''=\frac{3}{4}\frac{1}{\sqrt{x}}.$$

For $x=0$ we have $y=0$, $y'=0$. And so the given branch of the curve has a tangent $y=0$ at the origin. The second branch of the curve $y=-\sqrt{x^3}$ also passes through the origin and has the same tangent $y=0$. Thus, two different branches of the curve meet at the origin, have the same tangent, and are situated on different sides of the tangent. This kind of singular point is called a *cusp of the first kind* (Fig. 188).

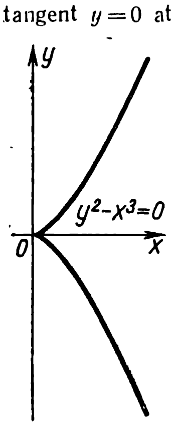


Fig. 188.

Note. The curve $y^2-x^3=0$ may be regarded as a limiting case of the curve $y^2=x(x-a)^2=0$ (considered in Example 1) as $a \to 0$; that is, when the loop of the curve is contracted into a point.

Example 3. Investigate the curve

$$(y-x^2)^2-x^5=0.$$

Solution. The coordinates of the singular points are defined by the following set of equations:

$$-4x(y-x^2)-5x^4=0;\quad 2(y-x^2)=0,$$

which has only one solution: $x=0$, $y=0$. Hence, the origin is a singular point.

Rewrite the given equation in the form

$$y=x^2 \pm \sqrt{x^5}.$$

From this equation it follows that x can take on values from 0 to $+\infty$.

Let us determine the first and second derivatives:

$$y'=2x \pm \frac{5}{2}\sqrt{x^3};\quad y''=2 \pm \frac{15}{4}\sqrt{x}.$$

Investigate, separately, the branches of the curve corresponding to plus and minus. In both cases, when $x=0$ we have $y=0$, $y'=0$, which means that for both branches the x-axis is a tangent.

Let us first consider the branch

$$y=x^2+\sqrt{x^5}.$$

As x increases from 0 to ∞, y increases from 0 to ∞.

The second branch

$$y=x^2-\sqrt{x^5}$$

cuts the x-axis at the points (0, 0) and (1, 0).

For $x=\frac{16}{25}$ the function $y=x^2-\sqrt{x^5}$ has a maximum. If $x \to +\infty$, then $y \to -\infty$.

Thus, in this case the two branches of the curve meet at the origin; both branches have the same tangent and are situated on the same side of the tangent near the point of tangency. This kind of singular point is called a *cusp of the second kind*. The graph of this function is shown in Fig. 189.

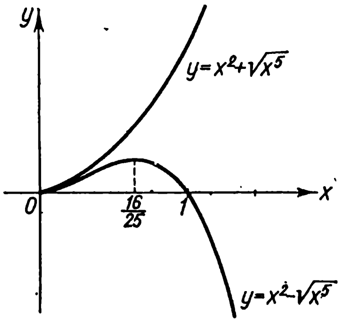


Fig. 189.

Example 4. Investigate the curve

$$y^2-x^4+x^6=0.$$

Solution. The origin is a singular point. To investigate the curve near this point rewrite the equation of the curve in the form

$$y=\pm x^2\sqrt{1-x^2}.$$

Since the equation of the curve contains only even powers of the variables, the curve is symmetric about the coordinate axes and, consequently, it is sufficient to investigate that part of the curve which corresponds to the positive values of x and y. From the latter equation it follows that x can vary over the interval from 0 to 1, that is, $0\leqslant x\leqslant 1$.

Let us evaluate the first derivative for that branch of the curve which is a graph of the function $y=+x^2\sqrt{1-x^2}$:

$$y'=\frac{x(2-3x^2)}{\sqrt{1-x^2}}.$$

For $x=0$ we have $y=0$, $y'=0$. Thus, the curve touches the x-axis at the origin.

For $x=1$ we have $y=0$, $y'=\infty$; consequently, at the point (1, 0) the tangent is parallel to the y-axis. For $x=\sqrt{\frac{2}{3}}$ the function has a maximum (Fig. 190).

At the origin (at the singular point) the two branches of the curve corresponding to plus and minus in front of the radical sign are mutually tangent. A singular point of this kind is called a *point of osculation* (also known as **tacnode** or **double cusp**).

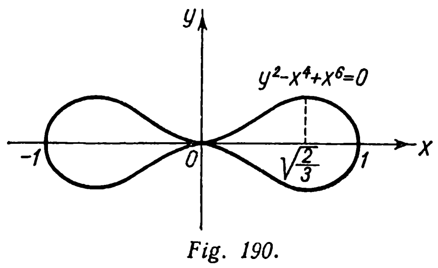


Fig. 190.

Example 5. Investigate the curve

$$y^2-x^2(x-1)=0.$$

Solution. Let us write the system of equations defining the singular points:

$$y^2-x^2(x-1)=0;$$
$$-3x^2+2x=0,\ 2y=0.$$

This system has the solution $x=0$, $y=0$. Therefore, the point (0, 0) is a singular point of the curve. Let us rewrite the given equation in the form

$$y=\pm x\sqrt{x-1}.$$

It is obvious that x can vary from 1 to ∞ and also take the value 0 (in which case $y=0$).

Let us investigate the branch of the curve corresponding to the plus sign in front of the radical. As x increases from 1 to ∞, y increases from 0 to ∞. The derivative

$$y'=\frac{3x-2}{2\sqrt{x-1}}.$$

When $x=1$ we have $y'=\infty$; hence, at the point (1, 0) the tangent is parallel to the y-axis.

The second branch of the curve corresponding to the minus sign is symmetric with the first about the x-axis.

The point (0, 0) has coordinates that satisfy the equation and, consequently, belongs to the curve, but near it there are no other points of the curve (Fig. 191). This kind of singular point is called an *isolated singular point.*

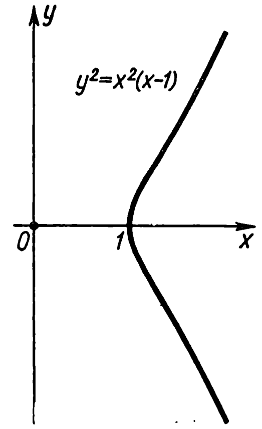


Fig. 191.

Exercises on Chapter VIII

Find the partial derivatives of the following functions:

1. $z=x^2\sin^2 y$. *Ans.* $\frac{\partial z}{\partial x}=2x\sin^2 y$; $\frac{\partial z}{\partial y}=x^2\sin 2y$. **2.** $z=x^{y^2}$. *Ans.* $\frac{\partial z}{\partial x}=y^2x^{y^2-1}$; $\frac{\partial z}{\partial y}=x^{y^2}\cdot 2y\ln x$. **3.** $u=e^{x^2+y^2+z^2}$. *Ans.* $\frac{\partial u}{\partial x}=2xe^{x^2+y^2+z^2}$; $\frac{\partial u}{\partial y}=2ye^{x^2+y^2+z^2}$; $\frac{\partial u}{\partial z}=2ze^{x^2+y^2+z^2}$; **4.** $u=\sqrt{x^2+y^2+z^2}$. *Ans.* $\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}$. **5.** $z=\arctan(xy)$. *Ans.* $\frac{\partial z}{\partial x}=\frac{y}{1+x^2y^2}$; $\frac{\partial z}{\partial y}=\frac{x}{1+x^2y^2}$. **6.** $z=\arctan\frac{y}{x}$. *Ans.* $\frac{\partial z}{\partial x}=\frac{-y}{x^2+y^2}$; $\frac{\partial z}{\partial y}=\frac{x}{x^2+y^2}$. **7.** $z=\ln\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}$. *Ans.* $\frac{\partial z}{\partial x}=-\frac{2}{\sqrt{x^2+y^2}}$. $\frac{\partial z}{\partial y}=\frac{2x}{y\sqrt{x^2+y^2}}$. **8.** $u=e^{\frac{x}{y}}+e^{\frac{z}{y}}$. *Ans.* $\frac{\partial u}{\partial x}=\frac{1}{y}e^{\frac{x}{y}}$; $\frac{\partial u}{\partial y}=-\frac{x}{y^2}e^{\frac{x}{y}}-\frac{z}{y^2}e^{\frac{z}{y}}$; $\frac{\partial u}{\partial z}=\frac{1}{y}e^{\frac{z}{y}}$. **9.** $z=\arcsin(x+y)$. *Ans.* $\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-(x+y)^2}}=\frac{\partial z}{\partial y}$. **10.** $z=\arctan\sqrt{\frac{x^2-y^2}{x^2+y^2}}$. Ans. $\frac{\partial z}{\partial x}=\frac{y^2}{x\sqrt{x^4-y^4}}$; $\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{x^4-y^4}}$.

Find the total differentials of the following functions: **11.** $z=x^2+xy^2+\sin y$. *Ans.* $dz=(2x+y^2)\,dx+(2xy+\cos y)\,dy$. **12.** $z=\ln(xy)$. *Ans.* $dz=\frac{dx}{x}+\frac{dy}{y}$.

13. $z=e^{x^2+y^2}$. *Ans.* $dz=2e^{x^2+y^2}(x\,dx+y\,dy)$. 14. $u=\tan(3x-y)+6^{y+z}$. *Ans.* $du=\frac{3dx}{\cos^2(3x-y)}+\left(-\frac{1}{\cos^2(3x-y)}+6^{y+z}\ln 6\right)dy+6^{y+z}\ln 6\,dz$. 15. $w=\arcsin\frac{x}{y}$. *Ans.* $dw=\frac{y\,dx-x\,dy}{|y|\sqrt{y^2-x^2}}$.

16. Evaluate $f'_x(2, 3)$ and $f'_y(2, 3)$ if $f(x, y)=x^2+y^3$. *Ans.* $f'_x(2, 3)=4$, $f'_y(2, 3)=27$.

17. Evaluate $df(x, y)$ for $x=1, y=0; dx=\frac{1}{2}, dy=\frac{1}{4}$ if $f(x, y)=\sqrt{x^2+y^2}$. *Ans.* $\frac{1}{2}$.

18. Form a formula which, for small absolute values of the quantities x, y and z, yields an approximate expression for $\sqrt{\frac{1+x}{(1+y)(1+z)}}$. *Ans.* $1+\frac{1}{2}(x-y-z)$.

19. Do the same for $\sqrt{\frac{1+x}{1+y+z}}$. *Ans.* $1+\frac{1}{2}(x-y-z)$.

20. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if $z=u+v^2$; $u=x^2+\sin y$, $v=\ln(x+y)$. *Ans.* $\frac{\partial z}{\partial x}=2x+2v\frac{1}{x+y}$; $\frac{\partial z}{\partial y}=\cos y+2v\frac{1}{x+y}$.

21. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z=\sqrt{\frac{1+u}{1+v}}$; $u=-\cos x$; $v=\cos x$. *Ans.* $\frac{\partial z}{\partial x}=\frac{1}{2\cos^2\frac{x}{2}}$; $\frac{\partial z}{\partial y}=0$.

22. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z=e^{u-2v}$, $u=\sin x$, $v=x^3+y^2$. *Ans.* $\frac{\partial z}{\partial x}=e^{u-2v}(\cos x-6x^2)$, $\frac{\partial z}{\partial y}=e^{u-2v}(0-2\cdot 2y)=-4ye^{u-2v}$.

23. Find the total derivatives of the given functions: $z=\arcsin(u+v)$; $u=\sin x\cos\alpha$; $v=\cos x\sin\alpha$. *Ans.* $\frac{\partial z}{\partial x}=1$ if $2k\pi-\frac{\pi}{2}<x+\alpha<2k\pi+\frac{\pi}{2}$, $\frac{\partial z}{\partial x}=-1$ if $2k\pi+\frac{\pi}{2}<x+\alpha<(2k+1)\pi+\frac{\pi}{2}$. 24. $u=\frac{e^{ax}(y-z)}{a^2+1}$; $y=a\sin x$; $z=\cos x$. *Ans.* $\frac{\partial u}{\partial x}=e^{ax}\sin x$. 25. $z=\ln(1-x^4)$; $x=\sqrt{\sin\theta}$; $\frac{\partial z}{\partial \theta}=-2\tan\theta$.

Find the derivatives of implicit functions of x given by the following equations: 26. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. *Ans.* $\frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y}$. 27. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. *Ans.* $\frac{dy}{dx}=\frac{b^2}{a^2}\frac{x}{y}$. 28. $y^x=x^y$. *Ans.* $\frac{dy}{dx}=\frac{yx^{y-1}-y^x\ln y}{xy^{x-1}-x^y\ln x}$. 29. $\sin(xy)-e^{xy}-x^2y=0$.

Ans. $\frac{dy}{dx}=\frac{y[\cos(xy)-e^{xy}-2x]}{x[x+e^{xy}-\cos(xy)]}$. **30.** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. *Ans.* $\frac{\partial z}{\partial x}=-\frac{c^2x}{a^2z}$; $\frac{\partial z}{\partial y}=-\frac{c^2y}{b^2z}$. **31.** $u-v\tan aw=0$; find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$. *Ans.* $\frac{\partial w}{\partial u}=$ $=\frac{\cos^2 aw}{av}$; $\frac{\partial w}{\partial v}=-\frac{\sin 2aw}{2av}$. **32.** $z^2+\frac{2}{x}=\sqrt{y^2-z^2}$; show that $x^2\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{1}{z}$.

33. $\frac{z}{x}=F\left(\frac{y}{z}\right)$; show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$, no matter what the differentiable function F.

Compute the second-order partial derivatives:

34. $z=x^3-4x^2y+5y^2$. *Ans.* $\frac{\partial^2 z}{\partial x^2}=6x-8y$; $\frac{\partial^2 z}{\partial y\partial x}=-8x$; $\frac{\partial^2 z}{\partial y^2}=10$.

35. $z=e^x\ln y+\sin y\ln x$. *Ans.* $\frac{\partial^2 z}{\partial x^2}=e^x\ln y-\frac{\sin y}{x^2}$, $\frac{\partial^2 z}{\partial x\partial y}=\frac{e^x}{y}+\frac{\cos y}{x}$; $\frac{\partial^2 z}{\partial y^2}=-\frac{e^x}{y^2}-\sin y\ln x$.

36. Prove that if $u=\frac{1}{\sqrt{x^2+y^2+z^2}}$, then $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$.

37. Prove that if $z=\frac{x^2y^2}{x+y}$, then $x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=2\frac{\partial z}{\partial x}$.

38. Prove that if $z=\ln(x^2+y^2)$, then $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$.

39. Prove that if $z=\varphi(y+ax)+\psi(y-ax)$, then $a^2\frac{\partial^2 z}{\partial y^2}-\frac{\partial^2 z}{\partial x^2}=0$ for any doubly differentiable φ and ψ.

40. Find the derivative of the function $z=3x^4-xy+y^3$ at the point $M(1, 2)$ in the direction that makes an angle of 60° with the x-axis. *Ans.* $5+\frac{11\sqrt{3}}{2}$.

41. Find the derivative of the function $z=5x^2-3x-y-1$ at the point $M(2, 1)$ in the direction from this point to the point $N(5, 5)$. *Ans.* $\frac{47}{5}=9.4$.

42. Find the derivative of the function $f(x, y)$ in the direction of: 1) the bisector of the quadrantal angle Oxy. *Ans.* $\frac{1}{\sqrt{2}}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)$; 2) the negative semi-axis Ox. *Ans.* $-\frac{\partial f}{\partial x}$.

43. $f(x, y)=x^3+3x^2+4xy+y^2$. Show that at the point $M\left(\frac{2}{3}, -\frac{4}{3}\right)$ the derivative in any direction is equal to zero (the "function is stationary").

44. Of all triangles with the same perimeter $2p$, determine the triangle with greatest area. *Ans.* Equilateral triangle.

45. Find a rectangular parallelepiped of greatest volume for a given total surface S. *Ans.* A cube with edge $\sqrt{\frac{S}{6}}$.

46. Find the distance between two straight lines in space whose equations are $\frac{x-1}{1}=\frac{y}{2}=\frac{z}{1}$, $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$. *Ans.* $\frac{\sqrt{2}}{2}$.

Test for maximum and minimum the functions:

47. $z=x^3y^2(a-x-y)$. *Ans.* Maximum z at $x=\frac{a}{2}$; $y=\frac{a}{3}$.

48. $z=x^2+xy+y^2+\frac{1}{x}+\frac{1}{y}$. *Ans.* Minimum z at $x=y=\frac{1}{\sqrt[3]{3}}$.

49. $z=\sin x+\sin y+\sin(x+y)$ $\left(0\leqslant x\leqslant\frac{\pi}{2};\ 0\leqslant y\leqslant\frac{\pi}{2}\right)$. *Ans.* Maximum z at $x=y=\frac{\pi}{3}$.

50. $z=\sin x\sin y\sin(x+y)$ $(0\leqslant x\leqslant\pi;\ 0\leqslant y\leqslant\pi)$. *Ans.* Maximum z at $x=y=\frac{\pi}{3}$.

Find the singular points of the following curves, investigate their character and form equations of the tangents at these points:

51. $x^3+y^3-3axy=0$. *Ans.* $M_0(0,0)$ is a node; $x=0$, $y=0$ are the equations of the tangents.

52. $a^4y^2=x^4(a^2-x^2)$. *Ans.* A double cusp at the origin; the double tangent $y^2=0$.

53. $y^2=\frac{x^3}{2a-x}$. *Ans.* $M_0(0,0)$ is a cusp of the first kind; $y^2=0$ is a tangent.

54. $y^2=x^2(9-x^2)$. *Ans.* $M_0(0,0)$ is a node; $y=\pm 3x$ are the equations of the tangents.

55. $x^4-2ax^2y-axy^2+a^2x^2=0$. *Ans.* $M_0(0,0)$ is a cusp of the second kind; $y^2=0$ is a double tangent.

56. $y^2(a^2+x^2)=x^2(a^2-x^2)$. *Ans.* $M_0(0,0)$ is a node; $y=\pm x$ are the equations of the tangents.

57. $b^2x^2+a^2y^2=x^2y^2$. *Ans.* $M_0(0,0)$ is an isolated point.

58. Show that the curve $y=x\ln x$ has an end point at the coordinate origin and a tangent which is the y-axis.

59. Show that the curve $y=\frac{x}{1+e^{\frac{1}{x}}}$ has a nodal point at the origin and that the tangents at this point are: on the right $y=0$, on the left $y=x$.

CHAPTER IX

APPLICATIONS OF DIFFERENTIAL CALCULUS TO SOLID GEOMETRY

SEC. 1. THE EQUATIONS OF A CURVE IN SPACE

Let us consider the vector $\overline{OA} = \boldsymbol{r}$ whose origin is coincident with the coordinate origin and whose terminus is a certain point $A(x, y, z)$ (Fig. 192). A vector of this kind is called a *radius vector*.

Let us express this vector in terms of the projections on the coordinate axes:

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}. \qquad (1)$$

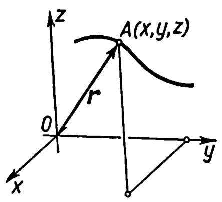


Fig. 192.

Let the projections of the vector $\boldsymbol{r}$ be functions of some parameter t:

$$\left.\begin{aligned} x &= \varphi(t), \\ y &= \psi(t), \\ z &= \chi(t). \end{aligned}\right\} \qquad (2)$$

Then formula (1) may be rewritten as follows:

$$\boldsymbol{r} = \varphi(t)\boldsymbol{i} + \psi(t)\boldsymbol{j} + \chi(t)\boldsymbol{k} \qquad (1')$$

or, in abbreviated form,

$$\boldsymbol{r} = \boldsymbol{r}(t). \qquad (1'')$$

As t varies, x, y, and z vary; and the point A (the terminus of the vector $\boldsymbol{r}$) will trace out a line in space that is called the *hodograph* of the vector $\boldsymbol{r} = \boldsymbol{r}(t)$. Equation (1′) or (1″) is called the *vector equation* of the line in space. Equations (2) are known as the *parametric equations* of the line in space. With the aid of these equations, the coordinates x, y, z of the corresponding point of the curve are determined for each value of t.

Note. A curve in space can also be defined as the locus of points of the intersection of two surfaces. It can therefore be given by two equations of two surfaces:

$$\left.\begin{aligned} \Phi_1(x, y, z) &= 0, \\ \Phi_2(x, y, z) &= 0. \end{aligned}\right\} \qquad (3)$$

Thus, for example, the equations

$$x^2 + y^2 + z^2 = 4, \quad z = 1$$

are the equations of a circle obtained at the intersection of a sphere and a plane (Fig. 193).

Thus, a curve in space may be represented either by parametric equations (2) or by two equations of surfaces (3).

If we eliminate the parameter t from equations (2) and get two equations connecting x, y, z, we will thus make the transition from the parametric method of representing a line to the surface

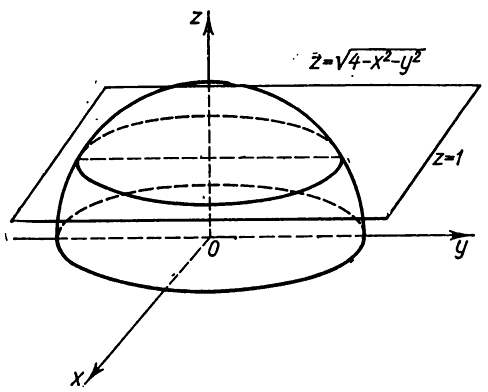


Fig. 193.

method. And conversely, if we put $x=\varphi(t)$, where $\varphi(t)$ is an arbitrary function, and find y and z as functions of t from equations

$$\Phi_1[\varphi(t), y, z]=0, \quad \Phi_2[\varphi(t), y, z]=0,$$

we will then make the transition from representation of a line by means of surfaces to its parametric representation.

Example 1. The equations

$$x=4t-1, \quad y=3t, \quad z=t+2$$

are parametric equations of a straight line. Eliminating the parameter t we get two equations, each of which is an equation of a plane. For instance, if from the first equation we subtract, termwise, the second and third, we get $x-y-z=-3$. But subtracting (from the first) four times the second we get $x-4z=-9$. Thus, the given straight line is the line of intersection of the planes $x-y-z+3=0$ and $x-4z+9=0$.

Example 2. Let us consider a right circular cylinder of radius a, whose axis coincides with the z-axis (Fig. 194). Onto this cylinder we wind a right triangle C_1AC so that the vertex A of the triangle lies at the point of intersection of the generator of the cylinder with the x-axis, while the leg AC_1 is wound onto the circular section of the cylinder lying in the xy-plane. Then the hypotenuse will generate on the cylinder a line that is called a helix.

Let us write the equation of the helix, denoting by x, y, and z the coordinates of its variable point M and by t the angle AOP (see Fig. 194). Then

$$x = a\cos t,\quad y = a\sin t,\quad z = PM = \widehat{AP}\ \tan\theta,$$

where θ denotes the acute angle of the triangle C_1AC. Noting that $\widehat{AP} = at$, since $\widehat{AP}$ is an arc of the circle of radius a corresponding to the central angle t, and designating $\tan\theta$ in terms of m, we get the parametric equations of the helix in the form

$$x = a\cos t,\quad y = a\sin t,\quad z = amt$$

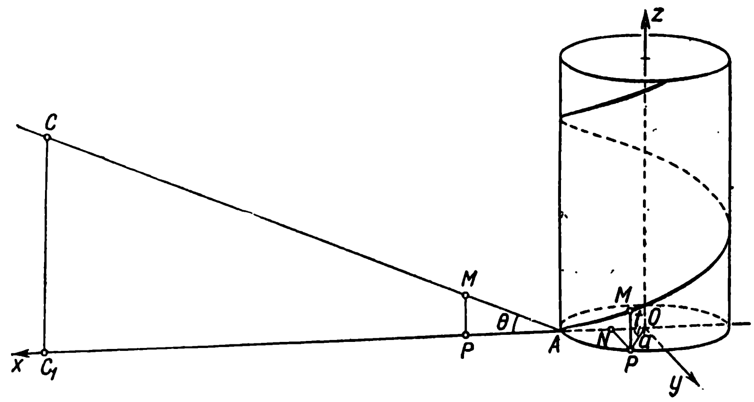


Fig. 194.

(here t is the parameter), or in the vector form:

$$\mathbf{r} = \mathbf{i}a\cos t + \mathbf{j}a\sin t + \mathbf{k}amt.$$

It is not difficult to eliminate the parameter t from the parametric equations of the helix: square the first two equations and add. We find $x^2 + y^2 = a^2$. This is the equation of the cylinder on which the helix lies. Then, dividing termwise the second equation by the first and substituting into the obtained equation the value of t found from the third equation, we find the equation of another surface on which the helix lies:

$$\frac{y}{x} = \tan\frac{z}{am}.$$

This is the so-called helicoid. It is generated as the trace of a half-line parallel to the xy-plane if the end point of this half-line lies on the z-axis and if the half-line itself rotates about the z-axis at a constant angular velocity, and rises with constant velocity so that its extremity is translated along the z-axis. The helix is the line of intersection of these two surfaces, and so can be represented by two equations:

$$x^2 + y^2 = a^2,\quad \frac{y}{x} = \tan\frac{z}{am}.$$

SEC. 2. THE LIMIT AND DERIVATIVE OF THE VECTOR FUNCTION OF A SCALAR ARGUMENT. THE EQUATION OF A TANGENT TO A CURVE. THE EQUATION OF A NORMAL PLANE

Reverting to the formulas (1′) and (1″) of the preceding section, we have

$$\boldsymbol{r}=\varphi(t)\,\boldsymbol{i}+\psi(t)\,\boldsymbol{j}+\chi(t)\,\boldsymbol{k}$$

or

$$\boldsymbol{r}=\boldsymbol{r}(t).$$

When t varies, the vector $\boldsymbol{r}$ varies in the general case both in magnitude and direction. We say that $\boldsymbol{r}$ is *a vector function* of the scalar argument t. Let us suppose that

$$\lim_{t\to t_0}\varphi(t)=\varphi_0,$$
$$\lim_{t\to t_0}\psi(t)=\psi_0,$$
$$\lim_{t\to t_0}\chi(t)=\chi_0.$$

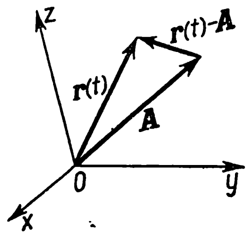


Fig. 195.

Then we say that the vector $\boldsymbol{r}_0=\varphi_0\boldsymbol{i}+\psi_0\boldsymbol{j}+$ $+\chi_0\boldsymbol{k}$ is the limit of the vector $\boldsymbol{r}=\boldsymbol{r}(t)$ and we write (Fig. 195)

$$\lim_{t\to t_0}\boldsymbol{r}(t)=\boldsymbol{r}_0.$$

From the latter equation follow the obvious equations

$$\lim_{t\to t_0}|\boldsymbol{r}(t)-\boldsymbol{r}_0|=\lim_{t\to t_0}\sqrt{[\varphi(t)-\varphi_0]^2+[\psi(t)-\psi_0]^2+[\chi(t)-\chi_0]^2}=0$$

and

$$\lim_{t\to t_0}|\boldsymbol{r}(t)|=|\boldsymbol{r}_0|.$$

Let us now take up the question of the derivative of the vector function of a scalar argument,

$$\boldsymbol{r}(t)=\varphi(t)\,\boldsymbol{i}+\psi(t)\,\boldsymbol{j}+\chi(t)\,\boldsymbol{k},\qquad(1)$$

assuming that the origin of the vector $\boldsymbol{r}(t)$ lies at the coordinate origin. We know that the latter equation is the equation of some space curve.

Let us take some fixed value t corresponding to a definite point M on the curve, and let us change t by the increment Δt; we then get the vector

$$\boldsymbol{r}(t+\Delta t)=\varphi(t+\Delta t)\,\boldsymbol{i}+\psi(t+\Delta t)\,\boldsymbol{j}+\chi(t+\Delta t)\,\boldsymbol{k},$$

which defines a certain point M_1 on the curve (Fig. 196). Let us find the increment of the vector

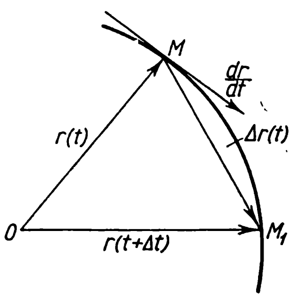


Fig. 196.

$$\Delta \boldsymbol{r} = \boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t) = \\ = [\varphi(t+\Delta t) - \varphi(t)]\,\boldsymbol{i} + \\ + [\psi(t+\Delta t) - \psi(t)]\,\boldsymbol{j} + \\ + [\chi(t+\Delta t) - \chi(t)]\,\boldsymbol{k}.$$

In Fig. 196, where $\overline{OM} = \boldsymbol{r}(t)$, $\overline{OM_1} = \boldsymbol{r}(t+\Delta t)$, this increment is shown by the vector $\overline{MM_1} = \Delta\boldsymbol{r}(t)$.

Let us consider the ratio $\frac{\Delta \boldsymbol{r}(t)}{\Delta t}$ of the increment of a vector function to the increment of a scalar argument; this is obviously a vector collinear with the vector $\Delta\boldsymbol{r}(t)$, since it is obtained from the latter by multiplication with the scalar factor $\frac{1}{\Delta t}$. We can write this vector as follows:

$$\frac{\Delta \boldsymbol{r}(t)}{\Delta t} = \frac{\varphi(t+\Delta t) - \varphi(t)}{\Delta t}\boldsymbol{i} + \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t}\boldsymbol{j} + \frac{\chi(t+\Delta t) - \chi(t)}{\Delta t}\boldsymbol{k}.$$

If the functions $\varphi(t)$, $\psi(t)$, $\chi(t)$ have derivatives for the chosen value of t, the factors of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ will in the limit become the derivatives $\varphi'(t)$, $\psi'(t)$, $\chi'(t)$ as $\Delta t \to 0$. Therefore, in this case the limit of $\frac{\Delta \boldsymbol{r}}{\Delta t}$ as $\Delta t \to 0$ exists and is equal to the vector $\varphi'(t)\boldsymbol{i} + \psi'(t)\boldsymbol{j} + \chi'(t)\boldsymbol{k}$:

$$\lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \varphi'(t)\,\boldsymbol{i} + \psi'(t)\,\boldsymbol{j} + \chi'(t)\,\boldsymbol{k}.$$

The vector defined by the latter equation is called the *derivative* of the vector $\boldsymbol{r}(t)$ with respect to the scalar argument t. The derivative is denoted by the symbol $\frac{d\boldsymbol{r}}{dt}$ or $\boldsymbol{r}'$.

Thus,

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{r}' = \varphi'(t)\,\boldsymbol{i} + \psi'(t)\,\boldsymbol{j} + \chi'(t)\,\boldsymbol{k} \tag{2}$$

or

$$\frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}. \tag{2'}$$

Let us determine the direction of the vector $\frac{d\boldsymbol{r}}{dt}$.

Since as $\Delta t \to 0$ the point M_1 approaches M, the direction of the secant MM_1 yields, in the limit, the direction of the tangent.

Hence, the vector of the derivative $\frac{dr}{dt}$ lies along the tangent to the curve at M. The length of the vector $\frac{dr}{dt}$ is defined by the formula *)

$$\left|\frac{dr}{dt}\right| = \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2 + [\chi'(t)]^2}. \qquad (3)$$

From the results obtained it is easy to write the equation of the tangent to the curve

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

at the point $M(x, y, z)$, bearing in mind that in the equation of the curve $x = \varphi(t)$, $y = \psi(t)$, $z = \chi(t)$.

The equation of the straight line passing through the point $M(x, y, z)$ is of the form

$$\frac{X-x}{m} = \frac{Y-y}{n} = \frac{Z-z}{p},$$

where X, Y, Z are the coordinates of the variable point of the straight line, while m, n, and p are quantities proportional to the direction cosines of this straight line (that is to say, to the projections of the directional vector of the straight line).

On the other hand, we have established that the vector

$$\frac{dr}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$$

is directed along the tangent. For this reason, the projections of this vector are numbers that are proportional to the direction cosines of the tangent, hence also to the numbers m, n, p. Thus, the equation of the tangent will be of the form

$$\frac{X-x}{\frac{dx}{dt}} = \frac{Y-y}{\frac{dy}{dt}} = \frac{Z-z}{\frac{dz}{dt}}. \qquad (4)$$

Example 1. Write the equation of a tangent to the helix

$$x = a\cos t, \quad y = a\sin t, \quad z = amt$$

for an arbitrary value of t and for $t = \frac{\pi}{4}$.

Solution.

$$\frac{dx}{dt} = -a\sin t, \quad \frac{dy}{dt} = a\cos t, \quad \frac{dz}{dt} = am.$$

*) We shall assume that at the points under consideration $\left|\frac{dr}{dt}\right| \neq 0$.

From formula (4) we have

$$\frac{X - a\cos t}{-a\sin t} = \frac{Y - a\sin t}{a\cos t} = \frac{Z - amt}{am}.$$

In particular, for $t = \frac{\pi}{4}$ we get

$$\frac{X - \frac{a\sqrt{2}}{2}}{-\frac{a\sqrt{2}}{2}} = \frac{Y - \frac{a\sqrt{2}}{2}}{\frac{a\sqrt{2}}{2}} = \frac{Z - am\frac{\pi}{4}}{am}.$$

Just as in the case of a plane curve, a straight line perpendicular to a tangent and passing through the point of tangency is called a *normal* to the space curve at the given point. Obviously, one can draw an infinitude of normals to a given space curve at a given point. They all lie in the plane perpendicular to the tangent line. This plane is the *normal plane.*

From the condition of perpendicularity of a normal plane to a tangent (4), we get the equation of the normal plane:

$$\frac{dx}{dt}(X - x) + \frac{dy}{dt}(Y - y) + \frac{dz}{dt}(Z - z) = 0. \qquad (5)$$

Example 2. Write the equation of a normal plane to a helix at a point for which $t = \frac{\pi}{4}$.

Solution. From Example 1 and formula (5) we get

$$-\frac{\sqrt{2}}{2}\left(X - \frac{a\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}\left(Y - \frac{a\sqrt{2}}{2}\right) + m\left(Z - am\frac{\pi}{4}\right) = 0.$$

Let us now derive the equation of a tangent line and the normal plane of a space curve for the case when this curve is given by the equations

$$\Phi_1(x, y, z) = 0, \quad \Phi_2(x, y, z) = 0. \qquad (6)$$

Let us express the coordinates x, y, z of this curve as functions of some parameter t:

$$x = \varphi(t), \ y = \psi(t), \ z = \chi(t). \qquad (7)$$

We shall assume that $\varphi(t)$, $\psi(t)$, $\chi(t)$ are differentiable functions of t.

Substituting into equations (6), in place of x, y, z, their values for the points of the curve expressed in terms of t, we get two identities in t:

$$\Phi_1[\varphi(t), \psi(t), \chi(t)] = 0, \qquad (8a)$$
$$\Phi_2[\varphi(t), \psi(t), \chi(t)] = 0. \qquad (8b)$$

Differentiating the identities (8a) and (8b) with respect to t, we

get

$$\left.\begin{aligned}\frac{\partial\Phi_1}{\partial x}\frac{dx}{dt}+\frac{\partial\Phi_1}{\partial y}\frac{dy}{dt}+\frac{\partial\Phi_1}{\partial z}\frac{dz}{dt}=0,\\ \frac{\partial\Phi_2}{\partial x}\frac{dx}{dt}+\frac{\partial\Phi_2}{\partial y}\frac{dy}{dt}+\frac{\partial\Phi_2}{\partial z}\frac{dz}{dt}=0.\end{aligned}\right\} \tag{9}$$

From these equations it follows that

$$\frac{\frac{dx}{dt}}{\frac{dz}{dt}}=\frac{\frac{\partial\Phi_1}{\partial y}\frac{\partial\Phi_2}{\partial z}-\frac{\partial\Phi_1}{\partial z}\frac{\partial\Phi_2}{\partial y}}{\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_2}{\partial y}-\frac{\partial\Phi_1}{\partial y}\frac{\partial\Phi_2}{\partial x}};\quad \frac{\frac{dy}{dt}}{\frac{dz}{dt}}=\frac{\frac{\partial\Phi_1}{\partial z}\frac{\partial\Phi_2}{\partial z}-\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_2}{\partial z}}{\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_2}{\partial y}-\frac{\partial\Phi_1}{\partial y}\frac{\partial\Phi_2}{\partial x}}. \tag{10}$$

Here, we naturally assume that the expression $\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_2}{\partial y}-\frac{\partial\Phi_1}{\partial y}\frac{\partial\Phi_2}{\partial x}\neq$ $\neq 0$; however, it may be proved that the final formulas (11) and (12) (see below) hold also for the case when this expression is equal to zero, provided that at least one of the determinants in the final formulas differs from zero.

From equations (10) we have

$$\frac{\frac{dx}{dt}}{\frac{\partial\Phi_1}{\partial y}\frac{\partial\Phi_2}{\partial z}-\frac{\partial\Phi_1}{\partial z}\frac{\partial\Phi_2}{\partial y}}=\frac{\frac{dy}{dt}}{\frac{\partial\Phi_1}{\partial z}\frac{\partial\Phi_2}{\partial x}-\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_2}{\partial z}}=\frac{\frac{dz}{dt}}{\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_2}{\partial y}-\frac{\partial\Phi_1}{\partial y}\frac{\partial\Phi_2}{\partial x}}.$$

Consequently, from formula (4) the equation of the tangent line will have the form

$$\frac{X-x}{\frac{\partial\Phi_1}{\partial y}\frac{\partial\Phi_2}{\partial z}-\frac{\partial\Phi_1}{\partial z}\frac{\partial\Phi_2}{\partial y}}=\frac{Y-y}{\frac{\partial\Phi_1}{\partial z}\frac{\partial\Phi_2}{\partial x}-\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_2}{\partial z}}=\frac{Z-z}{\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_2}{\partial y}-\frac{\partial\Phi_1}{\partial y}\frac{\partial\Phi_2}{\partial x}},$$

or, using determinants,

$$\frac{X-x}{\begin{vmatrix}\frac{\partial\Phi_1}{\partial y} & \frac{\partial\Phi_1}{\partial z}\\ \frac{\partial\Phi_2}{\partial y} & \frac{\partial\Phi_2}{\partial z}\end{vmatrix}}=\frac{Y-y}{\begin{vmatrix}\frac{\partial\Phi_1}{\partial z} & \frac{\partial\Phi_1}{\partial x}\\ \frac{\partial\Phi_2}{\partial z} & \frac{\partial\Phi_2}{\partial x}\end{vmatrix}}=\frac{Z-z}{\begin{vmatrix}\frac{\partial\Phi_1}{\partial x} & \frac{\partial\Phi_1}{\partial y}\\ \frac{\partial\Phi_2}{\partial x} & \frac{\partial\Phi_2}{\partial y}\end{vmatrix}}. \tag{11}$$

The normal plane is represented by the equation

$$(X-x)\begin{vmatrix}\frac{\partial\Phi_1}{\partial y} & \frac{\partial\Phi_1}{\partial z}\\ \frac{\partial\Phi_2}{\partial y} & \frac{\partial\Phi_2}{\partial z}\end{vmatrix}+(Y-y)\begin{vmatrix}\frac{\partial\Phi_1}{\partial z} & \frac{\partial\Phi_1}{\partial x}\\ \frac{\partial\Phi_2}{\partial z} & \frac{\partial\Phi_2}{\partial x}\end{vmatrix}+(Z-z)\begin{vmatrix}\frac{\partial\Phi_1}{\partial x} & \frac{\partial\Phi_1}{\partial y}\\ \frac{\partial\Phi_2}{\partial x} & \frac{\partial\Phi_2}{\partial y}\end{vmatrix}=0. \tag{12}$$

These formulas are meaningful only when at least one of the determinants involved is different from zero. But if at some point

of the curve all three determinants

$$\begin{vmatrix} \frac{\partial \Phi_1}{\partial y} & \frac{\partial \Phi_1}{\partial z} \\ \frac{\partial \Phi_2}{\partial y} & \frac{\partial \Phi_2}{\partial z} \end{vmatrix}, \quad \begin{vmatrix} \frac{\partial \Phi_1}{\partial z} & \frac{\partial \Phi_1}{\partial x} \\ \frac{\partial \Phi_2}{\partial z} & \frac{\partial \Phi_2}{\partial x} \end{vmatrix}, \quad \begin{vmatrix} \frac{\partial \Phi_1}{\partial x} & \frac{\partial \Phi_1}{\partial y} \\ \frac{\partial \Phi_2}{\partial x} & \frac{\partial \Phi_2}{\partial y} \end{vmatrix}$$

vanish, this point is called a *singular point* of the space curve. At this point the curve may not have a tangent at all, as was the case with singular points in plane curves (see Sec. 19, Ch. VIII).

Example 3. Find the equations of a tangent line and a normal plane to the line of intersection of the sphere $x^2+y^2+z^2=4r^2$ and the cylinder $x^2+y^2=2ry$ at the point $M(r, r, r\sqrt{2})$ (Fig. 197).

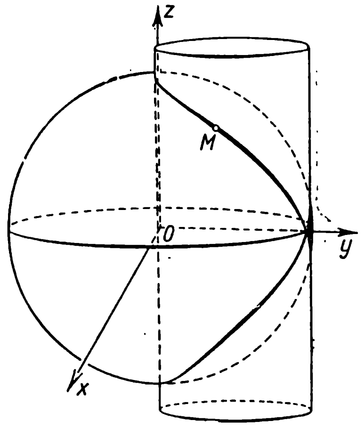


Fig. 197.

Solution.

$$\Phi_1(x, y, z)=x^2+y^2+z^2-4r^2,$$
$$\Phi_2(x, y, z)=x^2+y^2-2ry,$$

$$\frac{\partial \Phi_1}{\partial x}=2x, \quad \frac{\partial \Phi_1}{\partial y}=2y, \quad \frac{\partial \Phi_1}{\partial z}=2z,$$
$$\frac{\partial \Phi_2}{\partial x}=2x, \quad \frac{\partial \Phi_2}{\partial y}=2y-2r, \quad \frac{\partial \Phi_2}{\partial z}=0.$$

The values of the derivatives at the given point M will be

$$\frac{\partial \Phi_1}{\partial x}=2r, \quad \frac{\partial \Phi_1}{\partial y}=2r, \quad \frac{\partial \Phi_1}{\partial z}=2r\sqrt{2},$$
$$\frac{\partial \Phi_2}{\partial x}=2r, \quad \frac{\partial \Phi_2}{\partial y}=0, \quad \frac{\partial \Phi_2}{\partial z}=0.$$

For this reason the equation of the tangent line has the form

$$\frac{X-r}{0}=\frac{Y-r}{\sqrt{2}}=\frac{Z-r\sqrt{2}}{-1}.$$

The equation of the normal plane is

$$\sqrt{2}(Y-r)-(Z-r\sqrt{2})=0.$$

SEC. 3. RULES FOR DIFFERENTIATING VECTORS (VECTOR FUNCTIONS)

As we have seen, the derivative of a vector

$$\boldsymbol{r}(t)=\varphi(t)\boldsymbol{i}+\psi(t)\boldsymbol{j}+\chi(t)\boldsymbol{k}, \tag{1}$$

is, by definition, equal to

$$\boldsymbol{r}'(t)=\varphi'(t)\boldsymbol{i}+\psi'(t)\boldsymbol{j}+\chi'(t)\boldsymbol{k}. \tag{2}$$

Whence it straightway follows that the basic rules for differentiating functions hold for vectors as well. Here, we shall derive the formulas for differentiating a sum and a scalar product of vectors; the other formulas we shall write down and leave their derivation for the student.

I. *The derivative of a sum of vectors is equal to the sum of the derivatives of the vectors.*

Indeed, let there be two vectors:

$$\left.\begin{aligned} \boldsymbol{r}_1(t) &= \varphi_1(t)\,\boldsymbol{i} + \psi_1(t)\,\boldsymbol{j} + \chi_1(t)\,\boldsymbol{k}, \\ \boldsymbol{r}_2(t) &= \varphi_2(t)\,\boldsymbol{i} + \psi_2(t)\,\boldsymbol{j} + \chi_2(t)\,\boldsymbol{k}; \end{aligned}\right\} \tag{3}$$

their sum is

$$\boldsymbol{r}_1(t) + \boldsymbol{r}_2(t) = [\varphi_1(t) + \varphi_2(t)]\,\boldsymbol{i} + [\psi_1(t) + \psi_2(t)]\,\boldsymbol{j} + [\chi_1(t) + \chi_2(t)]\,\boldsymbol{k}.$$

By the definition of a derivative of a variable vector, we have

$$\frac{d\,[\boldsymbol{r}_1(t) + \boldsymbol{r}_2(t)]}{dt} = [\varphi_1(t) + \varphi_2(t)]'\,\boldsymbol{i} + [\psi_1(t) + \psi_2(t)]'\boldsymbol{j} + [\chi_1(t) + \chi_2(t)]'\boldsymbol{k}$$

or

$$\frac{d\,[\boldsymbol{r}_1(t) + \boldsymbol{r}_2(t)]}{dt} = [\varphi_1'(t) + \varphi_2'(t)]\,\boldsymbol{i} + [\psi_1'(t) + \psi_2'(t)]\,\boldsymbol{j} + [\chi_1'(t) + \chi_2'(t)]\boldsymbol{k} =$$

$$= \varphi_1'(t)\,\boldsymbol{i} + \psi_1'(t)\,\boldsymbol{j} + \chi_1'(t)\,\boldsymbol{k} + \varphi_2'(t)\,\boldsymbol{i} + \psi_2'(t)\,\boldsymbol{j} + \chi_2'(t)\,\boldsymbol{k} = \boldsymbol{r}_1' + \boldsymbol{r}_2'.$$

Hence,

$$\frac{d\,[\boldsymbol{r}_1(t) + \boldsymbol{r}_2(t)]}{dt} = \frac{d\boldsymbol{r}_1}{dt} + \frac{d\boldsymbol{r}_2}{dt}. \tag{I}$$

II. *The derivative of a scalar product of vectors is expressed by the formula*

$$\frac{d\,(\boldsymbol{r}_1\boldsymbol{r}_2)}{dt} = \frac{d\boldsymbol{r}_1}{dt}\,\boldsymbol{r}_2 + \boldsymbol{r}_1\,\frac{d\boldsymbol{r}_2}{dt}. \tag{II}$$

Indeed, if $\boldsymbol{r}_1(t)$, $\boldsymbol{r}_2(t)$ are defined by formulas (3), then, as we know, the scalar product of these vectors is equal to

$$\boldsymbol{r}_1(t)\,\boldsymbol{r}_2(t) = \varphi_1\varphi_2 + \psi_1\psi_2 + \chi_1\chi_2.$$

For this reason

$$\frac{d\,(\boldsymbol{r}_1\boldsymbol{r}_2)}{dt} = \varphi_1'\varphi_2 + \varphi_1\varphi_2' + \psi_1'\psi_2 + \psi_1\psi_2' + \chi_1'\chi_2 + \chi_1\chi_2' =$$

$$= (\varphi_1'\varphi_2 + \psi_1'\psi_2 + \chi_1'\chi_2) + (\varphi_1\varphi_2' + \psi_1\psi_2' + \chi_2\chi_2') =$$

$$= (\varphi_1'\boldsymbol{i} + \psi_1'\boldsymbol{j} + \chi_1'\boldsymbol{k})(\varphi_2\boldsymbol{i} + \psi_2\boldsymbol{j} + \chi_2\boldsymbol{k}) + (\varphi_1\boldsymbol{i} + \psi_1\boldsymbol{j} + \chi_1\boldsymbol{k})(\varphi_2'\boldsymbol{i} + \psi_2'\boldsymbol{j} + \chi_2'\boldsymbol{k}) =$$

$$= \frac{d\boldsymbol{r}_1}{dt}\,\boldsymbol{r}_2 + \boldsymbol{r}_1\,\frac{d\boldsymbol{r}_2}{dt}.$$

The theorem is proved.

From formula (II) we have the following important corollary.

Corollary. *If the vector* $\boldsymbol{e}$ *is a unit vector, that is,* $|\boldsymbol{e}| = 1$, *then its derivative is a vector perpendicular to it.*

Proof. It $\boldsymbol{e}$ is a unit vector, then

$$\boldsymbol{ee}=1.$$

Let us take the derivative, with respect to t, of both sides of the latter equation:

$$\boldsymbol{e}\frac{d\boldsymbol{e}}{dt}+\frac{d\boldsymbol{e}}{dt}\boldsymbol{e}=0,$$

or

$$2\boldsymbol{e}\frac{d\boldsymbol{e}}{dt}=0,$$

that is, the scalar product

$$\boldsymbol{e}\frac{d\boldsymbol{e}}{dt}=0,$$

and this means that the vector $\frac{d\boldsymbol{e}}{dt}$ is perpendicular to the vector $\boldsymbol{e}$.

III. *The constant numerical factor may be taken outside the sign of the derivative:*

$$\frac{da\boldsymbol{r}(t)}{dt}=a\frac{d\boldsymbol{r}(t)}{dt}=a\boldsymbol{r}'(t). \qquad \text{(III)}$$

IV. *The derivative of a vector product of vectors* $\boldsymbol{r}_1$ *and* $\boldsymbol{r}_2$ *is determined by the formula*

$$\frac{d[\boldsymbol{r}_1\times\boldsymbol{r}_2]}{dt}=\frac{d\boldsymbol{r}_1}{dt}\times\boldsymbol{r}_2+\boldsymbol{r}_1\times\frac{d\boldsymbol{r}_2}{dt}. \qquad \text{(IV)}$$

SEC. 4. THE FIRST AND SECOND DERIVATIVES OF A VECTOR WITH RESPECT TO THE ARC LENGTH. THE CURVATURE OF A CURVE. THE PRINCIPAL NORMAL

The arc length *) of a space curve $\widehat{M_0A}=s$ (Fig. 198) is determined just as in the case of curves in a plane. When a variable point $A(x, y, z)$ moves along a curve, the arc length s varies; conversely, when s varies, the coordinates x, y, z of a variable point A lying on the curve also vary. Therefore, the coordinates x, y, z of a variable point A of the curve may be regarded as functions of the arc length s:

$$x=\varphi(s),$$
$$y=\psi(s),$$
$$z=\chi(s).$$

*) The arc length of a space curve is defined in exactly the same way as the arc length of a plane curve (see Sec. 1, Ch. VI and Sec. 3, Ch. XII).

In these parametric equations of the curve, the arc length s is the parameter. The vector $\overline{OA}=\boldsymbol{r}$ is, accordingly, expressed as

$$\boldsymbol{r}=\varphi(s)\,\boldsymbol{i}+\psi(s)\,\boldsymbol{j}+\chi(s)\,\boldsymbol{k}$$

or

$$\boldsymbol{r}=\boldsymbol{r}(s). \tag{1}$$

Thus the vector $\boldsymbol{r}$ is a function of the arc length s.

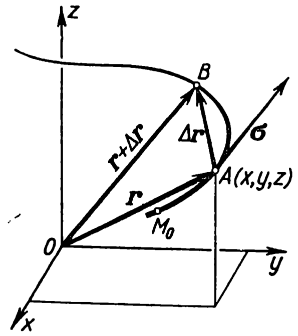


Fig. 198.

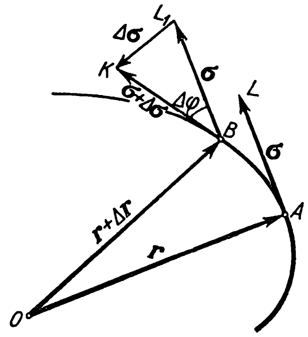


Fig. 199.

Let us find out the geometrical meaning of the derivative $\frac{d\boldsymbol{r}}{ds}$. As is evident from Fig. 198, we have the following equations:

$$\overset{\frown}{M_0A}=s,\quad \overset{\frown}{AB}=\Delta s,\quad \overset{\frown}{M_0B}=s+\Delta s,$$

$$\overline{OA}=\boldsymbol{r}(s),\quad \overline{OB}=\boldsymbol{r}(s+\Delta s),$$

$$\overline{AB}=\Delta\boldsymbol{r}=\boldsymbol{r}(s+\Delta s)-\boldsymbol{r}(s),$$

$$\frac{\Delta\boldsymbol{r}}{\Delta s}=\frac{\overline{AB}}{\overset{\frown}{AB}}.$$

We have already seen in Sec. 2 that the vector $\frac{d\boldsymbol{r}}{ds}=\lim\limits_{\Delta s\to 0}\frac{\Delta\boldsymbol{r}}{\Delta s}$ is in the direction of the tangent to the curve at the point A towards increasing s. On the other hand, we have the equality $\lim\left|\frac{\overline{AB}}{\overset{\frown}{AB}}\right|=1$ [the limit of the ratio of the chord length to the arc length *)].

*) In Sec. 1, Ch. VI, we mentioned this relation for a plane curve. It also holds for a space curve: $\boldsymbol{r}(t)=\varphi(t)\,\boldsymbol{i}+\psi(t)\,\boldsymbol{j}+\chi(t)\,\boldsymbol{k}$ if the functions $\varphi(t)$, $\psi(t)$ and $\chi(t)$ have continuous derivatives that do not vanish simultaneously.

Hence, $\frac{d\boldsymbol{r}}{ds}$ is a **unit** vector in the direction of the tangent; let us denote it by $\boldsymbol{\sigma}$:

$$\frac{d\boldsymbol{r}}{ds}=\boldsymbol{\sigma}. \tag{2}$$

If the vector $\boldsymbol{r}$ is represented by the projections

$$\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k},$$

then

$$\boldsymbol{\sigma}=\frac{dx}{ds}\boldsymbol{i}+\frac{dy}{ds}\boldsymbol{j}+\frac{dz}{ds}\boldsymbol{k}, \tag{3}$$

and

$$\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2}=1.$$

Let us now examine the **second** derivative of the vector function $\frac{d^2\boldsymbol{r}}{ds^2}$, that is, the derivative with respect to $\frac{d\boldsymbol{r}}{ds}$, and determine its geometric significance.

From formula (2) it follows that

$$\frac{d^2\boldsymbol{r}}{ds^2}=\frac{d}{ds}\left[\frac{d\boldsymbol{r}}{ds}\right]=\frac{d\boldsymbol{\sigma}}{ds}.$$

Consequently, we have to find $\lim\limits_{\Delta s\to 0}\frac{\Delta\boldsymbol{\sigma}}{\Delta s}$.

From Fig. 199 we have $AB=\Delta s$, $\overline{AL}=\boldsymbol{\sigma}$, $\overline{BK}=\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}$. Draw from the point B the vector $\overline{BL_1}=\boldsymbol{\sigma}$. From the triangle BKL_1 we find

$$\overline{BK}=\overline{BL_1}+\overline{L_1K}$$

or

$$\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}=\boldsymbol{\sigma}+\overline{L_1K}.$$

Thus, $L_1K=\Delta\boldsymbol{\sigma}$. Since, by what has been proved, the length of the vector $\boldsymbol{\sigma}$ does not change, $|\boldsymbol{\sigma}|=|\boldsymbol{\sigma}+\Delta\boldsymbol{\sigma}|$; hence, the triangle BKL_1 is an isosceles triangle.

The angle $\Delta\varphi$ at the vertex of the triangle is the angle through which the tangent to the curve turns from the point A to the point B; in other words, it corresponds to the increment in the arc length Δs. From the triangle BKL_1 we find

$$L_1K=|\Delta\boldsymbol{\sigma}|=2|\boldsymbol{\sigma}|\left|\sin\frac{\Delta\varphi}{2}\right|=2\left|\sin\frac{\Delta\varphi}{2}\right|$$

(since $|\boldsymbol{\sigma}|=1$).

Divide both sides of the latter equation by Δs:

$$\left|\frac{\Delta\sigma}{\Delta s}\right| = 2\left|\frac{\sin\frac{\Delta\varphi}{2}}{\Delta s}\right| = \left|\frac{\sin\frac{\Delta\varphi}{2}}{\frac{\Delta\varphi}{2}}\right|\left|\frac{\Delta\varphi}{\Delta s}\right|.$$

Let us now pass to the limit on both sides of the latter equation as $\Delta s \longrightarrow 0$. On the left side we have

$$\lim_{\Delta s\to 0}\left|\frac{\Delta\sigma}{\Delta s}\right| = \left|\frac{d\sigma}{ds}\right|.$$

Then

$$\lim_{\Delta s\to 0}\left|\frac{\sin\frac{\Delta\varphi}{2}}{\frac{\Delta\varphi}{2}}\right| = 1,$$

since in this case we consider curves such that there exists a limit $\lim\limits_{\Delta s\to 0}\frac{\Delta\varphi}{\Delta s}$ and, consequently, $\Delta\varphi \longrightarrow 0$ as $\Delta s \longrightarrow 0$. Thus, after passing to the limit we have

$$\left|\frac{d\sigma}{ds}\right| = \lim_{\Delta s\to 0}\left|\frac{\Delta\varphi}{\Delta s}\right|. \tag{4}$$

The ratio of the angle of turn $\Delta\varphi$ of the tangent, when the point A goes to the point B, to the length Δs of the arc AB (in absolute value) is called (just as it is in the case of a plane curve) the *average curvature* of the given line on the segment AB:

$$\text{average curvature} = \left|\frac{\Delta\varphi}{\Delta s}\right|.$$

The limit of the average curvature as $\Delta s \longrightarrow 0$ is called the *curvature* of the line at the point A and is denoted by K:

$$K = \lim_{\Delta s\to 0}\left|\frac{\Delta\varphi}{\Delta s}\right|.$$

But then from (4) it follows that $\frac{d\sigma}{ds} = K$; which means that the length of the derivative of a unit vector*) of a tangent with respect to the arc length is equal to the curvature of the line at the given point. Since the vector σ is a unit vector, its derivative $\frac{d\sigma}{ds}$ is perpendicular to it (see Sec. 3, Ch. IX, Corollary).

*) It should be remembered that the derivative of a **vector** is a **vector** and for this reason we can speak of the **length** of the derivative.

Thus, the vector $\frac{d\sigma}{ds}$ is equal, in length, to the curvature of the curve, and, in direction, is perpendicular to the vector of the tangent.

Definition. The straight line that has the same direction as the vector $\frac{d\sigma}{ds}$ and passes through the corresponding point of the curve is called the *principal normal* of the curve at the given point. We denote by $\boldsymbol{n}$ the unit vector of this direction.

Since the length of the vector $\frac{d\sigma}{ds}$ is equal to K, which is the curvature of the curve, we have

$$\frac{d\sigma}{ds}=K\boldsymbol{n}.$$

The reciprocal of the curvature is called the *radius of curvature* of the line at the given point and is denoted by R, $\frac{1}{K}=R$. So we can write

$$\frac{d^2\boldsymbol{r}}{ds^2}=\frac{d\sigma}{ds}=\frac{\boldsymbol{n}}{R}. \tag{5}$$

From this formula it follows that

$$\frac{1}{R^2}=\left(\frac{d^2\boldsymbol{r}}{ds^2}\right)^2. \tag{6}$$

But

$$\frac{d^2\boldsymbol{r}}{ds^2}=\frac{d^2x}{ds^2}\boldsymbol{i}+\frac{d^2y}{ds^2}\boldsymbol{j}+\frac{d^2z}{ds^2}\boldsymbol{k}.$$

Hence,

$$\frac{1}{R}=\sqrt{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2}. \tag{6'}$$

This formula enables us to compute the curvature of a line at any point provided that this line is represented by parametric equations in which the parameter is the arc length s (in other words, if the radius vector of the variable point of the given line is expressed as a function of the arc length).

Let us consider the case when the radius vector $\boldsymbol{r}$ is expressed as a function of an arbitrary parameter t:

$$\boldsymbol{r}=\boldsymbol{r}(t).$$

In this case the arc length s will be regarded as a function of the parameter t. Then the curvature is computed as follows:

$$\frac{d\boldsymbol{r}}{dt}=\frac{d\boldsymbol{r}}{ds}\frac{ds}{dt}. \tag{7}$$

Since

$$\left|\frac{d\mathbf{r}}{ds}\right| = 1, \text{*)}$$

we have

$$\left(\frac{d\mathbf{r}}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2. \tag{8}$$

Differentiating the right and left sides of (8) and reducing by two, we get

$$\frac{d\mathbf{r}}{dt}\frac{d^2\mathbf{r}}{dt^2} = \frac{ds}{dt}\frac{d^2s}{dt^2}. \tag{9}$$

Further, from formula (7) it follows that

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{1}{\frac{ds}{dt}}.$$

Differentiate, with respect to s, both sides of this equation:

$$\frac{d^2\mathbf{r}}{ds^2} = \frac{d^2\mathbf{r}}{dt^2}\frac{1}{\left(\frac{ds}{dt}\right)^2} - \frac{d\mathbf{r}}{dt}\frac{\frac{d^2s}{dt^2}}{\left(\frac{ds}{dt}\right)^3}.$$

Substituting into formula (6) the expression obtained for $\frac{d^2\mathbf{r}}{ds^2}$ we get

$$\frac{1}{R^2} = \left[\frac{d^2\mathbf{r}}{dt^2}\frac{1}{\left(\frac{ds}{dt}\right)^2} - \frac{d\mathbf{r}}{dt}\frac{\frac{d^2s}{dt^2}}{\left(\frac{ds}{dt}\right)^3}\right]^2 =$$

$$= \frac{\left(\frac{d^2\mathbf{r}}{dt^2}\right)^2\left(\frac{ds}{dt}\right)^2 - 2\frac{d^2\mathbf{r}}{dt^2}\frac{d\mathbf{r}}{dt}\frac{ds}{dt}\frac{d^2s}{dt} + \left(\frac{d\mathbf{r}}{dt}\right)^2\left(\frac{d^2s}{dt^2}\right)^2}{\left(\frac{ds}{dt}\right)^6}.$$

*) This equation follows from the fact that $\left|\frac{d\mathbf{r}}{ds}\right| = \lim_{\Delta s \to 0}\left|\frac{\Delta \mathbf{r}}{\Delta s}\right|$. But $\Delta \mathbf{r}$ is a chord subtending an arc of length Δs. Therefore $\frac{\Delta r}{\Delta s}$ approaches 1 as $\Delta s \to 0$.

Expressing $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ by formulas (8) and (9) in terms of the derivatives of $\boldsymbol{r}(t)$, we get *)

$$\frac{1}{R^2}=\frac{\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)^2\left(\frac{d\boldsymbol{r}}{dt}\right)^2-\left(\frac{d^2\boldsymbol{r}}{dt^2}\frac{d\boldsymbol{r}}{dt}\right)^2}{\left\{\left(\frac{d\boldsymbol{r}}{dt}\right)^2\right\}^3}. \tag{10}$$

Formula (10) may be rewritten as follows: **)

$$K^2=\frac{1}{R^2}=\frac{\left[\frac{d\boldsymbol{r}}{dt}\times\frac{d^2\boldsymbol{r}}{dt^2}\right]^2}{\left\{\left(\frac{d\boldsymbol{r}}{dt}\right)^2\right\}^3}. \tag{11}$$

We have obtained a formula that enables us to calculate the curvature of a given line at any point for an **arbitrary** parametric representation of this curve.

If in a particular case the curve is a plane curve and lies in the xy-plane, then its parametric equations have the form

$$x=\varphi(t),$$
$$y=\psi(t),$$
$$z=0.$$

Putting these expressions of x, y, z into formula (11), we get the earlier derived (in Ch. VI) formula that yields the curvature of a **plane** curve represented parametrically:

$$K=\frac{|\varphi'(t)\psi''(t)-\psi'(t)\varphi''(t)|}{\{[\varphi'(t)]^2+[\psi'(t)]^2\}^{3/2}}.$$

Example. Compute the curvature of the helix

$$\boldsymbol{r}=\boldsymbol{i}\,a\cos t+\boldsymbol{j}\,a\sin t+\boldsymbol{k}\,amt$$

at an arbitrary point.

*) We transform the denominator as follows: $\left(\frac{ds}{dt}\right)^6=\left\{\left(\frac{ds}{dt}\right)^2\right\}^3=\left\{\left(\frac{d\boldsymbol{r}}{dt}\right)^2\right\}^3$. Here we cannot write $\left(\frac{d\boldsymbol{r}}{dt}\right)^6$. By $\left(\frac{d\boldsymbol{r}}{dt}\right)^2$ we mean the scalar square of the vector $\frac{d\boldsymbol{r}}{dt}$; by $\left\{\left(\frac{d\boldsymbol{r}}{dt}\right)^2\right\}^3$, the third power of $\left(\frac{d\boldsymbol{r}}{dt}\right)^2$. The expression $\left(\frac{d\boldsymbol{r}}{dt}\right)^6$ is meaningless.

**) We utilised the identity $\boldsymbol{a}^2\boldsymbol{b}^2-(\boldsymbol{ab})^2=(\boldsymbol{a}\times\boldsymbol{b})^2$ whose validity is readily recognisable if one rewrites the identity as follows: $a^2b^2-(ab\cos\varphi)^2=(ab\sin\varphi)^2$.

Solution.

$$\frac{d\boldsymbol{r}}{dt} = -\boldsymbol{i}\, a \sin t + \boldsymbol{j}\, a \cos t + \boldsymbol{k}\, am,$$

$$\frac{d^2\boldsymbol{r}}{dt^2} = -\boldsymbol{i}\, a \cos t - \boldsymbol{j}\, a \sin t,$$

$$\frac{d\boldsymbol{r}}{dt} \times \frac{d^2\boldsymbol{r}}{dt^2} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -a \sin t & a \cos t & am \\ -a \cos t & -a \sin t & 0 \end{vmatrix} = \boldsymbol{i} a^2 m \sin t - \boldsymbol{j}\, a^2 m \cos t + \boldsymbol{k} a^2,$$

$$\left(\frac{d\boldsymbol{r}}{dt} \times \frac{d^2\boldsymbol{r}}{dt^2}\right)^2 = a^4 (m^2 + 1),$$

$$\left(\frac{d\boldsymbol{r}}{dt}\right)^2 = a^2 \sin^2 t + a^2 \cos^2 t + a^2 m^2 = a^2 (1 + m^2).$$

Consequently,

$$\frac{1}{R^2} = \frac{a^4 (m^2 + 1)}{[a^2 (1 + m^2)]^3} = \frac{1}{a^2 (1 + m^2)^2},$$

whence

$$R = a (1 + m^2) = \text{const}.$$

Thus, the helix has a constant radius of curvature.

Note. If a curve lies in a plane, then without violating generality, we can assume that it lies in the xy-plane (this can always be achieved by transforming the coordinates). Now if the curve lies in the xy-plane, then $z = 0$; but then $\frac{d^2 z}{ds^2} = 0$ also and, consequently, the vector $\boldsymbol{n}$ likewise lies in the xy-plane. We thus conclude that if a curve lies in a plane then its principal normal lies in the same plane.

SEC. 5. OSCULATING PLANE. BINORMAL. TORSION

Definition 1. The plane passing through the tangent line and the principal normal to a given curve at the point A is called an *osculating plane* at the point A.

For the plane of a curve, the osculating plane coincides with the plane of the curve. But if the curve is not a plane curve, and if we take two points on it, P and P_1, we get two different osculating planes that form a dihedral angle μ. The bigger the angle μ, the more the curve differs in shape from a plane curve. To make this more precise, let us introduce another definition.

Definition 2. The normal (to a curve) perpendicular to an osculating plane is called a *binormal*.

On the binormal let us take a unit vector $\boldsymbol{b}$ and make its direction such that the vectors $\boldsymbol{\sigma}$, $\boldsymbol{n}$, $\boldsymbol{b}$ form a triple with the

same orientation as the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ lying on the coordinate axes (Figs. 200, 201).

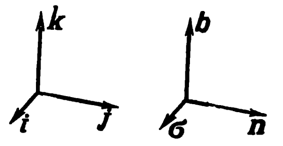


Fig. 200.

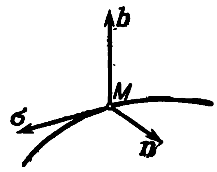


Fig. 201.

By virtue of the definition of a vector and scalar product of vectors we have

$$\boldsymbol{b}=\boldsymbol{\sigma}\times\boldsymbol{n};\quad \boldsymbol{bb}=1. \tag{1}$$

We find the derivative of $\frac{d\boldsymbol{b}}{ds}$. By formula (IV), Sec. 3,

$$\frac{d\boldsymbol{b}}{ds}=\frac{d(\boldsymbol{\sigma}\times\boldsymbol{n})}{ds}=\frac{d\boldsymbol{\sigma}}{ds}\times\boldsymbol{n}+\boldsymbol{\sigma}\times\frac{d\boldsymbol{n}}{ds}. \tag{2}$$

But $\frac{d\boldsymbol{\sigma}}{ds}=\frac{\boldsymbol{n}}{R}$ (see Sec. 4), therefore

$$\frac{d\boldsymbol{\sigma}}{ds}\times\boldsymbol{n}=\frac{1}{R}\,\boldsymbol{n}\times\boldsymbol{n}=0,$$

and formula (2) takes the form

$$\frac{d\boldsymbol{b}}{ds}=\boldsymbol{\sigma}\times\frac{d\boldsymbol{n}}{ds}. \tag{3}$$

From this it follows (by the definition of a vector product) that $\frac{d\boldsymbol{b}}{ds}$ is a vector perpendicular to the vector of the tangent $\boldsymbol{\sigma}$. On the other hand, since $\boldsymbol{b}$ is a unit vector, $\frac{d\boldsymbol{b}}{ds}$ is perpendicular to $\boldsymbol{b}$ (see Sec. 3, Corollary).

This means that the vector $\frac{d\boldsymbol{b}}{ds}$ is perpendicular both to $\boldsymbol{\sigma}$ and to $\boldsymbol{b}$; that is, it is collinear with the vector $\boldsymbol{n}$.

Let us denote the length of the vector $\frac{d\boldsymbol{b}}{ds}$ by $\frac{1}{T}$; we put

$$\left|\frac{d\boldsymbol{b}}{ds}\right|=\frac{1}{T};$$

then

$$\frac{d\boldsymbol{b}}{ds}=\frac{1}{T}\,\boldsymbol{n}. \tag{4}$$

The quantity $\frac{1}{T}$ is the *torsion* of the given curve.

The dihedral angle μ between the osculating planes that correspond to two points of the curve is equal to the angle between the binormals. By analogy with formula (4), Sec. 4, Ch. IX, one can write

$$\left|\frac{d\boldsymbol{b}}{ds}\right| = \lim_{\Delta s \to 0} \frac{\mu}{|\Delta s|}.$$

To summarise, then, the torsion of a curve at a point A is equal, in absolute value, to the limit which is approached (as $\Delta s \to 0$), by the ratio of the angle μ between the osculating planes at the point A and the neighbouring point B to the length $|\Delta s|$ of the arc AB.

If the curve is **plane** then the osculating plane does not change its direction and, consequently, the torsion is equal to zero.

From the definition of torsion it is clear that it is a measure of the deviation of a space curve from a plane curve.

The quantity T is called the *radius of torsion* of the curve.

Let us find a formula for computing torsion. From (3) and (4) it follows that

$$\frac{1}{T}\boldsymbol{n} = \boldsymbol{\sigma} \times \frac{d\boldsymbol{n}}{ds}.$$

Multiplying scalarly both sides by $\boldsymbol{n}$, we get

$$\frac{1}{T}\boldsymbol{n}\boldsymbol{n} = \boldsymbol{n}\left[\boldsymbol{\sigma} \times \frac{d\boldsymbol{n}}{ds}\right].$$

On the right side of this equation we have the so-called mixed (or triple) product of three vectors $\boldsymbol{n}$, $\boldsymbol{\sigma}$ and $\frac{d\boldsymbol{n}}{ds}$. In a product of this kind the factors, as we know, may be circularly permuted. In addition, taking into consideration that $\boldsymbol{n}\boldsymbol{n} = 1$, we rewrite the latter equation in the following form:

$$\frac{1}{T} = \boldsymbol{\sigma}\left[\frac{d\boldsymbol{n}}{ds} \times \boldsymbol{n}\right]$$

or

$$\frac{1}{T} = -\boldsymbol{\sigma}\left[\boldsymbol{n} \times \frac{d\boldsymbol{n}}{ds}\right]. \qquad (5)$$

But since $\boldsymbol{n} = R\frac{d^2\boldsymbol{r}}{ds^2}$, we have

$$\frac{d\boldsymbol{n}}{ds} = R\frac{d^3\boldsymbol{r}}{ds^3} + \frac{dR}{ds}\frac{d^2\boldsymbol{r}}{ds^2}$$

and

$$\left[\boldsymbol{n}\times\frac{d\boldsymbol{n}}{ds}\right]=R\,\frac{d^2\boldsymbol{r}}{ds^2}\times\left\{R\,\frac{d^3\boldsymbol{r}}{ds^3}+\frac{dR}{ds}\,\frac{d^2\boldsymbol{r}}{ds^2}\right\}=$$
$$=R^2\left[\frac{d^2\boldsymbol{r}}{ds^2}\times\frac{d^3\boldsymbol{r}}{ds^3}\right]+R\,\frac{dR}{ds}\left[\frac{d^2\boldsymbol{r}}{ds^2}\times\frac{d^2\boldsymbol{r}}{ds^2}\right].$$

But since the vector product of a vector into itself is equal to zero,

$$\left[\frac{d^2\boldsymbol{r}}{ds^2}\times\frac{d^2\boldsymbol{r}}{ds^2}\right]=0.$$

Thus,

$$\left[\boldsymbol{n}\times\frac{d\boldsymbol{n}}{ds}\right]=R^2\left[\frac{d^2\boldsymbol{r}}{ds^2}\times\frac{d^3\boldsymbol{r}}{ds^3}\right].$$

Noting that $\boldsymbol{\sigma}=\frac{d\boldsymbol{r}}{ds}$ and reverting to (5), we get

$$\frac{1}{T}=-R^2\,\frac{d\boldsymbol{r}}{ds}\left[\frac{d^2\boldsymbol{r}}{ds^2}\times\frac{d^3\boldsymbol{r}}{ds^3}\right]. \tag{6}$$

If the factor $\boldsymbol{r}$ is expressed as a function of an arbitrary parameter t, it may be shown, *) much like was done in the preceding

---

*) Indeed,

$$\frac{d\boldsymbol{r}}{dt}=\frac{d\boldsymbol{r}}{ds}\frac{ds}{dt}.$$

Differentiating this equality once again with respect to t, we get

$$\frac{d^2\boldsymbol{r}}{dt^2}=\frac{d}{ds}\left(\frac{d\boldsymbol{r}}{ds}\right)\frac{ds}{dt}\frac{ds}{dt}+\frac{d\boldsymbol{r}}{ds}\frac{d^2s}{dt^2}=\frac{d^2\boldsymbol{r}}{ds^2}\left(\frac{ds}{dt}\right)^2+\frac{d\boldsymbol{r}}{ds}\,\frac{d^2s}{dt^2}.$$

Differentiate it once more with respect to t:

$$\frac{d^3\boldsymbol{r}}{dt^3}=\frac{d}{ds}\left(\frac{d^2\boldsymbol{r}}{ds}\right)\frac{ds}{dt}\left(\frac{ds}{dt}\right)^2+\frac{d^2\boldsymbol{r}}{ds^2}\cdot 2\frac{ds}{dt}\frac{d^2s}{dt^2}+\frac{d}{ds}\left(\frac{d\boldsymbol{r}}{ds}\right)\frac{ds}{dt}\frac{d^2s}{dt^2}+\frac{d\boldsymbol{r}}{ds}\,\frac{d^3s}{dt^3}=$$
$$=\frac{d^3\boldsymbol{r}}{ds^3}\left(\frac{ds}{dt}\right)^3+3\,\frac{d^2\boldsymbol{r}}{ds^2}\,\frac{ds}{dt}\frac{d^2s}{dt^2}+\frac{d\boldsymbol{r}}{ds}\frac{d^3s}{dt^3}.$$

Let us now form a triple product:

$$\frac{d\boldsymbol{r}}{dt}\left(\frac{d^2\boldsymbol{r}}{dt^2}\times\frac{d^3\boldsymbol{r}}{dt^3}\right)=$$
$$=\frac{d\boldsymbol{r}}{ds}\frac{ds}{dt}\left\{\left[\frac{d^2\boldsymbol{r}}{ds^2}\left(\frac{ds}{dt}\right)^2+\frac{d\boldsymbol{r}}{ds}\,\frac{d^2s}{dt^2}\right]\times\left[\frac{d^3\boldsymbol{r}}{ds^3}\left(\frac{ds}{ds^3}\right)^3+3\,\frac{d^2\boldsymbol{r}}{ds^2}\,\frac{ds}{dt}\frac{d^2s}{dt^2}+\frac{d\boldsymbol{r}}{ds}\,\frac{d^3\boldsymbol{r}}{dt^3}\right]\right\}.$$

Opening the brackets of this product by the rule of multiplying polynomials, and disregarding those terms that contain even two identical vector factors (since the triple product of three factors where at least two are equal is zero), we get

$$\frac{d\boldsymbol{r}}{dt}\left(\frac{d^2\boldsymbol{r}}{dt^2}\times\frac{d^3\boldsymbol{r}}{dt^3}\right)=\frac{d\boldsymbol{r}}{ds}\left(\frac{d^2\boldsymbol{r}}{ds^2}\times\frac{d^3\boldsymbol{r}}{ds^3}\right)\left(\frac{ds}{dt}\right)^6.$$

section, that

$$\frac{dr}{ds}\left[\frac{d^2r}{ds^2}\times\frac{d^3r}{ds^3}\right]=\frac{\frac{dr}{dt}\left[\frac{d^2r}{dt^2}\times\frac{d^3r}{dt^3}\right]}{\left\{\left(\frac{dr}{dt}\right)^2\right\}^3}.$$

Putting this expression into formula (6) and replacing R^2 by its expression from formula (11), Sec. 4, we finally get

$$\frac{1}{T}=-\frac{\frac{dr}{dt}\left[\frac{d^2r}{dt^2}\times\frac{d^3r}{dt^3}\right]}{\left[\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right]^2}. \qquad (7)$$

This formula makes it possible to compute the torsion of the curve at any point if the curve is represented by parametric equations with an arbitrary parameter t.

Concluding this section, we note that the formulas which express the derivatives of the vectors $\boldsymbol{\sigma}$, $\boldsymbol{b}$, $\boldsymbol{n}$ are called *Serret-Frenet formulas:*

$$\frac{d\boldsymbol{\sigma}}{ds}=\frac{\boldsymbol{n}}{R},\quad \frac{d\boldsymbol{b}}{ds}=\frac{\boldsymbol{n}}{T},\quad \frac{d\boldsymbol{n}}{ds}=-\frac{\boldsymbol{\sigma}}{R}-\frac{\boldsymbol{b}}{T}.$$

The last one of them is obtained as follows:

$$\boldsymbol{n}=\boldsymbol{b}\times\boldsymbol{\sigma},$$

$$\frac{d\boldsymbol{n}}{ds}=\frac{d(\boldsymbol{b}\times\boldsymbol{\sigma})}{ds}=\frac{d\boldsymbol{b}}{ds}\times\boldsymbol{\sigma}+\boldsymbol{b}\times\frac{d\boldsymbol{\sigma}}{ds}=\frac{\boldsymbol{n}}{T}\times\boldsymbol{\sigma}+\boldsymbol{b}\times\frac{\boldsymbol{n}}{R}=$$

$$=\frac{1}{T}\boldsymbol{n}\times\boldsymbol{\sigma}+\frac{1}{R}\boldsymbol{b}\times\boldsymbol{n};$$

but

$$\boldsymbol{n}\times\boldsymbol{\sigma}=-\boldsymbol{b};\quad \boldsymbol{b}\times\boldsymbol{n}=-\boldsymbol{\sigma},$$

therefore

$$\frac{d\boldsymbol{n}}{ds}=-\frac{\boldsymbol{b}}{T}-\frac{\boldsymbol{\sigma}}{R}.$$

Finally, noting that

$$\left(\frac{ds}{dt}\right)^2=\left(\frac{dr}{dt}\right)^2,$$

or

$$\left(\frac{ds}{dt}\right)^6=\left\{\left(\frac{dr}{dt}\right)^2\right\}^3,$$

we obtain the required equality.

Example. Compute the torsion of the helix

$$\mathbf{r}=\mathbf{i}a\cos t+\mathbf{j}a\sin t+\mathbf{k}\,amt.$$

Solution.

$$\frac{d\mathbf{r}}{dt}\left[\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^3\mathbf{r}}{dt^3}\right]=\begin{vmatrix}-a\sin t & a\cos t & am\\ -a\cos t & -a\sin t & 0\\ a\sin t & -a\cos t & 0\end{vmatrix}=a^3m,$$

$$\left[\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right]^2=a^4(1+m^2) \text{ (see Example, Sec. 4).}$$

Consequently,

$$T=-\frac{a^4(1+m^2)}{a^3m}=-\frac{a(1+m^2)}{m}.$$

SEC. 6. A TANGENT PLANE AND NORMAL TO A SURFACE

Let there be a surface given by an equation of the form

$$F(x,\ y,\ z)=0. \tag{1}$$

We introduce the following definition.

Definition 1. A straight line is a *tangent* to a surface at some point $P(x,\ y,\ z)$ if it is a tangent to some curve lying on the surface and passing through P.

Since an infinitude of different curves lying on the surface pass through the point P, then, generally speaking, there will also be an infinitude of tangents to the surface passing through this point.

We introduce the concept of singular and ordinary points of a surface $F(x,\ y,\ z)=0$.

If at the point $M(x,\ y,\ z)$ all three derivatives $\frac{\partial F}{\partial x},\ \frac{\partial F}{\partial y},\ \frac{\partial F}{\partial z}$ are equal to zero or at least one of these derivatives does not exist, then M is called a *singular* point of the surface. If at $M(x,\ y,\ z)$ all three derivatives $\frac{\partial F}{\partial x},\ \frac{\partial F}{\partial y},\ \frac{\partial F}{\partial z}$ exist and are continuous, and at least one of them differs from zero, then M is an *ordinary* point of the surface.

We can now formulate the following theorem.

Theorem. *All tangent lines to a given surface* (1) *at an ordinary point of it P lie in one plane.*

Proof. Let us consider, on a surface, a certain line L, (Fig. 202) passing through a given point P of the surface. Let this curve be represented by parametric equations:

$$x=\varphi(t);\quad y=\psi(t);\quad z=\chi(t). \tag{2}$$

A tangent to the curve will be a tangent to the surface. The equations of this tangent have the form

$$\frac{X-x}{\frac{dx}{dt}}=\frac{Y-y}{\frac{dy}{dt}}=\frac{Z-z}{\frac{dz}{dt}}.$$

If we put expressions (2) into equation (1), the latter will become an identity in t, since the curve (2) lies on the surface (1). Differentiating it with respect to t, we get*)

$$\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}=0. \qquad (3)$$

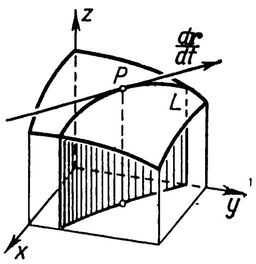


Fig. 202.

Let us further examine the vectors $\boldsymbol{N}$ and $\frac{d\boldsymbol{r}}{dt}$ that pass through P:

$$\boldsymbol{N}=\frac{\partial F}{\partial x}\boldsymbol{i}+\frac{\partial F}{\partial y}\boldsymbol{j}+\frac{\partial F}{\partial z}\boldsymbol{k}. \qquad (4)$$

The projections of this vector $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ depend on x, y, z, which are the coordinates of P; it will be noted that since P is an ordinary point, these projections at the point P do not simultaneously vanish and therefore

$$|\boldsymbol{N}|=\sqrt{\left(\frac{\partial F^2}{\partial x}\right)^2+\left(\frac{\partial F^2}{\partial y}\right)^2+\left(\frac{\partial F^2}{\partial z}\right)^2}\neq 0$$

The vector

$$\frac{d\boldsymbol{r}}{dt}=\frac{dx}{dt}\boldsymbol{i}+\frac{dy}{dt}\boldsymbol{j}+\frac{dz}{dt}\boldsymbol{k}, \qquad (5)$$

is tangent to the curve passing through the point P and lying on the surface. The projections of this vector are computed from equations (2) with the value of the parameter t corresponding to the point P. Let us compute the scalar product of the vectors $\boldsymbol{N}$ and $\frac{d\boldsymbol{r}}{dt}$, which product is equal to the sum of the products of like projections:

$$\boldsymbol{N}\frac{d\boldsymbol{r}}{dt}=\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}.$$

---

*) Here we apply the rule for differentiating a composite function of three variables. This rule is applicable here since all the partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ are, as stated, continuous.

On the basis of (3), the expression on the right is equal to zero; hence

$$N\frac{dr}{dt}=0.$$

From the latter equality it follows that the vector N and the tangent vector $\frac{dr}{dt}$ to the curve (2) at the point P are perpendicular. The foregoing reasoning holds for any curve (2) passing through the point P and lying on the surface. Therefore, every tangent to the surface at the point P is perpendicular to one and the same vector N and for this reason all these tangents lie in a single plane that is perpendicular to the vector N. The theorem is proved.

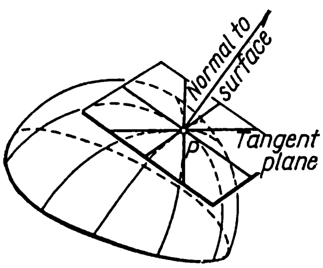


Fig. 203.

Definition 2. The plane in which lie all the tangent lines to the lines on the surface passing through the given point P is called the *tangent plane* to the surface at the point P (Fig. 203).

It should be noted that there may not exist a tangent plane at the singular points of the surface. At such points, the tangent lines to the surface may not lie in one plane. For instance, the vertex of a conical surface is a singular point. The tangents to the conical surface at this point do not lie in one plane (they themselves form a conical surface).

Let us write the equation of a tangent plane to a surface (1) at an ordinary point. Since this plane is perpendicular to the vector (4), its equation has the form

$$\frac{\partial F}{\partial x}(X-x)+\frac{\partial F}{\partial y}(Y-y)+\frac{\partial F}{\partial z}(Z-z)=0. \qquad (6)$$

If the equation of a surface is given in the form

$$z=f(x, y), \quad \text{or} \quad z-f(x, y)=0,$$

then

$$\frac{\partial F}{\partial x}=-\frac{\partial f}{\partial x}, \quad \frac{\partial F}{\partial y}=-\frac{\partial f}{\partial y}, \quad \frac{\partial F}{\partial z}=1,$$

and the equation of the tangent plane is then of the form

$$Z-z=\frac{\partial f}{\partial x}(X-x)+\frac{\partial f}{\partial y}(Y-y). \qquad (6')$$

Note. If in formula (6′) we put $X-x=\Delta x$; $Y-y=\Delta y$, then this formula will take the form

$$Z-z=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y;$$

its right side is the total differential of the function $z=f(x, y)$. Therefore, $Z-z=dz$. Thus, the total differential of a function of two variables at the point $M(x, y)$, which corresponds to the increments Δx and Δy of the independent variables x and y, is equal to the corresponding increment on the z-axis of the tangent plane to the surface which is a graph of the given function.

Definition 3. The straight line drawn through the point $P(x, y, z)$ of surface (1) perpendicular to the tangent plane is called the *normal* to the surface (Fig. 203).

Let us write the equations of the normal. Since its direction coincides with that of the vector $\boldsymbol{N}$, its equations will have the form

$$\frac{X-x}{\frac{\partial F}{\partial x}}=\frac{Y-y}{\frac{\partial F}{\partial y}}=\frac{Z-z}{\frac{\partial F}{\partial z}}. \tag{7}$$

If the equation of the surface is given in the form $z=f(x, y)$, or

$$z-f(x, y)=0,$$

then the equations of the normal have the form

$$\frac{X-x}{-\frac{\partial f}{\partial x}}=\frac{Y-y}{-\frac{\partial f}{\partial y}}=\frac{Z-z}{1}.$$

Note. Let the surface $F(x, y, z)=0$ be the level surface for some function of three variables $u=u(x, y, z)$; that is,

$$F(x, y, z)=u(x, y, z)-C=0.$$

Obviously, the vector $\boldsymbol{N}$ defined by formula (4) and in the direction of the normal to the level surface $F=u(x, y, z)-C=0$, will be

$$\boldsymbol{N}=\frac{\partial u}{\partial x}\boldsymbol{i}+\frac{\partial u}{\partial y}\boldsymbol{j}+\frac{\partial u}{\partial z}\boldsymbol{k},$$

that is,

$$\boldsymbol{N}=\operatorname{grad} u.$$

We have thus proved that the *gradient of the function $u(x, y, z)$ is in the direction of the normal to the level surface passing through the given point.*

Example. Write the equation of the tangent plane and the equations of the normal to the surface of the sphere $x^2+y^2+z^2=14$ at the point $P(1, 2, 3)$.

Solution.

$$F(x, y, z)=x^2+y^2+z^2-14=0; \quad \frac{\partial F}{\partial x}=2x; \quad \frac{\partial F}{\partial y}=2y; \quad \frac{\partial F}{\partial z}=2z;$$

for $x=1$, $y=2$, $z=3$ we have

$$\frac{\partial F}{\partial x}=2; \quad \frac{\partial F}{\partial y}=4; \quad \frac{\partial F}{\partial z}=6.$$

Therefore, the equation of the tangent plane will be

$$2(x-1)+4(y-2)+6(z-3)=0 \quad \text{or} \quad x+2y+3z-14=0.$$

The equations of the normal are

$$\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{6}$$

or

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}.$$

Exercises on Chapter IX

Find the derivatives of the vectors: **1.** $r=i\cot t+j\operatorname{arc}\tan t$. *Ans.* $r'=-\frac{1}{\sin^2 t}i+\frac{1}{1+t^2}j$. **2.** $r=ie^{-t}+j2t+k\ln t$. *Ans.* $r'=-ie^{-t}+2j+\frac{k}{t}$. **3.** $r=t^2i-\frac{j}{t}+\frac{k}{t^2}$. *Ans.* $r'=2ti+\frac{j}{t^2}-\frac{2k}{t^3}$.

4. Find the vector of a tangent, the equations of the tangent and the equations of the normal plane to the curve $r=ti+t^2j+t^3k$ at the point (3, 9, 27). *Ans.* $r'=i+6j+27k$; tangent: $\frac{x-3}{1}=\frac{y-9}{6}=\frac{z-27}{27}$; normal plane: $x+6y+27z=786$.

5. Find the vector of a tangent, the equations of the tangent and the equation of the normal plane to the curve $r=i\cos^2\frac{t}{2}+\frac{1}{2}j\sin t+k\sin\frac{t}{2}$. *Ans.* $r'=-\frac{1}{2}i\sin t+\frac{1}{2}j\cos t+\frac{1}{2}k\cos\frac{t}{2}$; the equation of the tangent $\frac{X-\cos^2\frac{t}{2}}{-\sin t}=\frac{Y-\frac{1}{2}\sin t}{\cos t}=\frac{Z-\sin\frac{t}{2}}{\cos\frac{t}{2}}$; the equation of the normal plane: $+X\sin t-Y\cos t-Z\cos\frac{t}{2}=-x\sin t+y\cos t+z\cos\frac{t}{2}$ where x, y, z are the coordinates of that point of the curve at which the normal plane is drawn $\left(\text{that is, } x=\cos^2\frac{t}{2},\ y=\frac{1}{2}\sin t,\ z=\sin\frac{t}{2}\right)$.

6. Find the equations of the tangent to the curve $x=t-\sin t$, $y=1-\cos t$, $z=4\sin\frac{t}{2}$ and the cosines of the angles that it makes with the coordinate

axes. *Ans.* $\frac{X-X_0}{\sin\frac{t_0}{2}}=\frac{Y-Y_0}{\cos\frac{t_0}{2}}=\frac{Z-Z_0}{\cot\frac{t_0}{2}}$, $\cos\alpha=\sin^2\frac{t_0}{2}$; $\cos\beta=\frac{1}{2}\sin t_0$; $\cos\gamma=\cos\frac{t_0}{2}$.

7. Find the equation of the normal plane to the curve $z=x^2-y^2$, $y=x$ at the origin. **Hint.** Write the equations of the curve in parametric form. *Ans.* $x+y=0$.

8. Find $\boldsymbol{\sigma}$, $\boldsymbol{n}$, $\boldsymbol{b}$ at the point $t=\frac{\pi}{2}$ for the curve $\boldsymbol{r}=\boldsymbol{i}(\cos t+\sin^2 t)+\boldsymbol{j}\sin t(1-\cos t)-\boldsymbol{k}\cos t$. *Ans.* $\boldsymbol{\sigma}=\frac{1}{\sqrt{3}}(-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$; $\boldsymbol{n}=\frac{-5\boldsymbol{i}-4\boldsymbol{j}-\boldsymbol{k}}{\sqrt{42}}$; $\boldsymbol{b}=\frac{\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}}{\sqrt{14}}$.

9. Find the equations of the principal normal and the binormal to the curve $x=\frac{t^4}{4}$; $y=\frac{t^3}{3}$; $z=\frac{t^2}{2}$ at the point (x_0, y_0, z_0). *Ans.* $\frac{x-x_0}{t_0^3+2t_0}=\frac{y-y_0}{1-t_0^4}=\frac{z-z_0}{-2t_0^3-t_0}$, $\frac{x-x_0}{1}=\frac{y-y_0}{2t_0}=\frac{z-z_0}{t_0^2}$.

10. Find the equation of the osculating plane to the curve $y^2=x$; $x^2=z$ at the point $M(1, 1, 1)$. *Ans.* $6x-8y-z+3=0$.

11. Find the radius of curvature for a curve represented by the equations $x^2+y^2+z^2-4=0$, $x+y-z=0$. *Ans.* $R=2$.

12. Find the radius of torsion of the curve: $\boldsymbol{r}=\boldsymbol{i}\cos t+\boldsymbol{j}\sin t+\boldsymbol{k}\frac{e^t-e^{-t}}{2}$. *Ans.* $T=\frac{(e^t-e^{-t})^2}{2(e^t-e^{-t})}$.

13. Find the radius of curvature and the torsion for the curve $\boldsymbol{r}=t^2\boldsymbol{i}+2t^3\boldsymbol{j}$. *Ans.* $R=\frac{2}{3}t(1+9t^2)^{3/2}$, $T=\infty$.

14. Prove that the curve $\boldsymbol{r}=(a_1t^2+b_1t+c_1)\boldsymbol{i}+(a_2t^2+b_2t+c_2)\boldsymbol{j}+(a_3t^2+b_3t+c_3)\boldsymbol{k}$ is plane. *Ans.* $\boldsymbol{r}'''\equiv 0$; therefore the torsion is equal to zero.

15. Find the curvature and torsion of the curve $x=e^t$, $y=e^{-t}$, $z=t\sqrt{2}$. *Ans.* The curvature is $\frac{\sqrt{2}}{(x+y)^2}$; the torsion is $\frac{-\sqrt{2}}{(x-y)^2}$.

16. Find the curvature and torsion of the curve $x=e^{-t}\sin t$, $y=e^{-t}\cos t$; $z=e^t$. *Ans.* The curvature is $\frac{\sqrt{2}}{3}e^t$; the torsion is $\frac{1}{3}e^t$.

17. Find the equation of the tangent plane to the hyperboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ at the point (x_1, y_1, z_1). *Ans.* $\frac{x_1x}{a^2}-\frac{y_1y}{b^2}-\frac{z_1z}{c^2}=1$.

18. Find the equation of the normal to the surface $x^2-4y^2+2z^2=6$ at the point $(2, 2, 3)$. *Ans.* $y+4x=10$; $3x-z=3$.

19. Find the equation of the tangent plane to the surface $z=2x^2+4y^2$ at the point $M(2, 1, 12)$. *Ans.* $8x+8y-z=12$.

20. Draw to the surface $x^2+2y^2+z^2=1$ a tangent plane parallel to the plane $x-y+2z=0$. *Ans.* $x-y+2z=\pm\sqrt{\frac{11}{2}}$.

CHAPTER X

INDEFINITE INTEGRALS

SEC. 1. ANTIDERIVATIVE AND THE INDEFINITE INTEGRAL

In Chapter III we considered a problem like the following: Given a function $F(x)$, find its derivative, that is, the function $f(x)=F'(x)$.

In this chapter we shall consider the reverse problem: Given the function $f(x)$, it is required to find a function $F(x)$ such that its derivative is equal to $f(x)$, that is,

$$F'(x)=f(x).$$

Definition 1. The function $F(x)$ is called the *antiderivative* of the function $f(x)$ on the interval [a, b] if at all points of this interval the equality $F'(x)=f(x)$ is fulfilled.

Example. Find the antiderivative of the function $f(x)=x^2$.

From the definition of an antiderivative it follows that the function $F(x)=\frac{x^3}{3}$ is an antiderivative, since $\left(\frac{x^3}{3}\right)'=x^2$.

It is easy to see that if for the given function $f(x)$ there exists an antiderivative, then this antiderivative is not the only one. In the foregoing example, we could take the following functions as antiderivatives: $F(x)=\frac{x^3}{3}+1$; $F(x)=\frac{x^3}{3}-7$ or, generally, $F(x)=\frac{x^3}{3}+C$ (where C is an arbitrary constant), since

$$\left(\frac{x^3}{3}+C\right)'=x^2.$$

On the other hand, it may be proved that functions of the form $\frac{x^3}{3}+C$ exhaust **all** antiderivatives of the function x^2. This follows from the following theorem.

Theorem. *If $F_1(x)$ and $F_2(x)$ are two antiderivatives of the function $f(x)$ on the interval $[a, b]$, then the difference between them is a constant.*

Proof. By virtue of the definition of an antiderivative we have

$$\left.\begin{aligned} F_1'(x)&=f(x),\\ F_2'(x)&=f(x)\end{aligned}\right\} \qquad (1)$$

for any value of x on the interval $[a, b]$.

Let us put

$$F_1(x) - F_2(x) = \varphi(x). \tag{2}$$

Then by (1) we have

$$F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

or

$$\varphi'(x) = [F_1(x) - F_2(x)]' \equiv 0$$

for any value of x on the interval $[a, b]$. But from $\varphi'(x) = 0$ it follows that $\varphi(x)$ is a constant.

Indeed, let us apply the Lagrange theorem (see Sec. 2, Ch. IV) to the function $\varphi(x)$, which, obviously, is continuous and differentiable on the interval $[a, b]$.

No matter what the point x on the interval $[a, b]$, we have, by virtue of the Lagrange theorem,

$$\varphi(x) - \varphi(a) = (x - a)\varphi'(\xi),$$

where $a < \xi < x$.

Since $\varphi'(\xi) = 0$,

$$\varphi(x) - \varphi(a) = 0$$

or

$$\varphi(x) = \varphi(a). \tag{3}$$

Thus, the function $\varphi(x)$ at any point x of the interval $[a, b]$ retains the value $\varphi(a)$, and this means that the function $\varphi(x)$ is constant on $[a, b]$. Denoting the constant $\varphi(a)$ by C, we get, from (2) and (3),

$$F_1(x) - F_2(x) = C.$$

From the proved theorem it follows that if for a given function $f(x)$ **some** one antiderivative $F(x)$ is found, then **any other** antiderivative of $f(x)$ has the form $F(x) + C$, where $C = \text{const}$.

Definition 2. If the function $F(x)$ is an antiderivative of $f(x)$, then the expression $F(x) + C$ is the *indefinite integral* of the function $f(x)$ and is denoted by the symbol $\int f(x)\,dx$. Thus, by definition

$$\int f(x)\,dx = F(x) + C,$$

if

$$F'(x) = f(x).$$

Here, the function $f(x)$ is called the *integrand*, $f(x)\,dx$ is the *element of integration* (the expression under the integral sign), and $\int$ is the *integral sign*.

Thus, an indefinite integral is a **family of functions** $y = F(x) + C$.

From the geometrical point of view, an indefinite integral is an assemblage (family) of curves, each of which is obtained by translating one of the curves parallel to itself upwards or downwards (that is, along the y-axis).

A natural question arises: do antiderivatives (and, hence, an indefinite integral) exist for every function $f(x)$? The answer is no. Let us note, however, without proof, that *if a function $f(x)$ is continuous on the interval* $[a, b]$, *then there is an antiderivative of this function* (and, hence, there *is also an indefinite integral*).

This chapter is devoted to working out methods by means of which we can find antiderivatives (and indefinite integrals) of certain classes of elementary functions.

The finding of an antiderivative of a given function $f(x)$ is called *integration* of the function $f(x)$.

Note the following: if the derivative of an elementary function is always an elementary function, then the antiderivative of the elementary function may not prove to be representable by a finite number of elementary functions. We shall return to this question at the end of the chapter.

From Definition 2 it follows that:

1. *The derivative of an indefinite integral is equal to the integrand, that is, if $F'(x) = f(x)$, then also*

$$\left(\int f(x)\,dx\right)' = (F(x) + C)' = f(x). \tag{4}$$

This equation should be understood in the sense that the derivative of any antiderivative is equal to the integrand.

2. *The differential of an indefinite integral is equal to the expression under the integral sign:*

$$d\left(\int f(x)\,dx\right) = f(x)\,dx. \tag{5}$$

This results from formula (4).

3. *The indefinite integral of the differential of some function is equal to this function plus an arbitrary constant:*

$$\int dF(x) = F(x) + C.$$

The truth of this equation may easily be checked by differentiation [the differentials of both sides are equal to $dF(x)$].

SEC. 2. TABLE OF INTEGRALS

Before starting on methods of integration, we give the following table of integrals of the simplest functions.

The table of integrals follows directly from Definition 2, Sec. 1, Ch. X, and from the table of derivatives (Sec. 15, Ch. III).

(The truth of the equations can easily be checked by differentiation: to establish that the derivative of the right side is equal to the integrand).

1. $\int x^{\alpha}dx = \frac{x^{\alpha+1}}{\alpha+1} + C\ (\alpha \neq -1)$. (Here and in the formulas that follow, C stands for an arbitrary constant.)

2. $\int \frac{dx}{x} = \ln|x| + C.$

3. $\int \sin x\,dx = -\cos x + C.$

4. $\int \cos x\,dx = \sin x + C.$

5. $\int \frac{dx}{\cos^2 x} = \tan x + C.$

6. $\int \frac{dx}{\sin^2 x} = -\cot x + C.$

7. $\int \tan x\,dx = -\ln|\cos x| + C.$

8. $\int \cot x\,dx = \ln|\sin x| + C.$

9. $\int e^x\,dx = e^x + C.$

10. $\int a^x\,dx = \frac{a^x}{\ln a} + C.$

11. $\int \frac{dx}{1+x^2} = \arctan x + C.$

11′. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan \frac{x}{a} + C.$

12. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C.$

13 $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C.$

13′ $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C.$

14. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C.$

Note. The table of derivatives (Sec. 15, Ch. III) does not have formulas corresponding to formulas 7, 8, 11′, 12, 13′ and 14. However, differentiation will readily prove the truth of these as well.

In the case of formula 7 we have

$$(-\ln|\cos x|)' = -\frac{-\sin x}{\cos x} = \tan x,$$

consequently, $\int \tan x\,dx = -\ln|\cos x| + C.$

In the case of formula 8

$$(\ln|\sin x|)' = \frac{\cos x}{\sin x} = \cot x,$$

consequently, $\int \cot x = \ln|\sin x| + C$.

In the case of formula 12,

$$\left(\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|\right)' = \frac{1}{2a}[\ln|a+x| - \ln|a-x|]' =$$

$$= \frac{1}{2a}\left[\frac{1}{a+x} + \frac{1}{a-x}\right] = \frac{1}{a^2-x^2},$$

therefore,

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C.$$

It should be noted that the latter formula will also follow from the general results of Sec. 9, Ch. X.

In the case of formula 14,

$$(\ln|x+\sqrt{x^2\pm a^2}|)' = \frac{1}{x+\sqrt{x^2\pm a^2}}\left(1+\frac{x}{\sqrt{x^2\pm a^2}}\right) = \frac{1}{\sqrt{x^2\pm a^2}},$$

hence,

$$\int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln|x+\sqrt{x^2\pm a^2}| + C.$$

This formula likewise will follow from the general results of Sec. 11.

Formulas 11′ and 13′ may be verified in similar fashion. These formulas will later be derived from formulas 11 and 13 (see Sec. 4, Examples 3 and 4).

SEC. 3. SOME PROPERTIES OF AN INDEFINITE INTEGRAL

Theorem 1. *The indefinite integral of an algebraic sum of two or several functions is equal to the sum of their integrals*

$$\int [f_1(x) + f_2(x)]\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx. \qquad (1)$$

For proof, let us find the derivatives of the left and right sides of this equation. On the basis of (4) of the preceding section we have

$$\left(\int [f_1(x) + f_2(x)]\,dx\right)' = f_1(x) + f_2(x),$$

$$\left(\int f_1(x)\,dx + \int f_2(x)\,dx\right)' =$$

$$= \left(\int f_1(x)\,dx\right)' + \left(\int f_2(x)\,dx\right)' = f_1(x) + f_2(x).$$

Thus, the derivatives of the left and right sides of (1) are equal; in other words, the derivative of any antiderivative on the left-hand side is equal to the derivative of any function on the right-hand side of the equation. Therefore, by the theorem of Sec. 1, Ch. X, any function on the left of (1) differs from any function on the right of (1) by a constant term. That is how we should understand (1).

Theorem 2. *The constant factor may be taken outside the integral sign; that is, if $a = \text{const}$, then*

$$\int af(x)\,dx = a\int f(x)\,dx. \tag{2}$$

To prove (2), let us find the derivatives of the left and right sides:

$$\left(\int af(x)\,dx\right)' = af(x),$$

$$\left(a\int f(x)\,dx\right)' = a\left(\int f(x)\,dx\right)' = af(x).$$

The derivatives of the right and left sides are equal, therefore, as in (1), the difference of any two functions on the left and right is a constant. That is how we should understand equation (2).

When evaluating indefinite integrals it is useful to bear in mind the following rules.

I. If

$$\int f(x)\,dx = F(x) + C,$$

then

$$\int f(ax)\,dx = \frac{1}{a}F(ax) + C. \tag{3}$$

Indeed, differentiating the left and right sides of (3), we get

$$\left(\int f(ax)\,dx\right)' = f(ax),$$

$$\left(\frac{1}{a}F(ax)\right)' = \frac{1}{a}(F(ax))'_x = \frac{1}{a}F'(ax)\,a = F'(ax) = f(ax).$$

The derivatives of the right and left sides are equal, which is what we set out to prove.

II. If

$$\int f(x)\,dx = F(x) + C,$$

then

$$\int f(x+b)\,dx = F(x+b) + C. \tag{4}$$

III. If

$$\int f(x)\,dx = F(x) + C,$$

then

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C. \tag{5}$$

Equations (4) and (5) are proved by differentiation of the right and left sides.

Example 1.

$$\int (2x^3 - 3\sin x + 5\sqrt{x})\,dx = \int 2x^3\,dx - \int 3\sin x\,dx + \int 5\sqrt{x}\,dx =$$

$$= 2\int x^3\,dx - 3\int \sin x\,dx + 5\int x^{\frac{1}{2}}\,dx =$$

$$= 2\frac{x^{3+1}}{3+1} - 3(-\cos x) + 5\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{1}{2}x^4 + 3\cos x + \frac{10}{3}x\sqrt{x} + C.$$

Example 2.

$$\int \left(\frac{3}{\sqrt[3]{x}} + \frac{1}{2\sqrt{x}} + x\sqrt[4]{x}\right)dx = 3\int x^{-\frac{1}{3}}\,dx + \frac{1}{2}\int x^{-\frac{1}{2}}\,dx + \int x^{\frac{5}{4}}\,dx =$$

$$= 3\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + \frac{1}{2}\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{5}{4}+1}}{\frac{5}{4}+1} + C = \frac{9}{2}\sqrt[3]{x^2} + \sqrt{x} + \frac{4}{9}x^2\sqrt[4]{x} + C.$$

Example 3.

$$\int \frac{dx}{x+3} = \ln|x+3| + C.$$

Example 4.

$$\int \cos 7x\,dx = \frac{1}{7}\sin 7x + C.$$

Example 5.

$$\int \sin(2x-6)\,dx = -\frac{1}{2}\cos(2x-6) + C.$$

SEC. 4. INTEGRATION BY SUBSTITUTION (CHANGE OF VARIABLE)

Let it be required to find the integral

$$\int f(x)\,dx;$$

we cannot directly select the antiderivative of $f(x)$ but we know that it exists.

Let us change the variable in the expression under the integral sign, putting

$$x=\varphi(t), \tag{1}$$

where $\varphi(t)$ is a continuous function with continuous derivative having an inverse function. Then $dx=\varphi'(t)\,dt$; we shall prove that in this case we have the following equation:

$$\int f(x)\,dx=\int f[\varphi(t)]\,\varphi'(t)\,dt. \tag{2}$$

Here we assume that after integration we substitute, on the right side, the expression of t in terms of x on the basis of (1).

To establish that the expressions to the right and left are the same in the sense indicated above, it is necessary to prove that their derivatives with respect to x are equal. Find the derivative of the left side:

$$\left(\int f(x)\,dx\right)'_x=f(x).$$

We differentiate the right side of (2) with respect to x as a composite function, where t is the intermediate argument. The dependence of t on x is expressed by (1); here, $\frac{dx}{dt}=\varphi'(t)$ and by the rule of differentiating an inverse function,

$$\frac{dt}{dx}=\frac{1}{\varphi'(t)}\,.$$

We thus have

$$\left(\int f[\varphi(t)]\,\varphi'(t)\,dt\right)'_x=\left(\int f[\varphi(t)]\,\varphi'(t)\,dt\right)'_t\frac{dt}{dx}=$$
$$=f[\varphi(t)]\,\varphi'(t)\frac{1}{\varphi'(t)}=f[\varphi(t)]=f(x).$$

Therefore, the derivatives, with respect to x, of the right and left side of (2) are equal, as required.

The function $x=\varphi(t)$ should be chosen so that one can evaluate the indefinite integral on the right side of (2).

Note. When integrating, it is sometimes better to choose a change of the variable in the form of $t=\psi(x)$ and not $x=\varphi(t)$. By way of illustration, let it be required to calculate an integral of the form

$$\int\frac{\psi'(x)\,dx}{\psi(x)}\,.$$

Here it is convenient to put

$$\psi(x)=t$$

then

$$\psi'(x)\,dx = dt,$$

$$\int \frac{\psi'(x)\,dx}{\psi(x)} = \int \frac{dt}{t} = \ln|t| + C = \ln|\psi(x)| + C.$$

The following are a number of instances of integration by substitution.

Example 1. $\int \sqrt{\sin x}\cos x\,dx = ?$ We make the substitution $t = \sin x$; then $dt = \cos x\,dx$ and, consequently, $\int \sqrt{\sin x}\cos x\,dx = \int \sqrt{t}\,dt = \int t^{\frac{1}{2}}\,dt = \frac{2t^{3/2}}{3} + C = \frac{2}{3}\sin^{3/2} x + C$.

Example 2. $\int \frac{x\,dx}{1+x^2} = ?$ We put $t = 1 + x^2$; then $dt = 2x\,dx$ and $\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln t + C = \frac{1}{2}\ln(1+x^2) + C$.

Example 3. $\int \frac{dx}{a^2+x^2} = \frac{1}{a^2}\int \frac{dx}{1+\left(\frac{x}{a}\right)^2}$. We put $t = \frac{x}{a}$; then $dx = a\,dt$,

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a^2}\int \frac{a\,dt}{1+t^2} = \frac{1}{a}\int \frac{dt}{1+t^2} = \frac{1}{a}\arctan t + C = \frac{1}{a}\arctan\frac{x}{a} + C.$$

Example 4. $\int \frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{a}\int \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$. We put $t = \frac{x}{a}$; then $dx = a\,dt$, $\int \frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{a}\int \frac{a\,dt}{\sqrt{1-t^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin\frac{x}{a} + C$ (it is assumed that $a > 0$).

The formulas 11′ and 13′ given in the Table of Integrals (see above, Sec. 2) are derived in Examples 3 and 4.

Example 5. $\int (\ln x)^3 \frac{dx}{x} = ?$ Put $t = \ln x$; then $dt = \frac{dx}{x}$, $\int (\ln x)^3 \frac{dx}{x} = \int t^3\,dt = \frac{t^4}{4} + C = \frac{1}{4}(\ln x)^4 + C$.

Example 6. $\int \frac{x\,dx}{1+x^4} = ?$ Put $t = x^2$; then $dt = 2x\,dx$, $\int \frac{x\,dx}{1+x^4} = \frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\arctan t + C = \frac{1}{2}\arctan x^2 + C$.

The method of substitution is one of the basic methods for calculating indefinite integrals. Even when we integrate by some other

method, we often resort to substitution in the intermediate stages of calculation. The success of integration depends largely on how appropriate the substitution is for simplifying the given integral. Essentially, the study of methods of integration reduces to finding out what kind of substitution has to be performed for a given element of integration. Most of this chapter is devoted to this problem.

SEC. 5. INTEGRALS OF FUNCTIONS CONTAINING A QUADRATIC TRINOMIAL

I. Let us consider the integral

$$I_1 = \int \frac{dx}{ax^2+bx+c}.$$

Let us first transform the trinomial in the denominator by representing it in the form of a sum or difference of squares:

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] =$$
$$= a\left[x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right] =$$
$$= a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right] = a\left[\left(x+\frac{b}{2a}\right)^2 \pm k^2\right],$$

where

$$\frac{c}{a}-\frac{b^2}{4a^2} = \pm k^2.$$

The plus or minus is taken depending on whether the expression on the left is positive or negative, that is, on whether the roots of the trinomial ax^2+bx+c are complex or real.

Thus, the integral I_1 will take the form

$$I_1 = \int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left[\left(x+\frac{b}{2a}\right)^2 \pm k^2\right]}.$$

In the latter integral let us change the variable:

$$x+\frac{b}{2a} = t, \quad dx = dt.$$

We then get

$$I_1 = \frac{1}{a}\int \frac{dt}{t^2 \pm k^2}.$$

These are tabular integrals (see formulas 11′ and 12).

Example 1. Calculate the integral

$$\int \frac{dx}{2x^2+8x+20}.$$

Solution.

$$I=\int\frac{dx}{2x^2+8x+20}=\frac{1}{2}\int\frac{dx}{x^2+4x+10}=$$

$$=\frac{1}{2}\int\frac{dx}{x^2+4x+4+10-4}=\frac{1}{2}\int\frac{dx}{(x+2)^2+6}.$$

Let us make the substitution $x+2=t$, $dx=dt$. Putting it into the integral we get the tabular integral

$$I=\frac{1}{2}\int\frac{dt}{t^2+6}=\frac{1}{2}\frac{1}{\sqrt{6}}\arctan\frac{t}{\sqrt{6}}+C.$$

Substituting in place of t its expression in terms of x, we finally get

$$I=\frac{1}{2\sqrt{6}}\arctan\frac{x+2}{\sqrt{6}}+C.$$

II. Let us consider an integral of a more general form:

$$I_2=\int\frac{Ax+B}{ax^2+bx+c}dx.$$

Perform an identical transformation of the integrand:

$$I_2=\int\frac{Ax+B}{ax^2+bx+c}dx=\int\frac{\frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right)}{ax^2+bx+c}dx.$$

Represent the latter integral in the form of a sum of two integrals. Taking the constant factors outside the integral sign, we get

$$I_2=\frac{A}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx+\left(B-\frac{Ab}{2a}\right)\int\frac{dx}{ax^2+bx+c}.$$

The latter integral is the integral I_1, which we are able to evaluate. In the first integral make a substitution:

$$ax^2+bx+c=t,\quad (2ax+b)\,dx=dt.$$

Thus,

$$\int\frac{(2ax+b)\,dx}{ax^2+bx+c}=\int\frac{dt}{t}=\ln|t|+C=\ln|ax^2+bx+c|C.$$

And we finally get

$$I_2=\frac{A}{2a}\ln|ax^2+bx+c|+\left(B-\frac{Ab}{2a}\right)I_1.$$

Example 2. Evaluate the integral

$$I=\int\frac{x+3}{x^2-2x-5}dx.$$

Applying the foregoing technique we have

$$I=\int\frac{x+3}{x^2-2x-5}=\int\frac{\frac{1}{2}(2x-2)+\left(3+\frac{1}{2}2\right)}{x^2-2x-5}dx=$$
$$=\frac{1}{2}\int\frac{(2x-2)\,dx}{x^2-2x-5}+4\int\frac{dx}{x^2-2x-5}=$$
$$=\frac{1}{2}\ln|x^2-2x-5|+4\int\frac{dx}{(x-1)^2-6}=$$
$$=\frac{1}{2}\ln|x^2-2x-5|+4\frac{1}{\sqrt{6}}\ln\left|\frac{\sqrt{6}-(x-1)}{\sqrt{6}+(x-1)}\right|+C.$$

III. Let us consider the integral

$$\int\frac{dx}{\sqrt{ax^2+bx+c}}.$$

By means of transformations considered in Item I, this integral reduces (depending on the sign of a) to tabular integrals of the form

$$\int\frac{dt}{\sqrt{t^2\pm k^2}}\text{ for }a>0\text{ or }\int\frac{dt}{\sqrt{k^2-t^2}}\text{ for }a<0,$$

which have already been examined in the Table of Integrals (see formulas 13′ and 14).

IV. An integral of the form

$$\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$$

is evaluated by means of the following transformations, which are similar to those considered in Item II:

$$\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx=\int\frac{\frac{A}{2a}(2ax+b)+\left(B-\frac{Ab}{2a}\right)}{\sqrt{ax^2+bx+c}}dx=$$
$$=\frac{A}{2a}\int\frac{2ax+b}{\sqrt{ax^2+bx+c}}dx+\left(B-\frac{Ab}{2a}\right)\int\frac{dx}{\sqrt{ax^2+bx+c}}.$$

Applying substitution to the first of the integrals obtained,

$$ax^2+bx+c=t,\quad (2ax+b)\,dx=dt,$$

we get

$$\int\frac{(2ax+b)\,dx}{\sqrt{ax^2+bx+c}}=\int\frac{dt}{\sqrt{t}}=2\sqrt{t}+C=2\sqrt{ax^2+bx+c}+C.$$

The second integral was considered in Item III of this section.

Example 3.

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx = \int \frac{\frac{5}{2}(2x+4)+(3-10)}{\sqrt{x^2+4x+10}}\,dx =$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}\,dx - 7\int \frac{dx}{\sqrt{(x+2)^2+6}} =$$

$$= 5\sqrt{x^2+4x+10} - 7\ln|x+2+\sqrt{(x+2)^2+6}| + C =$$

$$= 5\sqrt{x^2+4x+10} - 7\ln|x+2+\sqrt{x^2+4x+10}| + C.$$

SEC. 6. INTEGRATION BY PARTS

Let u and v be two differentiable functions of x. Then the differential of the product uv is found from the following formula:

$$d(uv) = u\,dv = v\,du.$$

Whence, by integration, we have

$$uv = \int u\,dv + \int v\,du$$

or

$$\int u\,dv = uv - \int v\,du. \qquad (1)$$

This formula is called the *formula of integration by parts*. It is most frequently used in the integration of expressions that may be represented in the form of a product of two factors u and dv in such a way that the finding of the function v from its differential dv, and the evaluation of the integral $\int v\,du$ should, taken together, be a simpler problem than the direct evaluation of the integral $\int u\,dv$. To become adept at breaking up a given element of integration into the factors u and dv, one has to solve problems; we shall show how this is done in a number of cases.

Example 1. $\int x\sin x\,dx = ?$ We let

$$u = x, \quad dv = \sin x\,dx;$$

then

$$du = dx, \quad v = -\cos x.$$

Hence,

$$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

Note. When determining the function v from the differential dv we can take any arbitrary constant, since it does not enter into the final result [this can be seen by putting the expression

$v+C$ into (1) in place of v]. It is therefore convenient to consider this constant equal to zero.

The rule for integration by parts is widely used. For example, integrals of the form

$$\int x^k \sin ax\,dx, \qquad \int x^k \cos ax\,dx,$$
$$\int x^k e^{ax} dx, \qquad \int x^k \ln x\,dx,$$

and certain integrals containing inverse trigonometric functions are evaluated by means of integration by parts.

Example 2. It is required to evaluate $\int \arctan x\,dx$. Letting $u=\arctan x$, $dv=dx$, we have $du=\frac{dx}{1+x^2}$, $v=x$. Thus,

$$\int \arctan x\,dx = x\arctan x - \int \frac{x\,dx}{1+x^2} = x\arctan x - \frac{1}{2}\ln|1+x^2|+C.$$

Example 3. It is required to evaluate $\int x^2 e^x\,dx$. Let us put $u=x^2$, $dv=e^x dx$; then $du=2x\,dx$, $v+e^x$,

$$\int x^2 e^x\,dx = x^2 e^x - 2\int xe^x\,dx.$$

The last integral we again integrate by parts, letting

$$u_1=x, \qquad du_1=dx,$$
$$dv_1=e^x\,dx, \qquad v_1=e^x.$$

Then

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Finally we get

$$\int x^2e^x\,dx = x^2e^x - 2(xe^x - e^x) + C = x^2e^x - 2xe^x + 2e^x + C = e^x(x^2-2x+2)+C.$$

Example 4. It is required to evaluate $\int (x^2+7x-5)\cos 2x\,dx$. We let $u=x^2+7x-5$; $dv=\cos 2x\,dx$; then

$$du=(2x+7)\,dx, \qquad v=\frac{\sin 2x}{2},$$

$$\int (x^2+7x-5)\cos 2x\,dx = (x^2+7x-5)\frac{\sin 2x}{2} - \int (2x+7)\frac{\sin 2x}{2}\,dx.$$

Apply integration by parts to the latter integral, letting $u_1=\frac{2x+7}{2}$, $dv_1=$ $=\sin 2x\,dx$; then

$$du_1=dx, \qquad v_1=-\frac{\cos 2x}{2};$$

$$\int \frac{2x+7}{2}\sin 2x\,dx = \frac{2x+7}{2}\left(-\frac{\cos 2x}{2}\right) - \int\left(-\frac{\cos 2x}{2}\right)dx =$$
$$= -\frac{(2x+7)\cos 2x}{4} + \frac{\sin 2x}{4} + C.$$

Therefore, we finally get

$$\int (x^2+7x-5)\cos 2x\,dx=(x^2+7x-5)\frac{\sin 2x}{2}+(2x+7)\frac{\cos 2x}{4}-\frac{\sin 2x}{4}+C.$$

Example 5. $I=\int \sqrt{a^2-x^2}\,dx=?$

Perform identical transformations. Multiply and divide the integrand by $\sqrt{a^2-x^2}$:

$$\int \sqrt{a^2-x^2}\,dx=\int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx=a^2\int \frac{dx}{\sqrt{a^2-x^2}}-\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}=$$

$$=a^2\arcsin\frac{x}{a}-\int x\frac{x\,dx}{\sqrt{a^2-x^2}}.$$

Integrate the latter integral by parts, letting

$$u=x,\qquad du=dx,$$

$$dv=\frac{x\,dx}{\sqrt{a^2-x^2}},\quad v=-\sqrt{a^2-x^2};$$

then

$$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}=\int x\frac{x\,dx}{\sqrt{a^2-x^2}}=-x\sqrt{a^2-x^2}+\int \sqrt{a^2-x^2}\,dx.$$

Putting the last result in the earlier obtained expression of the given integral, we will have

$$\int \sqrt{a^2-x^2}\,dx=a^2\arcsin\frac{x}{a}+x\sqrt{a^2-x^2}-\int \sqrt{a^2-x^2}\,dx.$$

Transposing the integral from right to left and performing elementary transformations, we finally get

$$\int \sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C.$$

Example 6. Evaluate the integrals

$$I_1=\int e^{ax}\cos bx\,dx\quad\text{and}\quad I_2=\int e^{ax}\sin bx\,dx.$$

Applying integration by parts to the first integral, we get

$$u=e^{ax},\qquad du=ae^{ax},$$

$$dv=\cos bx\,dx,\quad v=\frac{1}{b}\sin bx,$$

$$\int e^{ax}\cos bx\,dx=\frac{1}{b}e^{ax}\sin bx-\frac{a}{b}\int e^{ax}\sin bx\,dx.$$

Again apply the method of integration by parts to the last integral:

$$u=e^{ax},\qquad du=ae^{ax},$$

$$dv=\sin bx\,dx,\quad v=-\frac{1}{b}\cos bx,$$

$$\int e^{ax}\sin bx\,dx=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b}\int e^{ax}\cos bx\,dx.$$

Putting into the preceding equation the expression obtained gives us

$$\int e^{ax}\cos bx\,dx=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}e^{ax}\cos bx-\frac{a^2}{b^2}\int e^{ax}\cos bx\,dx.$$

From this equation let us find I_1:

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx=e^{ax}\left(\frac{1}{b}\sin bx+\frac{a}{b^2}\cos bx\right),$$

whence

$$I_1=\int e^{ax}\cos bx\,dx=\frac{e^{ax}(b\sin bx+a\cos bx)}{a^2+b^2}+C.$$

Similarly we find

$$I_2=\int e^{ax}\sin bx\,dx=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}+C.$$

SEC. 7. RATIONAL FRACTIONS. PARTIAL RATIONAL FRACTIONS AND THEIR INTEGRATION

As we shall see below, not every elementary function by far has an integral expressed in elementary functions. For this reason, it is very important to separate out those classes of functions whose integrals are expressed in terms of elementary functions. The simplest of these classes is the class of rational functions.

Every rational function may be represented in the form of a rational fraction, that is to say, as a ratio of two polynomials:

$$\frac{Q(x)}{f(x)}=\frac{B_0x^m+B_1x^{m-1}+\ldots+B_m}{A_0x^n+A_1x^{n-1}+\ldots+A_n}.$$

Without restricting the generality of our reasoning, we shall assume that these polynomials do not have common roots.

If the degree of the numerator is lower than that of the denominator, then the fraction is called *proper*, otherwise the fraction is called *improper*.

If the fraction is an improper one, then by dividing the numerator by the denominator (by the rule of division of polynomials), it is possible to represent the fraction as the sum of a polynomial and a proper fraction:

$$\frac{Q(x)}{f(x)}=M(x)+\frac{F(x)}{f(x)};$$

here $M(x)$ is a polynomial, and $\frac{F(x)}{f(x)}$ is a proper fraction.

Example 1. Given an improper rational fraction

$$\frac{x^4-3}{x^2+2x+1}.$$

Dividing the numerator by the denominator (by the rule of division of polynomials), we get

$$\frac{x^4-3}{x^2+2x+1}=x^2-2x+3-\frac{4x-6}{x^2+2x+1}.$$

Since integration of polynomials does not present any difficulties, the basic barrier when integrating rational fractions is the integration of *proper* rational fractions.

Definition. Proper rational fractions of the form:

I. $\frac{A}{x-a}$,

II. $\frac{A}{(x-a)^k}$ (k is a positive integer $\geqslant 2$),

III. $\frac{Ax+B}{x^2+px+q}$ (the roots of the denominator are complex, that is, $\frac{p^2}{4}-q<0$),

IV. $\frac{Ax+B}{(x^2+px+q)^k}$ (k is a positive integer $\geqslant 2$; the roots of the denominator are complex) are called *partial fractions of types* I, II, III, *and* IV.

It will be proved below (see Sec. 8) that every rational fraction may be represented as a sum of partial fractions. We shall therefore first consider integrals of partial fractions.

The integration of partial fractions of types I, II and III does not present any particular difficulties so we shall perform their integration without any remarks:

I. $\int\frac{A}{x-a}\,dx=A\ln|x-a|+C.$

II. $$\int\frac{A}{(x-a)^k}\,dx=A\int(x-a)^{-k}\,dx=A\frac{(x-a)^{-k+1}}{-k+1}+C=\frac{A}{(1-k)(x-a)^{k-1}}+C.$$

III. $$\int\frac{Ax+B}{x^2+px+q}\,dx=\int\frac{\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)}{x^2+px+q}\,dx=$$

$$=\frac{A}{2}\int\frac{2x+p}{x^2+px+q}\,dx+\left(B-\frac{Ap}{2}\right)\int\frac{dx}{x^2+px+q}=$$

$$=\frac{A}{2}\ln|x^2+px+q|+\left(B-\frac{Ap}{2}\right)\int\frac{dx}{\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)}=$$

$$=\frac{A}{2}\ln|x^2+px+q|+\frac{2B-Ap}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}}+C$$

(see Sec. 5).

The integration of partial fractions of type IV requires more involved computations. Suppose we have an integral of this type:

IV. $\int \frac{Ax+B}{(x^2+px+q)^k}\,dx.$

Perform the transformations:

$$\int \frac{Ax+B}{(x^2+px+q)^k}\,dx = \int \frac{\frac{A}{2}(2x+p)+\left(B-\frac{Ap}{2}\right)}{(x^2+px+q)^k}\,dx =$$
$$= \frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}\,dx + \left(B-\frac{Ap}{2}\right)\int \frac{dx}{(x^2+px+q)^k}.$$

The first integral is taken by substitution, $x^2+px+q=t$;

$$(2x+p)\,dx = dt:$$
$$\int \frac{2x+p}{(x^2+px+q)^k}\,dx = \int \frac{dt}{t^k} = \int t^{-k}\,dt = \frac{t^{-k+1}}{1-k}+C =$$
$$= \frac{1}{(1-k)(x^2+px+q)^{k-1}}+C.$$

We write the second integral (let us denote it by I_k) in the form

$$I_k = \int \frac{dx}{(x^2+px+q)^k} = \int \frac{dx}{\left[\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right]^k} = \int \frac{dt}{(t^2+m^2)^k}$$

assuming

$$x+\frac{p}{2}=t, \quad dx=dt, \quad q-\frac{p^2}{4}=m^2$$

(it is assumed that the roots of the denominator are complex, and hence, $q-\frac{p^2}{4}>0$). We then do as follows:

$$I_k = \int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2}\int \frac{(t^2+m^2)-t^2}{(t^2+m^2)^k}\,dt =$$
$$= \frac{1}{m^2}\int \frac{dt}{(t^2+m^2)^{k-1}} - \frac{1}{m^2}\int \frac{t^2}{(t^2+m^2)^k}\,dt. \tag{1}$$

Transform the last integral:

$$\int \frac{t^2\,dt}{(t^2+m^2)^k} = \int \frac{t\cdot t\,dt}{(t^2+m^2)^k} =$$
$$= \frac{1}{2}\int t\,\frac{d(t^2+m^2)}{(t^2+m^2)^k} = -\frac{1}{2(k-1)}\int t\,d\left(\frac{1}{(t^2+m^2)^{k-1}}\right).$$

Integrating by parts we get

$$\int \frac{t^2\,dt}{(t^2+m^2)^k} = -\frac{1}{2(k-1)}\left[t\,\frac{1}{(t^2+m^2)^{k-1}} - \int \frac{dt}{(t^2+m^2)^{k-1}}\right].$$

Putting this expression into (1), we have

$$I_k = \int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2}\int \frac{dt}{(t^2+m^2)^{k-1}} +$$

$$+\frac{1}{m^2}\frac{1}{2(k-1)}\left[\frac{t}{(t^2+m^2)^{k-1}} - \int \frac{dt}{(t^2+m^2)^{k-1}}\right] =$$

$$= \frac{t}{2m^2(k-1)(t^2+m^2)^{k-1}} + \frac{2k-3}{2m^2(k-1)}\int \frac{dt}{(t^2+m^2)^{k-1}}.$$

On the right side is an integral of the same type as I_k, but the exponent of the denominator of the integrand is less by unity $(k-1)$; we have thus expressed I_k in terms of I_{k-1}.

Continuing in the same manner we will arrive at the familiar integral

$$I_1 = \int \frac{dt}{t^2+m^2} = \frac{1}{m}\arctan\frac{t}{m} + C.$$

Then substituting everywhere in place of t and m their values, we get the expression of integral IV in terms of x and the given numbers A, B, p, q.

Example 2.

$$\int \frac{x-1}{(x^2+2x+3)^2}\,dx = \int \frac{\frac{1}{2}(2x+2)+(-1-1)}{(x^2+2x+3)^2}\,dx =$$

$$= \frac{1}{2}\int \frac{2x+2}{(x^2+2x+3)^2}\,dx - 2\int \frac{dx}{(x^2+2x+3)^2} =$$

$$= -\frac{1}{2}\,\frac{1}{(x^2+2x+3)} - 2\int \frac{dx}{(x^2+2x+3)^2}.$$

We apply the substitution $x+1=t$ to the last integral:

$$\int \frac{dx}{(x^2+2x+3)^2} = \int \frac{dx}{[(x+1)^2+2]^2} = \int \frac{dt}{(t^2+2)^2} = \frac{1}{2}\int \frac{(t^2+2)-t^2}{(t^2+2)^2}\,dt =$$

$$= \frac{1}{2}\int \frac{dt}{t^2+2} - \frac{1}{2}\int \frac{t^2}{(t^2+2)^2}\,dt =$$

$$= \frac{1}{2}\,\frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}} - \frac{1}{2}\int \frac{t^2\,dt}{(t^2+2)^2}.$$

Let us consider the last integral:

$$\int \frac{t^2\,dt}{(t^2+2)^2}=\frac{1}{2}\int \frac{t\,d(t^2+2)}{(t^2+2)^2}=-\frac{1}{2}\int t\,d\left(\frac{1}{t^2+2}\right)=$$
$$=-\frac{1}{2}\frac{t}{t^2+2}+\frac{1}{2}\int\frac{dt}{t^2+2}=$$
$$=-\frac{t}{2(t^2+2)}+\frac{1}{2\sqrt{2}}\arctan\frac{t}{\sqrt{2}}$$

(we do not yet write the arbitrary constant but will take it into account in the final result).

Consequently,

$$\int\frac{dx}{(x^2+2x+3)^2}=\frac{1}{2\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}-$$
$$-\frac{1}{2}\left[-\frac{x+1}{2(x^2+2x+3)}+\frac{1}{2\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}\right].$$

Finally we get

$$\int\frac{x-1}{(x^2+2x+3)^2}\,dx=-\frac{x+2}{2(x^2+2x+3)}-\frac{\sqrt{2}}{4}\arctan\frac{x+1}{\sqrt{2}}+C.$$

SEC 8. DECOMPOSITION OF A RATIONAL FRACTION INTO PARTIAL FRACTIONS

We shall now show that every proper rational fraction may be decomposed into a sum of partial fractions.

Suppose we have a proper rational fraction

$$\frac{F(x)}{f(x)}.$$

We shall assume that the coefficients of the polynomials are real numbers and that the given fraction is nonreducible (this means that the numerator and denominator do not have common roots).

Theorem 1. *Let $x=a$ be a root of the denominator of multiplicity k; that is $f(x)=(x-a)^k f_1(x)$ where $f_1(a)\neq 0$* (see Sec. 6, Ch. VII). *Then the given proper fraction $\frac{F(x)}{f(x)}$ may be represented in the form of a sum of two other proper fractions as follows:*

$$\frac{F(x)}{f(x)}=\frac{A}{(x-a)^k}+\frac{F_1(x)}{(x-a)^{k-1}f_1(x)},\tag{1}$$

where A is a constant not equal to zero, and $F_1(x)$ is a polynomial whose degree is less than the degree of the denominator $(x-a)^{k-1}f_1(x)$.

Proof. Let us write the identity

$$\frac{F(x)}{f(x)}=\frac{A}{(x-a)^k}+\frac{F(x)-Af_1(x)}{(x-a)^k f_1(x)}\tag{2}$$

(which is true for every A) and let us define the constant A so that the polynomial $F(x)-Af_1(x)$ can be divided by $x-a$. For this, by the remainder theorem, it is necessary and sufficient that the following equality be fulfilled:

$$F(a)-Af_1(a)=0.$$

Since $f_1(a)\neq 0$, $F(a)\neq 0$, A is uniquely defined by

$$A=\frac{F(a)}{f_1(a)}.$$

For such an A we shall have

$$F(x)-Af_1(x)=(x-a)F_1(x),$$

where $F_1(x)$ is a polynomial of degree less than that of the polynomial $(x-a)^{k-1}f_1(x)$. Cancelling $(x-a)$ from the fraction in formula (2), we get (1).

Corollary. Similar reasoning may be applied to the proper rational fraction

$$\frac{F_1(x)}{(x-a)^{k-1}f_1(x)},$$

in equation (1). Thus, if the denominator has a root $x=a$ of multiplicity k, one can write

$$\frac{F(x)}{f(x)}=\frac{A}{(x-a)^k}+\frac{A_1}{(x-a)^{k-1}}+\ldots+\frac{A_{k-1}}{x-a}+\frac{F_k(x)}{f_1(x)},$$

where $\frac{F_k(x)}{f_1(x)}$ is a proper nonreducible fraction. To it we can apply the theorem that has just been proved, provided $f_1(x)$ has other real roots.

Let us now consider the case of complex roots of the denominator. Recall that the complex roots of a polynomial with real coefficients are always conjugate in pairs (see Sec. 8, Ch. VII).

When factoring a polynomial into real factors, to each pair of complex roots of the polynomial there corresponds an expression of the form x^2+px+q. But if the complex roots are of multiplicity μ, they correspond to the expression $(x^2+px+q)^\mu$.

Theorem 2. *If $f(x)=(x^2+px+q)^\mu\varphi_1(x)$, where the polynomial $\varphi_1(x)$ is not divisible by x^2+px+q, then the proper rational fraction $\frac{F(x)}{f(x)}$ may be represented as a sum of two other proper fractions in the following manner:*

$$\frac{F(x)}{f(x)}=\frac{Mx+N}{(x^2+px+q)^\mu}+\frac{\Phi_1(x)}{(x^2+px+q)^{\mu-1}\varphi_1(x)},\qquad(3)$$

where $\Phi_1(x)$ is a polynomial of degree less than that of the polynomial $(x^2+px+q)^{\mu-1}\varphi_1(x)$.

Proof. Let us write the identity

$$\frac{F(x)}{f(x)}=\frac{F(x)}{(x^2+px+q)^{\mu}\varphi_1(x)}=\frac{Mx+N}{(x^2+px+q)^{\mu}}+\frac{F(x)-(Mx+N)\varphi_1(x)}{(x^2+px+q)^{\mu}\varphi_1(x)}, \quad (4)$$

which is true for all M and N, and let us define M and N so that the polynomial $F(x)-(Mx+N)\varphi_1(x)$ is divisible by x^2+px+q. To do this, it is necessary and sufficient that the equation

$$F(x)-(Mx+N)\varphi_1(x)=0$$

have the same roots $\alpha\pm i\beta$ as the polynomial x^2+px+q. Thus,

$$F(\alpha+i\beta)-[M(\alpha+i\beta)+N]\varphi_1(\alpha+i\beta)=0$$

or

$$M(\alpha+i\beta)+N=\frac{F(\alpha+i\beta)}{\varphi_1(\alpha+i\beta)}.$$

But $\frac{F(\alpha+i\beta)}{\varphi_1(\alpha+i\beta)}$ is a definite complex number which may be written in the form $K+iL$, where K and L are certain real numbers. Thus,

$$M(\alpha+i\beta)+N=K+iL;$$

whence

$$M\alpha+N=K, \quad M\beta=L$$

or

$$M=\frac{L}{\beta}, \quad N=\frac{K\beta-L\alpha}{\beta}.$$

With these values of the coefficients M and N the polynomial $F(x)-(Mx+N)\varphi_1(x)$ has the number $\alpha+i\beta$ for a root, and, hence, also the conjugate number $\alpha-i\beta$. But then the polynomial can be divided, without any remainder, by the differences $x-(\alpha+i\beta)$ and $x-(\alpha-i\beta)$, and, therefore, by their product, which is x^2+px+q. Denoting the quotient of this division by $\Phi_1(x)$, we get

$$F(x)-(Mx+N)\varphi_1(x)=(x^2+px+q)\Phi_1(x).$$

Cancelling x^2+px+q from the last fraction in (4), we get (3), and it is clear that the degree of $\Phi_1(x)$ is less than that of the denominator, which is what we set out to prove.

Now applying to the proper fraction $\frac{F(x)}{f(x)}$ the results of Theorems 1 and 2, we can obtain, successively, all the partial fractions

corresponding to all the roots of the denominator $f(x)$. Thus, from the foregoing there follows the result that

If

$$f(x)=(x-a)^{\alpha}(x-b)^{\beta}\dots(x^2+px+q)^{\mu}\dots(x^2+lx+s)^{\nu},$$

then the fraction $\frac{F(x)}{f(x)}$ *can be represented as follows:*

$$\left.\begin{aligned}\frac{F(x)}{f(x)}=\frac{A}{(x-a)^{\alpha}}+\frac{A_1}{(x-a)^{\alpha-1}}+\dots+\frac{A_{\alpha-1}}{x-a}+\\+\frac{B}{(x-b)^{\beta}}+\frac{B_1}{(x-b)^{\beta-1}}+\dots+\frac{B_{\beta-1}}{x-b}+\\ \dots\dots\dots\dots\dots\dots\dots\dots\\+\frac{Mx+N}{(x^2+px+q)^{\mu}}+\frac{M_1x+N_1}{(x^2+px+q)^{\mu-1}}+\dots+\frac{M_{\mu-1}x+N_{\mu-1}}{x^2+px+q}+\\ \dots\dots\dots\dots\dots\dots\dots\dots\\+\frac{Px+Q}{(x^2+lx+s)^{\nu}}+\frac{P_1x+Q_1}{(x^2+lx+s)^{\nu-1}}+\dots+\frac{P_{\nu-1}x+Q_{\nu-1}}{x^2+lx+s}.\end{aligned}\right\}\quad(5)$$

The coefficients A, $A_1, \dots, B$, B_1, ... may be determined by the following reasoning. This equality is an **identity**; and for this reason, by reducing the fractions to a common denominator we get identical polynomials in the numerators on the right and left. Equating the coefficients of the same degrees of x, we get a system of equations to determine the unknown coefficients A, $A_1, \dots, B, B_1, \dots$

In addition, to determine the coefficients we can take advantage of the following: since the polynomials obtained on the right and left sides of the equality must be identically equal after reducing to a common denominator, their values are equal for all particular values of x. Assigning particular values to x, we get equations for determining the coefficients.

We thus see that every proper rational fraction may be represented in the form of a sum of partial rational fractions.

Example. Let it be required to decompose the fraction $\frac{x^2+2}{(x+1)^3(x-2)}$ into partial fractions. From (5) we have

$$\frac{x^2+2}{(x+1)^3(x-2)}=\frac{A}{(x+1)^3}+\frac{A_1}{(x+1)^2}+\frac{A_2}{x+1}+\frac{B}{x-2}.$$

Reducing to a common denominator and equating the numerators, we have

$$x^2+2=A(x-2)+A_1(x+1)(x-2)+A_2(x+1)^2(x-2)+B(x+1)^3,\quad(6)$$

or

$$x^2+2=(A_2+B)x^3+(A_1+3B)x^2+$$
$$+(A-A_1-3A_2+3B)x+(-2A-2A_1-2A_2+B).$$

Equating the coefficients of x^3, x^2, x^1, x^0 (absolute term), we get a system of equations for determining the coefficients:

$$\begin{aligned}&0=A_2+B,\\&1=A_1+3B,\\&0=A-A_1-3A_2+3B,\\&2=-2A-2A_1-2A_2+B.\end{aligned}$$

Solving this system we find

$$A=-1;\quad A_1=\frac{1}{3};\quad A_2=-\frac{2}{9};\quad B=\frac{2}{9}.$$

It might also be possible to determine some of the coefficients of the equations that result for some particular values of x from equality (6), which is an identity in x.

Thus, setting $x=-1$ we have $3=-3A$ or $A=-1$; setting $x=2$, we have $6=27B$; $B=\frac{2}{9}$.

If to these two equations we add two equations that result from equating the coefficients of the same powers of x, we get four equations for determining the four unknown coefficients. As a result, we have the decomposition

$$\frac{x^2+2}{(x+1)^3(x-2)}=-\frac{1}{(x+1)^3}+\frac{1}{3(x+1)^2}-\frac{2}{9(x+1)}+\frac{2}{9(x-2)}.$$

SEC. 9. INTEGRATION OF RATIONAL FRACTIONS

Let it be required to evaluate the integral of a rational fraction $\frac{Q(x)}{f(x)}$; that is, the integral

$$\int \frac{Q(x)}{f(x)}\,dx.$$

If the given fraction is **improper**, we represent it as the sum of a polynomial $M(x)$ and the **proper** rational fraction $\frac{F(x)}{f(x)}$ (see Sec. 7). This latter we represent, applying formula (5), Sec. 8, as a sum of **partial** fractions. Thus, the integration of a rational fraction reduces to the integration of a polynomial and several **partial** fractions.

From the results of Sec. 8 it follows that the form of partial fractions is determined by the roots of the denominator $f(x)$. Here, the following cases are possible.

Case I. *The roots of the denominator are real and distinct*, that is

$$f(x)=(x-a)(x-b)\ldots(x-d).$$

Here, the fraction $\frac{F(x)}{f(x)}$ is decomposable into partial fractions of type I:

$$\frac{F(x)}{f(x)}=\frac{A}{x-a}+\frac{B}{x-b}+\ldots+\frac{D}{x-d},$$

and then

$$\int \frac{F(x)}{f(x)}dx = \int \frac{A}{x-a}dx + \int \frac{B}{x-b}dx + \ldots + \int \frac{D}{x-d}dx =$$
$$= A\ln|x-a| + B\ln|x-b| + \ldots + D\ln|x-d| + C.$$

Case II. *The roots of the denominator are real, and some of them are multiple:*

$$f(x) = (x-a)^{\alpha}(x-b)^{\beta}\ldots(x-d)^{\delta}.$$

In this case the fraction $\frac{F(x)}{f(x)}$ is decomposable into partial fractions of types I and II.

Example 1. (see example in Sec. 8, Ch. X).

$$\int \frac{x^2+2}{(x+1)^3(x-2)}dx = -\int \frac{dx}{(x+1)^3} + \frac{1}{3}\int \frac{dx}{(x+1)^2} - \frac{2}{9}\int \frac{dx}{x+1} +$$
$$+\frac{2}{9}\int \frac{dx}{x-2} = \frac{1}{2}\frac{1}{(x+1)^2} - \frac{1}{3(x+1)} - \frac{2}{9}\ln|x+1| + \frac{2}{9}\ln|x-2| + C =$$
$$= -\frac{2x-1}{6(x+1)^2} + \frac{2}{9}\ln\left|\frac{x-2}{x+1}\right| + C.$$

Case III. *Among the roots of the denominator there are complex nonrepeating* (that is, *distinct*) *roots*:

$$f(x) = (x^2+px+q)(x^2+lx+s)\ldots(x-a)^{\alpha}\ldots(x-d)^{\delta}.$$

In this case the fraction $\frac{F(x)}{f(x)}$ is decomposable into partial fractions of types I, II, and III.

Example 2. Evaluate the integral

$$\int \frac{x\,dx}{(x^2+1)(x-1)}.$$

Decompose the fraction under the integral sign into partial fractions [see (5), Sec. 8, Ch. X]

$$\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}.$$

Consequently,

$$x = (Ax+B)(x-1) + C(x^2+1).$$

Setting $x=1$, we get $1 = 2C$, $C = \frac{1}{2}$; setting $x=0$, we get $0 = -B + C$, $B = \frac{1}{2}$.

Equating the coefficients of x^2, we get $0=A+C$, whence $A=-\frac{1}{2}$. Thus,

$$\int \frac{x\,dx}{(x^2+1)(x-1)}=-\frac{1}{2}\int \frac{x-1}{x^2+1}\,dx+\frac{1}{2}\int \frac{dx}{x-1}=$$
$$=-\frac{1}{2}\int \frac{x\,dx}{x^2+1}+\frac{1}{2}\int \frac{dx}{x^2+1}+\frac{1}{2}\int \frac{dx}{x-1}=$$
$$=-\frac{1}{4}\ln|x^2+1|+\frac{1}{2}\arctan x+\frac{1}{2}\ln|x-1|+C.$$

Case IV. *Among the roots of the denominator there are complex multiple roots*

$$f(x)=(x^2+px+q)^{\mu}(x^2+lx+s)^{\nu}\ldots(x-a)^{\alpha}\ldots(x-d)^{\delta}.$$

In this case, decomposition of the fraction $\frac{F(x)}{f(x)}$ will also contain partial fractions of type IV.

Example 3. It is required to evaluate the integral

$$\int \frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}\,dx.$$

Solution. Decompose the fraction into partial fractions:

$$\frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}=\frac{Ax+B}{(x^2+2x+3)^2}+\frac{Cx+D}{(x^2+2x+3)}+\frac{E}{x+1},$$

whence

$$x^4+4x^3+11x^2+12x+8=$$
$$=(Ax+B)(x+1)+(Cx+D)(x^2+2x+3)(x+1)+E(x^2+2x+3)^2.$$

Combining the above-indicated methods of determining coefficients, we find

$$A=1,\quad B=-1,\quad C=0,\quad D=0,\quad E=1.$$

Thus, we get

$$\int \frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}\,dx=\int \frac{x-1}{(x^2+2x+3)^2}\,dx+\int \frac{dx}{x+1}=$$
$$=-\frac{x+2}{2(x^2+2x+3)}-\frac{\sqrt{2}}{4}\arctan\frac{x+1}{\sqrt{2}}+\ln|x+1|+C.$$

The first integral on the right was considered in Example 2, Sec. 7, Ch. X. The second integral is taken directly.

From the foregoing it follows that the integral of any rational function may be expressed in terms of elementary functions in final form, namely, in terms of:

1) logarithms in the case of partial fractions of type I;
2) rational functions in the case of partial fractions of type II;

3) logarithms and arc tangents in the case of partial fractions of type III;
4) rational functions and arc tangents in the case of partial fractions of type IV.

SEC. 10. OSTROGRADSKY'S METHOD

In the case of **multiple** roots in the denominator, the integral of a rational function may be evaluated by a different method that leads to simpler computations. This method permits separating out the **rational part** of the integral without decomposing the fraction into partial fractions, and then integrating the rational fraction whose denominator has only **simple** roots. It is easy to integrate such a fraction since it is decomposable into partial fractions of types I and III. This method belongs to the noted Russian mathematician M. V. Ostrogradsky (1801-1862) and is based on the following reasoning.

Let it be required to integrate the proper rational fraction $\frac{F(x)}{f(x)}$, where

$$f(x)=(x-a)^{\alpha}(x-b)^{\beta}\dots(x^2+px+q)^{\mu}.$$

Here, on the basis of (5), Sec. 8, everything is reduced to integrating proper rational fractions of four types (see Sec. 7). Here,

1) the integral of a fraction of the form $\frac{A}{(x-a)^{\alpha}}$ is a fraction of the form $\frac{A^*}{(x-a)^{\alpha-1}}$;

2) the integral of the fraction $\frac{Mx+N}{(x^2+px+q)^{\mu}}$ is a sum of fractions of the form $\frac{M^*x+N^*}{(x^2+px+q)^{\mu^*}}$ where $\mu^*\leqslant\mu-1$, and of an integral of the form

$$\int\frac{N^{**}}{x^2+px+q}\,dx.$$

We will not yet integrate fractions of types I and III.

Combining the rational fractions obtained after integrating fractions of types II and IV, we get a proper fraction of the form $\frac{Y(x)}{Q(x)}$, where the polynomial $Q(x)$ is equal to

$$Q(x)=(x-a)^{\alpha-1}(x-b)^{\beta-1}\dots(x^2+px+q)^{\mu-1}\dots(x^2+lx+s)^{\nu-1}.$$

$Y(x)$ is a polynomial of degree one less than that of the polynomial Q.

Combining the integrals of all the fractions of types I and III (including those integrals of the form $\int \frac{N^{**}}{x^2+px+q}\,dx$ which are obtained by integration of fractions of type IV), we get an integral of a proper fraction of the form $\frac{X(x)}{P(x)}$, where the polynomial $P(x)$ is

$$P(x) = (x-a)(x-b)\ldots(x^2+px+q)\ldots(x^2+lx+s).$$

We thus find that

$$\int \frac{F(x)\,dx}{f(x)} = \frac{Y(x)}{Q(x)} + \int \frac{X(x)}{P(x)}\,dx. \qquad (1)$$

Here $X(x)$ is a polynomial of degree one less than that of the polynomial $P(x)$.

Now let us determine the polynomials $X(x)$ and $Y(x)$ in the numerators. To do this, differentiate both sides of (1):

$$\frac{F(x)}{f(x)} = \frac{QY'-Q'Y}{Q^2} + \frac{X}{P}$$

or

$$F(x) = \frac{f(x)\,Y'}{Q} - \frac{f(x)\,Q'Y}{Q^2} + \frac{f(x)\,X}{P}. \qquad (2)$$

We shall show that the expression on the right is a polynomial. Noting that $f(x) = PQ$ we can rewrite (2) in the form

$$F(x) = PY' - \frac{PQ'Y}{Q} + QX. \qquad (2')$$

What remains now is to prove that the expression $-\frac{PQ'Y}{Q}$ is a polynomial or that PQ' is divisible by Q. We note that

$$\frac{Q'}{Q} = [\ln Q]' = [(\alpha-1)\ln(x-a) + (\beta-1)\ln(x-b) + \ldots$$
$$\ldots + (\mu-1)\ln(x^2+px+q) + \ldots + (\nu-1)\ln(x^2+lx+s)]' =$$
$$= \frac{\alpha-1}{x-a} + \frac{\beta-1}{x-b} + \ldots + \frac{(\mu-1)(2x+p)}{x^2+px+q} + \ldots + \frac{(\nu-1)(2x+l)}{x^2+lx+s}.$$

The polynomial P is the common denominator of the fractions on the right side. In the numerator there will be a certain polynomial of degree less than that of P. Let us denote it by T. Then,

$$\frac{Q'}{Q} = \frac{T}{P}.$$

Hence, the expression

$$P\frac{Q'}{Q}Y = P\frac{T}{P}Y = TY$$

is a polynomial. Equation (2′) takes the form

$$F(x) = PY' - TY + QX. \tag{3}$$

Comparing the coefficients of the same powers of the variable in (3), we get a system of equations from which we find the unknown coefficients of the polynomials X and Y.

Example. Evaluate

$$\int \frac{1}{(x^3-1)^2}\,dx.$$

Solution. In this case,

$$f(x) = (x-1)^2(x^2+x+1)^2,$$
$$P(x) = (x-1)\ (x^2+x+1) = x^3-1,$$
$$Q(x) = \qquad\qquad = x^3-1.$$

Equation (1) has the form

$$\int \frac{dx}{(x^3-1)^2} = \frac{Ax^2+Bx+C}{x^3-1} + \int \frac{Ex^2+Fx+G}{x^3-1}\,dx. \tag{4}$$

Differentiating both sides of (4) we get

$$\frac{1}{(x^3-1)^2} = \frac{(x^3-1)(2Ax+B)-(Ax^2+Bx+C)\,3x^2}{(x^3-1)^2} + \frac{Ex^2+Fx+G}{x^3-1}.$$

Clearing fractions, we have

$$1 = (x^3-1)(2Ax+B) - (Ax^2+Bx+C)\,3x^2 + (x^3-1)(Ex^2+Fx+G).$$

Equating the coefficients of identical powers of x on different sides of the equation, we get a system of six equations for determining the coefficients, A, B, C, E, F, G.

$$0 = E,$$
$$0 = -A+F,$$
$$0 = -2B+G,$$
$$0 = 3C-E,$$
$$0 = -2A-F,$$
$$1 = -B-G.$$

Solving this system we find

$$E=0, \quad A=0, \quad C=0, \quad B=-\frac{1}{3}, \quad F=0, \quad G=-\frac{2}{3}.$$

Putting the values of the coefficients thus found into (4), we get

$$\int \frac{dx}{(x^3-1)^2} = \frac{-\frac{1}{3}x}{x^3-1} + \int \frac{-\frac{2}{3}}{x^3-1}\,dx.$$

The denominator of the latter integral has only simple roots, thus making it easy to compute the integral. We finally obtain

$$\int \frac{dx}{(x^3-1)^2} = \frac{-x}{3(x^3-1)} + \int \left[\frac{-\frac{2}{9}}{x-1} + \frac{\frac{2}{9}x+\frac{4}{9}}{x^2+x+1} \right] dx =$$

$$= -\frac{x}{3(x^2-1)} - \frac{2}{9}\ln|x-1| + \frac{1}{9}\ln(x^2+x+1) + \frac{2\sqrt{3}}{9}\arctan\frac{2x+1}{\sqrt{3}} + C.$$

SEC. 11. INTEGRALS OF IRRATIONAL FUNCTIONS

It is impossible to express in terms of elementary functions the integral of every irrational function. In this and the following sections we shall consider irrational functions whose integrals are reduced (by means of substitution) to integrals of rational functions and, consequently, are integrated to the end.

1. We consider the integral $\int R\left(x, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}\right) dx$ where R is a rational function of its arguments.*)

Let k be a common denominator of the fractions $\frac{m}{n}, \ldots, \frac{r}{s}$. We make the substitution

$$x = t^k, \quad dx = kt^{k-1}\,dt.$$

Then each **fractional** power of x will be expressed in terms of an **integral** power of t and the integrand will thus be transformed into a **rational function of** t.

Example 1. It is required to compute the integral

$$\int \frac{x^{\frac{1}{2}}\,dx}{x^{\frac{3}{4}}+1}.$$

Solution. The common denominator of the fractions $\frac{1}{2}$, $\frac{3}{4}$ is 4; and so we substitute: $x=t^4$, $dx=4t^3dt$; then

$$\int \frac{x^{\frac{1}{2}}\,dx}{x^{\frac{3}{4}}+1} = 4\int \frac{t^2}{t^3+1}\,t^3\,dt = 4\int \frac{t^5}{t^3+1}\,dt = 4\int\left(t^2 - \frac{t^2}{t^3+1}\right)dt =$$

$$= 4\int t^2\,dt - 4\int \frac{t^2}{t^3+1}\,dt = 4\,\frac{t^3}{3} - \frac{4}{3}\ln|t^3+1| + C =$$

$$= \frac{4}{3}\left[x^{\frac{3}{4}} - \ln\left|x^{\frac{3}{4}}+1\right|\right] + C.$$

---

*) The notation $R\left(x, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}\right)$ indicates that only **rational** operations are performed on the quantities $x, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}$.

This is precisely the way that the following notations are henceforward to be understood: $R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \ldots\right)$, $R\,(x, \sqrt{ax^2+bx+c})$, $R\,(\sin x, \cos x)$, etc. For instance, the notation $R\,(\sin x, \cos x)$ indicates that rational operations are to be performed on $\sin x$ and $\cos x$.

II. Now consider an integral of the form

$$\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right] dx.$$

This integral reduces to the integral of a rational function by means of substitution:

$$\frac{ax+b}{cx+d} = t^k,$$

where k is the common denominator of the fractions $\frac{m}{n}, \ldots, \frac{r}{s}$.

Example 2. It is required to compute the integral

$$\int \frac{\sqrt{x+4}}{x} dx.$$

Solution. We make the substitution $x+4=t^2$, $x=t^2-4$; $dx=2t\,dt$; then

$$\int \frac{\sqrt{x+4}}{x} dx = 2\int \frac{t^2}{t^2-4} dt = 2\int \left(1+\frac{4}{t^2-4}\right) dt = 2\int dt + 8\int \frac{dt}{t^2-4} =$$

$$= 2t + 2\ln\left|\frac{t-2}{t+2}\right| + C = 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C.$$

SEC. 12. INTEGRALS OF THE FORM $\int R(x, \sqrt{ax^2+bx+c})\, dx$

Let us consider the integral

$$\int R\,(x, \sqrt{ax^2+bx+c})\, dx. \tag{1}$$

An integral of this kind reduces to the integral of a rational function of a new variable by means of the following Euler substitutions.

1. *First Euler substitution.* If $a>0$, then we put

$$\sqrt{ax^2+bx+c} = \pm\sqrt{a}x + t.$$

For the sake of definiteness we take the plus sign in front of $\sqrt{a}$. Then

$$ax^2+bx+c = ax^2 + 2\sqrt{a}xt + t^2,$$

whence x is determined as a rational function of t:

$$x = \frac{t^2-c}{x-2\sqrt{at}}$$

(thus, dx will also be expressed rationally in terms of t). Therefore,

$$\sqrt{ax^2+bx+c} = \sqrt{a}x + t = \sqrt{a}\frac{t^2-c}{b-2t\sqrt{a}} + t,$$

and $\sqrt{ax^2+bx+c}$ is a rational function of t.

Since $\sqrt{ax^2+bx+c}$, x and dx are expressed rationally in terms of t, the given integral (1) is transformed into an integral of a rational function of t.

Example 1. It is required to compute the integral

$$\int \frac{dx}{\sqrt{x^2+C}}.$$

Solution. Since here $a=1>0$, we put $\sqrt{x^2+C} = -x+t$; then

$$x^2+C = x^2-2xt+t^2,$$

whence

$$x = \frac{t^2-C}{2t}.$$

Consequently,

$$dx = \frac{t^2+C}{2t^2}\, dt,$$

$$\sqrt{x^2+C} = -x+t = -\frac{t^2-C}{2t}+t = \frac{t^2+C}{2t}.$$

Returning to the initial integral, we have

$$\int \frac{dx}{\sqrt{x^2+C}} = \int \frac{\frac{t^2+C}{2t^2}\, dt}{\frac{t^2+C}{2t}} = \int \frac{dt}{t} = \ln|t|+C_1 = \ln|x+\sqrt{x^2+C}|+C_1$$

(see formula 14 in the Table of Integrals).

2. *Second Euler substitution*. If $c>0$, we put

$$\sqrt{ax^2+bx+c} = xt \pm \sqrt{c};$$

then

$$ax^2+bx+c = x^2t^2+2xt\sqrt{c}+c.$$

(For the sake of definiteness we took the plus sign in front of the radical.) Then x is determined as a rational function of t:

$$x = \frac{2\sqrt{c}\,t-b}{a-t^2}.$$

Since dx and $\sqrt{ax^2+bx+c}$ are also expressed rationally in terms of t, by substituting the values of x, $\sqrt{ax^2+bx+c}$ and dx into

the integral $\int R(x, \sqrt{ax^2+bx+c})\,dx$, we reduce it to an integral of a rational function of t.

Example 2. It is required to compute the integral

$$\int \frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{1+x+x^2}}\,dx.$$

Solution. We set $\sqrt{1+x+x^2}=xt+1$; then

$$1+x+x^2=x^2t^2+2xt+1; \quad x=\frac{2t-1}{1-t^2}; \quad dx=\frac{2t^2-2t+2}{(1-t^2)^2}\,dt;$$

$$\sqrt{1+x+x^2}=xt+1=\frac{t^2-t+1}{1-t^2};$$

$$1-\sqrt{1+x+x^2}=\frac{-2t^2+t}{1-t^2}.$$

Putting the expressions obtained into the original integral, we find

$$\int \frac{(1-\sqrt{1+x+x^2})^2}{x^2\sqrt{1+x+x^2}}\,dx=\int \frac{(-2t^2+t)^2(1-t^2)^2(1-t^2)(2t^2-2t+2)}{(1-t^2)^2(2t-1)^2(t^2-t+1)(1-t^2)^2}\,dx=$$

$$=+2\int \frac{t^2}{1-t^2}\,dt+C=-2t+\ln\left|\frac{1+t}{1-t}\right|+C=$$

$$=-\frac{2(\sqrt{1+x+x^2}-1)}{x}+\ln\left|\frac{x+\sqrt{1+x+x^2}-1}{x-\sqrt{1+x+x^2}+1}\right|+C=$$

$$=-\frac{2(\sqrt{1+x+x^2}-1)}{x}+\ln|2x+2\sqrt{1+x+x^2}+1|+C.$$

3. *Third Euler substitution.* Let α and β be the real roots of the trinomial ax^2+bx+c. We put

$$\sqrt{ax^2+bx+c}=(x-\alpha)\,t.$$

Since $ax^2+bx+c=a(x-\alpha)(x-\beta)$, we have

$$\sqrt{a(x-\alpha)(x-\beta)}=(x-\alpha)\,t,$$
$$a(x-\alpha)(x-\beta)=(x-\alpha)^2t^2,$$
$$a(x-\beta)=(x-\alpha)\,t^2.$$

Whence we find x as a rational function of t:

$$x=\frac{a\beta-\alpha t^2}{a-t^2}.$$

Since dx and $\sqrt{ax^2+bx+c}$ also rationally depend upon t, the given integral is transformed into an integral of a rational function of t.

Note 1. The third Euler substitution is applicable not only for $a<0$, but also for $a>0$, provided the polynomial ax^2+bx+c has two real roots.

Example 3. It is required to compute the integral

$$\int \frac{dx}{\sqrt{x^2+3x-4}}.$$

Solution. Since $x^2+3x-4=(x+4)(x-1)$, we put

$$\sqrt{(x+4)(x-1)}=(x+4)t;$$

then

$$(x+4)(x-1)=(x+4)^2t^2, \quad x-1=(x+4)t^2,$$

$$x=\frac{1+4t^2}{1-t^2}, \quad dx=\frac{10t}{(1-t^2)^2}dt,$$

$$\sqrt{(x+4)(x-1)}=\left[\frac{1+4t^2}{1-t^2}+4\right]t=\frac{5t}{1-t^2}.$$

Returning to the original integral, we have

$$\int \frac{dx}{\sqrt{x^2+3x-4}}=\int \frac{10t(1-t^2)}{(1-t^2)^2 5t}dt=\int \frac{2}{1-t^2}dt=\ln\left|\frac{1+t}{1-t}\right|+C=$$

$$=\ln\left|\frac{1+\sqrt{\frac{x-1}{x+4}}}{1-\sqrt{\frac{x-1}{x+4}}}\right|+C=\ln\left|\frac{\sqrt{x+4}+\sqrt{x-1}}{\sqrt{x+4}-\sqrt{x-1}}\right|+C.$$

Note 2. It will be noted that to reduce integral (1) to an integral of a rational function, the first and third Euler substitutions are sufficient. Let us consider the trinomial ax^2+bx+c. If $b^2-4ac>0$, then the roots of the trinomial are real, and, hence, the third Euler substitution is applicable. If $b^2-4ac\leqslant 0$, then in this case

$$ax^2+bx+c=\frac{1}{4a}[(2ax+b)^2+(4ac-b^2)]$$

and therefore the trinomial has the same sign as that of a. For $\sqrt{ax^2+bx+c}$ to be real it is necessary that the trinomial be positive, and we must have $a>0$. In this case, the first substitution is applicable.

SEC. 13. INTEGRATION OF BINOMIAL DIFFERENTIALS

An expression of the form

$$x^m(a+bx^n)^p dx,$$

where m, n, p, a, b are constants is called a *binomial differential*.

Theorem. *The integral of a binomial differential*

$$\int x^m (a+bx^n)^p\,dx$$

if m, n, p are rational numbers, is reduced to an integral of a rational function and thus is expressed in terms of elementary functions in the following three cases:

1. *p is an integer* (positive, negative or zero);
2. $\frac{m+1}{n}$ *is an integer* (positive, negative or zero);
3. $\frac{m+1}{n}+p$ *is an integer* (positive, negative or zero).

Proof. Transform the given integral by substitution:

$$x=z^{\frac{1}{n}},\quad dx=\frac{1}{n}z^{\frac{1}{n}-1}dz.$$

Then

$$\int x^m (a+bx^n)^p\,dx=\frac{1}{n}\int z^{\frac{m+1}{n}-1}(a+bz)^p\,dz=\frac{1}{n}\int z^q (a+bz)^p\,dz, \quad (1)$$

where

$$q=\frac{m+1}{n}-1.$$

1. Let p be an integer. Since q is a rational number, we denote it by $\frac{r}{s}$. Integral (1) is then of the form

$$\int R\,(z^{\frac{r}{s}}, z)\,dx.$$

As was pointed out in Sec. 11, Ch. X, it reduces to an integral of a rational function by the substitution $z=t^s$.

2. Let $\frac{m+1}{n}$ be an integer. Then $q=\frac{m+1}{n}-1$ is also an integer. The number p is rational, $p=\frac{\lambda}{\mu}$. Here the integral (1) is of the form

$$\int R\,[x^q, (a+bz)^{\frac{\lambda}{\mu}}]\,dx.$$

This integral was considered in Sec. 11, Ch. X. It reduces to an integral of a rational function by the substitution $a+bz=t^\mu$.

3. Let $\frac{m+1}{n}+p$ be an integer. But then $\frac{m+1}{n}-1+p=q+p$ is an integer. We transform integral (1):

$$\int z^q (a+bz)^p\,dz=\int z^{q+p}\left(\frac{a+bz}{z}\right)^p dz,$$

where $q+p$ is an integer and $p=\frac{k}{l}$ is a rational number. The latter integral belongs to the class of integrals

$$\int R\left[z, \left(\frac{a+bz}{z}\right)^{\frac{k}{l}}\right] dz.$$

This integral was considered in Sec. 11, Ch. X. It reduces to an integral of a rational function by the substitution $\frac{a+bz}{z}=t^l$.

Let us examine examples of integration in all the three cases.

Example 1. $\int \frac{dx}{\sqrt[3]{x^2}(1+\sqrt[3]{x^2})}=\int x^{-\frac{2}{3}}(1+x^{\frac{2}{3}})^{-1} dx$. Here $p=-1$ (integer).

Putting $x^{\frac{2}{3}}=z$, we make the quantity in parentheses linear in z:

$$\int x^{-\frac{2}{3}}\left(1+x^{\frac{2}{3}}\right)^{-1} dx=\int z^{-1}(1+z)^{-1}\frac{3}{2}z^{\frac{1}{2}}dz=\frac{3}{2}\int z^{-\frac{1}{2}}(1+z)^{-1}dz.$$

Now make the substitution $z^{\frac{1}{2}}=t$. Then $z=t^2$, $dz=2t\,dt$ and

$$\int x^{-\frac{2}{3}}\left(1+x^{\frac{2}{3}}\right)^{-1} dx=\frac{3}{2}\int z^{-\frac{1}{2}}(1+z)^{-1}dz=\frac{3}{2}\int t^{-1}(1+t^2)^{-1}\,2t\,dt=$$

$$=3\int\frac{ds}{1+t^2}=3\arctan t+C=3\arctan\sqrt{z}+C=3\arctan\sqrt[3]{x}+C.$$

Example 2. $\int\frac{x^3}{\sqrt{1-x^2}}dx=\int x^3(1-x^2)^{-\frac{1}{2}}dx$. Here, $m=3$, $n=2$, $p=-\frac{1}{2}$, $\frac{m+1}{n}=2$ (integer!). We substitute $x^2=z$; then $x=z^{\frac{1}{2}}$, $dx=\frac{1}{2}z^{-\frac{1}{2}}dz$ and

$$\int\frac{x^3}{\sqrt{1-x^2}}dx=\int z^{\frac{3}{2}}(1-z)^{-\frac{1}{2}}\frac{1}{2}z^{-\frac{1}{2}}dz=\frac{1}{2}\int z(1-z)^{-\frac{1}{2}}dz.$$

In order to make the second parenthesis rational we put $(1-z)^{\frac{1}{2}}=t$; then $1-z=t^2$; $z=t^2-1$; $dz=2t\,dt$. Hence,

$$\int\frac{x^3}{\sqrt{1-x^2}}dx=\frac{1}{2}\int z(1-z)^{-\frac{1}{2}}dz=\frac{1}{2}\int(t^2-1)\,t^{-1}\,2t\,dt=\int(t^2-1)\,dt=$$

$$=\frac{t^3}{3}-t+C=\frac{t}{3}(t^2-3)+C=\frac{\sqrt{1-z}}{3}(-z-2)=C=\frac{\sqrt{1-x^2}}{3}(-x^2-2)+C.$$

Example 3. $\int \frac{dx}{x^2\sqrt{(1+x^2)^3}} = \int x^{-2}(1+x^2)^{-\frac{3}{2}}dx$. Here, $m=-2$, $n=2$, $p=-\frac{3}{2}$ and $\frac{m+1}{n}+p=-2$ (integer). We reduce the expression in the parentheses to a linear function:

$$x^2=z;\quad x=z^{\frac{1}{2}};\quad dx=\frac{1}{2}z^{-\frac{1}{2}}dz;$$

$$\int x^{-2}(1+x^2)^{-\frac{3}{2}}dx=\int z^{-1}(1+z)^{-\frac{3}{2}}\frac{1}{2}z^{-\frac{1}{2}}dz=$$

$$=\frac{1}{2}\int z^{-\frac{3}{2}}(1+z)^{-\frac{3}{2}}dz=\frac{1}{2}\int z^{-3}\left(\frac{1+z}{z}\right)^{-\frac{3}{2}}dz.$$

he first factor is a rational function. In order to make the second factor rational as well, we make the substitution:

$$\left(\frac{1+z}{z}\right)^{\frac{1}{2}}=t;$$

then

$$\frac{1+z}{z}=t^2;\quad z=\frac{1}{t^2-1};\quad dz=-\frac{2t\,dt}{(t^2-1)^2}.$$

Thus,

$$\int x^{-2}(1+x^2)^{-\frac{3}{2}}dx=\frac{1}{2}\int z^{-3}\left(\frac{1+z}{z}\right)^{-\frac{3}{2}}dz=$$

$$=\frac{1}{2}\int (t^2-1)^3\,t^{-3}\frac{-2t\,dt}{(t^2-1)^2}=-\int\frac{t^2-1}{t^2}dt=-t-\frac{1}{t}+C=$$

$$=-\left(\frac{1+z}{z}\right)^{\frac{1}{2}}-\left(\frac{z}{1+z}\right)^{\frac{1}{2}}+C=-\left(\frac{1+x^2}{x^2}\right)^{\frac{1}{2}}-\left(\frac{x^2}{1+x^2}\right)^{\frac{1}{2}}+C=$$

$$=-\frac{\sqrt{1+x^2}}{x}-\frac{x}{\sqrt{1+x^2}}+C.$$

Note. The noted Russian mathematician P. L. Chebyshev proved that **only** in the above three cases in an integral of binomial differentials with rational exponents expressed in terms of elementary functions (provided, of course, that $a\neq 0$ and $b\neq 0$). But if neither p, nor $\frac{m+1}{n}$, nor $\frac{m+1}{n}+p$ are integers, then the integral cannot be expressed in terms of elementary functions.

SEC. 14. INTEGRATION OF CERTAIN CLASSES OF TRIGONOMETRIC FUNCTIONS

Up to now we have made a systematic study only of the integrals of algebraic functions (rational and irrational). In this section we shall consider integrals of certain classes of nonalgebraic

functions, primarily trigonometric. Let us consider an integral of the form

$$\int R(\sin x, \cos x)\,dx. \qquad (1)$$

We shall show that this integral, by the substitution

$$\tan\frac{x}{2} = t \qquad (2)$$

always reduces to an integral of a rational function. Let us express $\sin x$ and $\cos x$ in terms of $\tan\frac{x}{2}$, and hence, in terms of t:

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{1} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}.$$

And

$$x = 2\arctan t, \quad dx = \frac{2dt}{1+t^2}.$$

In this way, $\sin x$, $\cos x$ and dx are expressed rationally in terms of t. Since a rational function of rational functions is a rational function, by substituting the expressions obtained into the integral (1) we get an integral of a rational function:

$$\int R(\sin x, \cos x)\,dx = \int R\left[\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right]\frac{2dt}{1-t^2}.$$

Example 1. Consider the integral

$$\int \frac{dx}{\sin x}.$$

On the basis of the foregoing formulas we have

$$\int \frac{dx}{\sin x} = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} = \ln|t| + C = \ln\left|\tan\frac{x}{2}\right| + C.$$

This substitution enables us to integrate any function of the form $R(\cos x, \sin x)$. For this reason it is sometimes called a "universal trigonometric substitution". However, in practice it frequently leads to extremely complex rational functions. It is

therefore convenient to know some other substitutions (in addition to the "universal" one) that sometimes lead more quickly to the desired end.

1) If an integral is of the form $\int R(\sin x)\cos x\,dx$ the substitution $\sin x = t$, $\cos x\,dx = dt$ reduces this integral to the form $\int R(t)\,dt$.

2) If the integral has the form $\int R(\cos x)\sin x\,dx$, it is reduced to an integral of a rational function by the substitution $\cos x = t$, $\sin x\,dx = -dt$.

3) If the integrand is dependent only on $\tan x$, then the substitution $\tan x = t$, $x = \text{arc}\tan t$, $dx = \frac{dt}{1+t^2}$ reduces this integral to an integral of a rational function:

$$\int R(\tan x)\,dx = \int R(t)\frac{dt}{1+t^2}.$$

4) If the integrand has the form $R(\sin x, \cos x)$, but $\sin x$ and $\cos x$ are involved only in **even** powers, then the same substitution is applied:

$$\tan x = t, \tag{2'}$$

because $\sin^2 x$ and $\cos^2 x$ are expressed rationally in terms of $\tan x$:

$$\cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2},$$

$$\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x} = \frac{t^2}{1+t^2},$$

$$dx = \frac{dt}{1+t^2}.$$

After the substitution we obtain an integral of a rational function.

Example 2. Compute the integral $\int \frac{\sin^3 x}{2+\cos x}\,dx$.

Solution. This integral is readily reduced to the form $\int R(\cos x)\sin x\,dx$. Indeed,

$$\int \frac{\sin^3 x}{2+\cos x}\,dx = \int \frac{\sin^2 x \sin x\,dx}{2+\cos x} = \int \frac{1-\cos^2 x}{2+\cos x}\sin x\,dx.$$

We make the substitution: $\cos x = z$. Then $\sin x\,dx = -dz$:

$$\int \frac{\sin^3 x}{2+\cos x}\,dx = \int \frac{1-z^2}{2+z}(-dz) = \int \frac{z^2-1}{z+2}\,dz = \int \left(z-2+\frac{3}{z+2}\right)dz =$$
$$= \frac{z^2}{2} - 2z + 3\ln(z+2) + C = \frac{\cos^2 x}{2} - 2\cos x + 3\ln(\cos x + 2) + C.$$

Example 3. Compute $\int \frac{dx}{2-\sin^2 x}$.
Make the substitution $\tan x = t$:

$$\int \frac{dx}{2-\sin^2 x} = \int \frac{dt}{\left(2-\frac{t^2}{1+t^2}\right)(1+t^2)} = \int \frac{dt}{2+t^2} = \frac{1}{\sqrt{2}} \arctan \frac{t}{\sqrt{2}} +$$
$$+C = \frac{1}{\sqrt{2}} \arctan \left(\frac{\tan x}{\sqrt{2}}\right) + C.$$

5) Now let us consider one more integral of the form $\int R(\sin x, \cos x)\,dx$, namely an integral under the sign of which is the product $\sin^m x \cos^n x\,dx$ (where m and n are integers). Here we shall have to consider three cases.

a) $\int \sin^m x \cos^n x\,dx$, where m and n are such that at least one of them is odd. For definiteness let us assume that n is odd. Put $n = 2p+1$ and transform the integral:

$$\int \sin^m x \cos^{2p+1} x\,dx = \int \sin^m x \cos^{2p} x \cos x\,dx =$$
$$= \int \sin^m x (1-\sin^2 x)^p \cos x\,dx.$$

Change the variable

$$\sin x = t, \quad \cos x\,dx = dt.$$

Putting the new variable into the given integral, we get

$$\int \sin^m x \cos^n x\,dx = \int t^m (1-t^2)^p\,dt,$$

which is an integral of a rational function of t.

Example 4.

$$\int \frac{\cos^3 x}{\sin^4 x}dx = \int \frac{\cos^2 x \cos x\,dx}{\sin^4 x} = \int \frac{(1-\sin^2 x)\cos x\,dx}{\sin^4 x}.$$

Denoting $\sin x = t$, $\cos x\,dx = dt$, we get

$$\int \frac{\cos^3 x}{\sin^4 x}dx = \int \frac{(1-t^2)\,dt}{t^4} = \int \frac{dt}{t^4} - \int \frac{dt}{t^2} = -\frac{1}{3t^3} + \frac{1}{t} + C =$$
$$= -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + C.$$

b) $\int \sin^m x \cos^n x\,dx$, where m and n are nonnegative and even numbers.

Put $m = 2p$, $n = 2q$. Write the familiar trigonometric formulas:

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x, \quad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x. \qquad (3)$$

Putting them into the integral we get

$$\int \sin^{2p} x \cos^{2q} x\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^p \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^q dx.$$

Powering and opening brackets, we get terms containing $\cos 2x$ in odd and even powers. The terms with odd powers are integrated as indicated in Case (a). We again reduce the even exponents by formulas (3). Continuing in this manner we arrive at terms of the form $\int \cos kx\, dx$, which can easily be integrated.

Example 5.

$$\int \sin^4 x\, dx = \frac{1}{2^2}\int (1-\cos 2x)^2\, dx = \frac{1}{4}\int (1-2\cos 2x + \cos^2 2x)\, dx =$$
$$= \frac{1}{4}\left[x - \sin 2x + \frac{1}{2}\int (1+\cos 4x)\, dx\right] = \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right] + C.$$

c) If both exponents are even, and at least one of them is negative, then the preceding technique does not give the desired result. Here, one should make the substitution $\tan x = t$ (or $\cot x = t$).

Example 6.

$$\int \frac{\sin^2 x\, dx}{\cos^6 x} = \int \frac{\sin^2 x(\sin^2 x + \cos^2 x)^2}{\cos^6 x} dx = \int \tan^2 x(1+\tan^2 x)^2\, dx.$$

Put $\tan x = t$; then $x = \arctan t$, $dx = \frac{dt}{1+t^2}$ and we get

$$\int \frac{\sin^2 x}{\cos^6 x} dx = \int t^2(1+t^2)^2 \frac{dt}{1+t^2} = \int t^2(1+t^2)\, dt = \frac{t^3}{3} + \frac{t^5}{5} + C =$$
$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C.$$

6) In conclusion let us consider integrals of the form

$$\int \cos mx \cos nx\, dx, \quad \int \sin mx \cos nx\, dx, \quad \int \sin mx \sin nx\, dx.$$

They are taken by means of the following*) formulas ($m \neq n$):

$$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x],$$

*) These formulas are easily derived as follows:

$$\cos(m+n)x = \cos mx \cos nx - \sin mx \sin nx;$$
$$\cos(m-n)x = \cos mx \cos nx + \sin mx \sin nx.$$

Combining these equations termwise and dividing them in half, we get the first of the three formulas. Subtracting termwise and dividing in half, we get the third formula. The second formula is similarly derived if we write analogous equations for $\sin(m+n)x$ and $\sin(m-n)x$ and then combine them termwise.

$$\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x],$$

$$\sin mx \sin nx = \frac{1}{2}[-\cos(m+n)x + \cos(m-n)x].$$

Substituting and integrating, we get

$$\int \cos mx \cos nx\, dx = \frac{1}{2}\int [\cos(m+n)x + \cos(m-n)x]\, dx =$$
$$= \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C.$$

The other two integrals are evaluated similarly.

Example 7.

$$\int \sin 5x \sin 3x\, dx = \frac{1}{2}\int [-\cos 8x + \cos 2x]\, dx = -\frac{\sin 8x}{16} + \frac{\sin 2x}{4} + C.$$

SEC. 15. INTEGRATION OF CERTAIN IRRATIONAL FUNCTIONS BY MEANS OF TRIGONOMETRIC SUBSTITUTIONS

Let us return to the integral considered in Sec. 12, Ch. X:

$$\int R(x, \sqrt{ax^2+bx+c})\, dx. \tag{1}$$

Here we shall give a method of transforming this integral into one of the form

$$\int \overline{R}(\sin z, \cos z)\, dz, \tag{2}$$

which was considered in the preceding section.

Transform the trinomial under the radical sign:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right).$$

Change the variable, putting

$$x + \frac{b}{2a} = t, \quad dx = dt.$$

Then

$$\sqrt{ax^2+bx+c} = \sqrt{at^2 + \left(c - \frac{b^2}{4a}\right)}.$$

Let us consider all possible cases.

1. Let $a > 0$, $c - \frac{b^2}{4a} > 0$. We introduce the designations: $a = m^2$, $c - \frac{b}{2a} = n^2$. In this case we have

$$\sqrt{ax^2+bx+c} = \sqrt{m^2t^2+n^2}.$$

2. Let $a>0$, $c-\frac{b^2}{4a}<0$. Then

$$a=m^2, \quad c-\frac{b^2}{4a}=-n^2.$$

Thus,

$$\sqrt{ax^2+bx+c}=\sqrt{m^2t^2-n^2}.$$

3. Let $a<0$, $c-\frac{b^2}{4a}>0$. Then

$$a=-m^2, \quad c-\frac{b^2}{4a}=n^2.$$

Hence,

$$\sqrt{ax^2+bx+c}=\sqrt{n^2-m^2t^2}.$$

4. Let $a<0$, $c-\frac{b^2}{4a}<0$. In this case $\sqrt{ax^2+bx+c}$ is a complex number for every value of x.

In this way, integral (1) is reduced to one of the following types of integrals:

$$\text{I.} \int R\left(t, \sqrt{m^2t^2+n^2}\right) dt. \tag{3.1}$$

$$\text{II.} \int R\left(t, \sqrt{m^2t^2-n^2}\right) dt. \tag{3.2}$$

$$\text{III.} \int R\left(t, \sqrt{n^2-m^2t^2}\right) dt. \tag{3.3}$$

Obviously, integral (3.1) is reduced to an integral of the form (2) by the substitution

$$t=\frac{n}{m}\tan z.$$

Integral (3.2) is reduced to the form (2) by the substitution

$$t=\frac{n}{m}\sec z.$$

Integral (3.3) is reduced to (2) by the substitution

$$t=\frac{n}{m}\sin t.$$

Example. Compute the integral

$$\int \frac{dx}{\sqrt{(a^2-x^2)^3}}.$$

Solution. This is an integral of type III. Make the substitution $x=a\sin z$, then

$$dx=a\cos z\, dz,$$

$$\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \int \frac{a \cos z\, dz}{\sqrt{(a^2-a^2 \sin^2 z)^3}} = \int \frac{a \cos z\, dz}{a^3 \cos^3 z} = \frac{1}{a^2} \int \frac{dz}{\cos^2 z} = \frac{1}{a^2} \tan z +$$

$$+ C = \frac{1}{a^2} \frac{\sin z}{\cos z} + C = \frac{1}{a^2} \frac{\sin z}{\sqrt{1-\sin^2 z}} + C = \frac{1}{a^2} \frac{x}{\sqrt{a^2-x^2}} + C.$$

SEC. 16. FUNCTIONS WHOSE INTEGRALS CANNOT BE EXPRESSED IN TERMS OF ELEMENTARY FUNCTIONS

In Sec. 1, Ch. X, we pointed out (without proof) that any function $f(x)$ continuous on the interval (a, b) has an antiderivative on this interval; in other words, there exists a function $F(x)$ such that $F'(x) = f(x)$. However, **not every antiderivative**, even when it exists, **is expressible, in final form, in terms of elementary functions.**

For instance, we have already pointed out that the antiderivatives of binomial differentials that do not belong to the three examined types cannot be expressed in terms of elementary functions in final form (Chebyshev's theorem). Such are the antiderivatives expressed by the integrals $\int e^{-x^2} dx$, $\int \frac{\sin x}{x} dx$, $\int \frac{\cos x}{x} dx$, $\int \sqrt{1-k^2 \sin^2 x}\, dx$, $\int \frac{dx}{\ln x}$ and many others.

In all such cases, the antiderivative is obviously some new function which does not reduce to a combination of a finite number of elementary functions.

For example, that one of the antiderivatives

$$\int e^{-x^2} dx + C,$$

which vanishes for $x = 0$ is called the Gauss function and is denoted by $\Phi(x)$. Thus,

$$\Phi(x) = \int e^{-x^2} dx + C_1,$$

if

$$\Phi(0) = 0.$$

This function has been studied in detail. Tables of its values for various values of x have been compiled. We shall see how this is done in Sec. 21, Ch. XVI. Figs. 204 and 205 show the graph of the integrand $y = e^{-x^2}$ and the graph of the Gauss function $y = \Phi(x)$. That one of the antiderivatives

$$\int \sqrt{1-k^2 \sin^2 x}\, dx + C \quad (k < 1),$$

which vanishes for $x = 0$ is called an "elliptic integral" and is

denoted by $E(x)$,

$$E(x)=\int\sqrt{1-k^2\sin^2 x}\,dx+C_2,$$

if

$$E(0)=0.$$

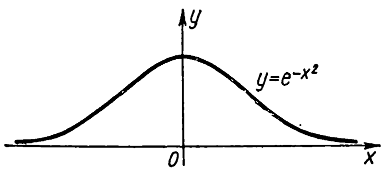


Fig. 204.

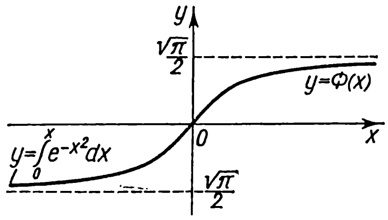


Fig. 205.

Tables of the values of this function have also been compiled for various values of x.

Exercises on Chapter X

I. Compute the integrals: **1.** $\int x^5\,dx$. *Ans.* $\frac{x^6}{6}+C$. **2.** $\int (x+\sqrt{x})\,dx$. *Ans.* $\frac{x^2}{2}+\frac{2x\sqrt{x}}{3}+C$. **3.** $\int\left(\frac{3}{\sqrt{x}}-\frac{x\sqrt{x}}{4}\right)dx$. *Ans.* $6\sqrt{x}-\frac{1}{10}x^2\sqrt{x}+C$. **4.** $\int\frac{x^2\,dx}{\sqrt{x}}$. *Ans.* $\frac{2}{5}x^2\sqrt{x}+C$. **5.** $\int\left(\frac{1}{x^2}+\frac{4}{x\sqrt{x}}+2\right)dx$. *Ans.* $-\frac{1}{x}-\frac{8}{\sqrt{x}}+2x+C$. **6.** $\int\frac{dx}{\sqrt[4]{x}}$. *Ans.* $\frac{4}{3}\sqrt[4]{x^3}+C$. **7.** $\int\left(x^2+\frac{1}{\sqrt[3]{x}}\right)^2dx$. *Ans.* $\frac{x^5}{5}+\frac{3}{4}x^2\sqrt[3]{x^2}+3\sqrt[3]{x}+C$.

Integration by substitution: **8.** $\int e^{5x}\,dx$. *Ans.* $\frac{1}{5}e^{5x}+C$. **9.** $\int\cos 5x\,dx$. *Ans.* $\frac{\sin 5x}{5}+C$. **10.** $\int\sin ax\,dx$. *Ans.* $-\frac{\cos ax}{a}+C$. **11.** $\int\frac{\ln x}{x}\,dx$. *Ans.* $\frac{1}{2}\ln^2 x+C$. **12.** $\int\frac{dx}{\sin^2 3x}$. *Ans.* $-\frac{\cot 3x}{3}+C$. **13.** $\int\frac{dx}{\cos^2 7x}$. *Ans.* $\frac{\tan 7x}{7}+C$. **14.** $\int\frac{dx}{3x-7}$. *Ans.* $\frac{1}{3}\ln|3x-7|+C$. **15.** $\int\frac{dx}{1-x}$. *Ans.* $-\ln|1-x|+C$. **16.** $\int\frac{dx}{5-2x}$. *Ans.* $-\frac{1}{2}\ln|5-2x|+C$. **17.** $\int\tan 2x\,dx$. *Ans.* $-\frac{1}{2}\ln|\cos 2x|+C$. **18.** $\int\cot(5x-7)\,dx$. *Ans.* $\frac{1}{5}\ln|\sin(5x-7)|+C$.

19. $\int \frac{dy}{\cot 3y}$. *Ans.* $-\frac{1}{3}\ln|\cos 3y|+C$. 20. $\int \cot\frac{x}{3}\,dx$. *Ans.* $3\ln\left|\sin\frac{x}{3}\right|+C$.

21. $\int \tan\varphi\cdot\sec^2\varphi\,d\varphi$. *Ans.* $\frac{1}{2}\tan^2\varphi+C$. 22. $\int(\cot e^x)\,e^x\,dx$. *Ans.* $\ln|\sin e^x|+C$. 23. $\int\left(\tan 4S-\cot\frac{S}{4}\right)dS$. *Ans.* $-\frac{1}{4}\ln|\cos 4S|-4\ln\left|\sin\frac{S}{4}\right|+C$. 24. $\int\sin^2 x\cos x\,dx$. *Ans.* $\frac{\sin^3 x}{3}+C$. 25. $\int\cos^3 x\sin x\,dx$. *Ans.* $-\frac{\cos^4 x}{4}+C$. 26. $\int\sqrt{x^2+1}\,x\,dx$. *Ans.* $\frac{1}{3}\sqrt{(x^2+1)^3}+C$.

27. $\int\frac{x\,dx}{\sqrt{2x^2+3}}$. *Ans.* $\frac{1}{2}\sqrt{2x^2+3}+C$. 28. $\int\frac{x^2\,dx}{\sqrt{x^3+1}}$. *Ans.* $\frac{2}{3}\sqrt{x^3+1}+C$.

29. $\int\frac{\cos x\,dx}{\sin^2 x}$. *Ans.* $-\frac{1}{\sin x}+C$. 30. $\int\frac{\sin x\,dx}{\cos^3 x}$. *Ans.* $\frac{1}{2\cos^2 x}+C$.

31. $\int\frac{\tan x}{\cos^2 x}dx$. *Ans.* $\frac{\tan^2 x}{2}+C$. 32. $\int\frac{\cot x}{\sin^2 x}dx$. *Ans.* $-\frac{\cot^2 x}{2}+C$.

33. $\int\frac{dx}{\cos^2 x\sqrt{\tan x-1}}$. *Ans.* $2\sqrt{\tan x-1}+C$. 34. $\int\frac{\ln(x+1)}{x+1}dx$. *Ans.* $\frac{\ln^2(x+1)}{2}+C$. 35. $\int\frac{\cos x\,dx}{\sqrt{2\sin x+1}}$. *Ans.* $\sqrt{2\sin x+1}+C$.

36. $\int\frac{\sin 2x\,dx}{(1+\cos 2x)^2}$. *Ans.* $\frac{1}{2(1+\cos 2x)}+C$. 37. $\int\frac{\sin 2x\,dx}{\sqrt{1+\sin^2 x}}$. *Ans.* $2\sqrt{1+\sin^2 x}+C$. 38. $\int\frac{\sqrt{\tan x+1}}{\cos^2 x}dx$. *Ans.* $\frac{2}{3}\sqrt{(\tan x+1)^3}+C$.

39. $\int\frac{\cos 2x\,dx}{(2+3\sin 2x)^3}$. *Ans.* $-\frac{1}{12}\frac{1}{(2+3\sin 2x)^2}+C$. 40. $\int\frac{\sin 3x\,dx}{\sqrt[3]{\cos^4 3x}}$. *Ans.* $\frac{1}{\sqrt[3]{\cos 3x}}+C$. 41. $\int\frac{\ln^2 x\,dx}{x}$. *Ans.* $\frac{\ln^3 x}{3}+C$. 42. $\int\frac{\arcsin x\,dx}{\sqrt{1-x^2}}$. *Ans.* $\frac{\arcsin^2 x}{2}+C$. 43. $\int\frac{\arctan x\,dx}{1+x^2}$. *Ans.* $\frac{\arctan^2 x}{2}+C$. 44. $\int\frac{\arccos^2 x}{\sqrt{1-x^2}}dx$. *Ans.* $-\frac{\arccos^3 x}{3}+C$. 45. $\int\frac{\operatorname{arc\,cot} x}{1+x^2}dx$. *Ans.* $-\frac{\operatorname{arc\,cot}^2 x}{2}+C$.

46. $\int\frac{x\,dx}{x^2+1}$. *Ans.* $\frac{1}{2}\ln(x^2+1)+C$. 47. $\int\frac{x+1}{x^2+2x+3}dx$. *Ans.* $\frac{1}{2}\ln(x^2+2x+3)+C$. 48. $\int\frac{\cos x\,dx}{2\sin x+3}$. *Ans.* $\frac{1}{2}\ln(2\sin x+3)+C$.

49. $\int\frac{dx}{x\ln x}$. *Ans.* $\ln\ln x+C$. 50. $\int 2x(x^2+1)^4\,dx$. *Ans.* $\frac{(x^2+1)^5}{5}=C$.

51. $\int\tan^4 x\,dx$. *Ans.* $\frac{\tan^3 x}{3}-\tan x+x+C$. 52. $\int\frac{dx}{(1+x^2)\arctan x}$. *Ans.* $\ln|\arctan x|+C$. 53. $\int\frac{dx}{\cos^2 x(3\tan x+1)}$. *Ans.* $\frac{1}{3}\ln(3\tan x+1)+C$.

54. $\int \frac{\tan^3 x}{\cos^2 x}\,dx$. *Ans.* $\frac{\tan^4 x}{4}+C$. 55. $\int \frac{dx}{\sqrt{1-x^2}\arcsin x}$. *Ans.* $\ln|\arcsin x|+C$. 56. $\int \frac{\cos 2x}{2+3\sin 2x}\,dx$. *Ans.* $\frac{1}{6}\ln|2+3\sin 2x|+C$. 57. $\int \cos(\ln x)\frac{dx}{x}$. *Ans.* $\sin(\ln x)+C$. 58. $\int \cos(a+bx)\,dx$. *Ans.* $\frac{1}{b}\sin(a+bx)+C$. 59. $\int e^{2x}\,dx$. *Ans.* $\frac{1}{2}e^{2x}+C$. 60. $\int e^{\frac{x}{3}}\,dx$. *Ans.* $3e^{\frac{x}{3}}+C$. 61. $\int e^{\sin x}\cos x\,dx$. *Ans.* $e^{\sin x}+C$. 62. $\int a^{x^2}x\,dx$. *Ans.* $\frac{a^{x^2}}{2\ln a}+C$. 63. $\int e^{\frac{x}{a}}\,dx$. *Ans.* $ae^{\frac{x}{a}}+C$. 64. $\int (e^{2x})^2\,dx$. *Ans.* $\frac{1}{4}e^{4x}+C$. 65. $\int 3^x e^x\,dx$. *Ans.* $\frac{3^x e^x}{\ln 3+1}+C$. 66. $\int e^{-3x}\,dx$. *Ans.* $-\frac{1}{3}e^{-3x}+C$. 67. $\int (e^{5x}+a^{5x})\,dx$. *Ans.* $\frac{1}{5}\left(e^{5x}+\frac{a^{5x}}{\ln a}+C\right)$. 68. $\int e^{x^2+4x+3}(x+2)\,dx$. *Ans.* $\frac{1}{2}e^{x^2+4x+3}+C$. 69. $\int \frac{(a^x-b^x)^2}{a^x b^x}\,dx$. *Ans.* $\frac{\left(\frac{a}{b}\right)^x-\left(\frac{b}{a}\right)^x}{\ln a-\ln b}-2x+C$.

70. $\int \frac{e^x\,dx}{3+4e^x}$. *Ans.* $\frac{1}{4}\ln(3+4e^x)+C$. 71. $\int \frac{e^{2x}\,dx}{2+e^{2x}}$. *Ans.* $\frac{1}{2}\ln(2+e^{2x})+C$. 72. $\int \frac{dx}{1+2x^2}$. *Ans.* $\frac{1}{\sqrt{2}}\arctan(\sqrt{2}x)+C$. 73. $\int \frac{dx}{\sqrt{1-3x^2}}$. *Ans.* $\frac{1}{\sqrt{3}}\arcsin(\sqrt{3}x)+C$. 74. $\int \frac{dx}{\sqrt{16-9x^2}}$. *Ans.* $\frac{1}{3}\arcsin\frac{3x}{4}+C$. 75. $\int \frac{dx}{\sqrt{9-x^2}}$. *Ans.* $\arcsin\frac{x}{3}+C$. 76. $\int \frac{dx}{4+x^2}$. *Ans.* $\frac{1}{2}\arctan\frac{x}{2}+C$. 77. $\int \frac{dx}{9x^2+4}$. *Ans.* $\frac{1}{6}\arctan\frac{3x}{2}+C$. 78. $\int \frac{dx}{4-9x^2}$. *Ans.* $\frac{1}{12}\ln\left|\frac{2+3x}{2-3x}\right|+C$. 79. $\int \frac{dx}{\sqrt{x^2+9}}$. *Ans.* $\ln|x+\sqrt{x^2+9}|+C$. 80. $\int \frac{dx}{\sqrt{b^2x^2-a^2}}$. *Ans.* $\frac{1}{b}\ln|bx+\sqrt{b^2x^2-a^2}|+C$. 81. $\int \frac{dx}{\sqrt{b^2+a^2x^2}}$. *Ans.* $\frac{1}{a}\ln|ax+\sqrt{b^2+a^2x^2}|+C$. 82. $\int \frac{dx}{a^2x^2-c^2}$. *Ans.* $\frac{1}{2ac}\ln\left|\frac{ax-c}{ax+c}\right|+C$. 83. $\int \frac{x^2\,dx}{5-x^6}$. *Ans.* $\frac{1}{6\sqrt{5}}\ln\left|\frac{x^3+\sqrt{5}}{x^3-\sqrt{5}}\right|+C$. 84. $\int \frac{x\,dx}{\sqrt{1-x^4}}$. *Ans.* $\frac{1}{2}\arcsin x^2+C$.

85. $\int \frac{x\,dx}{x^4+a^4}$. *Ans.* $\frac{1}{2a^2}\arctan\frac{x^2}{a^2}+C$. 86. $\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$. *Ans.* $\arcsin e^x+C$.

87. $\int \frac{dx}{\sqrt{3-5x^2}}$. *Ans.* $\frac{1}{\sqrt{5}} \arcsin \sqrt{\frac{5}{3}}\, x + C$. 88. $\int \frac{\cos x\, dx}{a^2+\sin^2 x}$.

Ans. $\frac{1}{a} \arctan \left(\frac{\sin x}{a}\right) + C$. 89. $\int \frac{dx}{x\sqrt{1-\ln^2 x}}$. *Ans.* $\arcsin (\ln x) + C$.

90. $\int \frac{\arccos x - x}{\sqrt{1-x^2}}\, dx$. *Ans.* $-\frac{1}{2} (\arccos x)^2 + \sqrt{1-x^2} + C$.

91. $\int \frac{x - \arctan x}{1+x^2}\, dx$. *Ans.* $\frac{1}{2} \ln (1+x^2) - \frac{1}{2} (\arctan x)^2 + C$.

92. $\int \frac{\sqrt{1+\ln x}}{x}\, dx$. *Ans.* $\frac{2}{3} \sqrt{(1+\ln x)^3} + C$. 93. $\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, dx$.

Ans. $\frac{4}{3} \sqrt{(1+\sqrt{x})^3} + C$. 94. $\int \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{x}}}$. *Ans.* $4\sqrt{1+\sqrt{x}} + C$.

95. $\int \frac{e^x\, dx}{1+e^{2x}}$. *Ans.* $\arctan e^x + C$. 96. $\int \frac{\cos x\, dx}{\sqrt[3]{\sin^2 x}}$. *Ans.* $3\sqrt[3]{\sin x} + C$.

97. $\int \sqrt{1+3\cos^2 x}\, \sin 2x\, dx$. *Ans.* $-\frac{2}{9} \sqrt{(1+3\cos^2 x)^3} + C$. 98. $\int \frac{\sin 2x\, dx}{\sqrt{1+\cos^2 x}}$.

Ans. $-2\sqrt{1+\cos^2 x} + C$. 99. $\int \frac{\cos^3 x}{\sin^4 x}\, dx$. *Ans.* $\frac{1}{\sin x} - \frac{1}{3\sin^3 x} + C$.

100. $\int \frac{\sqrt[3]{\tan^2 x}}{\cos^2 x}\, dx$. *Ans.* $\frac{3}{5} \sqrt[3]{\tan^5 x} + C$.

101. $\int \frac{dx}{2\sin^2 x + 3\cos^2 x}$. *Ans.* $\frac{1}{\sqrt{6}} \arctan \left(\sqrt{\frac{2}{3}} \tan x\right) + C$.

Integrals of the form $\int \frac{Ax+B}{ax^2+bx+c}\, dx$:

102. $\int \frac{dx}{x^2+2x+5}$. *Ans.* $\frac{1}{2} \arctan \frac{x+1}{2} + C$. 103. $\int \frac{dx}{3x^2-2x+4}$.

Ans. $\frac{1}{\sqrt{11}} \arctan \frac{3x-1}{\sqrt{11}} + C$. 104. $\int \frac{dx}{x^2+3x+1}$. *Ans.* $\frac{1}{\sqrt{5}} \ln \left| \frac{2x+3-\sqrt{5}}{2x+3+\sqrt{5}} \right| + C$.

105. $\int \frac{dx}{x^2-6x+5}$. *Ans.* $\frac{1}{4} \ln \left| \frac{x-5}{x-1} \right| + C$. 106. $\int \frac{dz}{2z^2-2z+1}$.

Ans. $\arctan (2z-1) + C$. 107. $\int \frac{dx}{3x^2-2x+2}$. *Ans.* $\frac{1}{\sqrt{5}} \arctan \frac{3x-1}{\sqrt{5}} + C$.

108. $\int \frac{(6x-7)\, dx}{3x^2-7x+11}$. *Ans.* $\ln |3x^2-7x+11| + C$. 109. $\int \frac{(3x-2)\, dx}{5x^2-3x+2}$.

Ans. $\frac{3}{10} \ln (5x^2-3x+2) - \frac{11}{5\sqrt{31}} \arctan \frac{10x-3}{\sqrt{31}} + C$. 110. $\int \frac{3x-1}{x^2-x+1}\, dx$.

Ans. $\frac{3}{2} \ln (x^2-x+1) + \frac{1}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}} + C$. 111. $\int \frac{7x+1}{6x^2+x-1}\, dx$.

Ans. $\frac{2}{3}\ln(3x-1)+\frac{1}{2}\ln(2x+1)+C$. **112.** $\int \frac{2x-1}{5x^2-x+2}\,dx$.

Ans. $\frac{1}{5}\ln(5x^2-x+2)+\frac{8}{5\sqrt{39}}\arctan\frac{10x-1}{\sqrt{39}}+C$. **113.** $\int \frac{6x^4-5x^3+4x^2}{2x^2-x+1}\,dx$.

Ans. $x^3-\frac{x^2}{2}+\frac{1}{4}\ln|2x^2-x+1|+\frac{1}{2\sqrt{7}}\arctan\frac{4x-1}{\sqrt{7}}+C$.

114. $\int \frac{dx}{2\cos^2 x+\sin x\cos x+\sin^2 x}$. *Ans.* $\frac{2}{\sqrt{7}}\arctan\frac{2\tan x+1}{\sqrt{7}}+C$.

Integrals of the form $\int \frac{Ax+B}{\sqrt{ax^2+bx+C}}\,dx$:

115. $\int \frac{dx}{\sqrt{2-3x-4x^2}}$. *Ans.* $\frac{1}{2}\arcsin\frac{8x+3}{\sqrt{41}}+C$. **116.** $\int \frac{dx}{\sqrt{1+x+x^2}}$.

Ans. $\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+C$. **117.** $\int \frac{dS}{\sqrt{2aS+S^2}}$. *Ans.* $\ln|S+a+$ $+\sqrt{2aS+S^2}|+C$. **118.** $\int \frac{dx}{\sqrt{5-7x-3x^2}}$. *Ans.* $\frac{1}{\sqrt{3}}\arcsin\frac{6x+7}{\sqrt{109}}+C$.

119. $\int \frac{dx}{\sqrt{x(3x+5)}}$. *Ans.* $\frac{1}{\sqrt{3}}\ln\left|6x+5+\sqrt{12x(3x+5)}\right|+C$.

120. $\int \frac{dx}{\sqrt{2-3x-x^2}}$. *Ans.* $\arcsin\frac{2x+3}{\sqrt{17}}+C$. **121.** $\int \frac{dx}{\sqrt{5x^2-x-1}}$.

Ans. $\frac{1}{\sqrt{5}}\ln(10x-1+\sqrt{20(5x^2-x-1)})+C$. **122.** $\int \frac{2ax+b}{\sqrt{ax^2+bx+C}}\,dx$.

Ans. $2\sqrt{ax^2+bx+c}+C$. **123.** $\int \frac{(x+3)\,dx}{\sqrt{4x^2+4x+3}}$. *Ans.* $\frac{1}{4}\sqrt{4x^2+4x+3}+$ $+\frac{5}{4}\ln|2x+1+\sqrt{4x^2+4x+3}|+C$. **124.** $\int \frac{(x-3)\,dx}{\sqrt{3+66x-11x^2}}$.

Ans. $-\frac{1}{11}\sqrt{3+66x-11x^2}+C$. **125.** $\int \frac{(x+3)\,dx}{\sqrt{3+4x-4x^2}}$. *Ans.* $-\frac{1}{4}\sqrt{3+4x-4x^2}+$ $+\frac{7}{4}\arcsin\frac{2x-1}{2}+C$. **126.** $\int \frac{3x+5}{\sqrt{x(2x-1)}}\,dx$. *Ans.* $\frac{3}{2}\sqrt{2x^2-x}+\frac{23}{4\sqrt{2}}\times$ $\times\ln(4x-1+\sqrt{8(2x^2-x)})+C$.

II. Integration by parts:

127. $\int xe^x\,dx$. *Ans.* $e^x(x-1)+C$. **128.** $\int x\ln x\,dx$. *Ans.* $\frac{1}{2}x^2\times$ $\times\left(\ln x-\frac{1}{2}\right)+C$. **129.** $\int x\sin x\,dx$. *Ans.* $\sin x-x\cos x+C$. **130.** $\int \ln x\,dx$.

Ans. $x(\ln x-1)+C$. **131.** $\int \arcsin x\,dx$. *Ans.* $x\arcsin x+\sqrt{1-x^2}+C$.

132. $\int \ln(1-x)\,dx$. *Ans.* $-x-(1-x)\ln(1-x)+C$. 133. $\int x^n \ln x\,dx$. *Ans.* $\frac{x^{n+1}}{n+1}\left(\ln x-\frac{1}{n+1}\right)+C$. 134. $\int x \arctan x\,dx$. *Ans.* $\frac{1}{2}[(x^2+1)\times$ $\times\arctan x-x]+C$. 135. $\int x \arcsin x\,dx$. *Ans.* $\frac{1}{4}[(2x^2-1)\arcsin x+$ $+x\sqrt{1-x^2}]+C$. 136. $\int \ln(x^2+1)\,dx$. *Ans.* $x\ln(x+1)-2x+2\arctan x+C$. 137. $\int \arctan\sqrt{x}\,dx$. *Ans.* $(x+1)\arctan\sqrt{x}-\sqrt{x}+C$. 138. $\int \frac{\arcsin\sqrt{x}}{\sqrt{x}}dx$. *Ans.* $2\sqrt{x}\arcsin\sqrt{x}+2\sqrt{1-x}+C$. 139. $\int \arcsin\sqrt{\frac{x}{x+1}}\,dx$. *Ans.* $x\arcsin\sqrt{\frac{x}{x+1}}-\sqrt{x}+\arctan\sqrt{x}+C$. 140. $\int x\cos^2 x\,dx$. *Ans.* $\frac{x^2}{4}+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+C$. 141. $\int \frac{x\arcsin x}{\sqrt{1-x^2}}dx$. *Ans.* $x-\sqrt{1-x^2}\times$ $\times\arcsin x+C$. 142. $\int \frac{x\arctan x}{(x^2+1)^2}dx$. *Ans.* $\frac{x}{4(1+x^2)}+\frac{1}{4}\arctan x-$ $-\frac{1}{2}\frac{\arctan x}{1+x^2}+C$. 143. $\int x\arctan\sqrt{x^2-1}\,dx$. *Ans.* $\frac{1}{2}x^2\arctan\sqrt{x^2-1}-$ $-\frac{1}{2}\sqrt{x^2-1}+C$. 144. $\int \frac{\arcsin x}{x^2}dx$. *Ans.* $\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|-\frac{1}{x}\arcsin x+C$. 145. $\int \ln(x+\sqrt{1+x^2})dx$. *Ans.* $x\ln|x+\sqrt{1+x^2}|-\sqrt{1+x^2}+C$. 146. $\int \arcsin x\frac{x\,dx}{\sqrt{(1-x^2)^3}}$. *Ans.* $\frac{\arcsin x}{\sqrt{1-x^2}}+\frac{1}{2}\ln\left|\frac{1-x}{1+x}\right|$.

Use trigonometric substitutions in the following examples:

147. $\int \frac{\sqrt{a^2-x^2}}{x^2}dx$. *Ans.* $-\frac{\sqrt{a^2-x^2}}{x}-\arcsin\frac{x}{a}+C$. 148. $\int x^2\sqrt{4-x^2}\,dx$. *Ans.* $2\arcsin\frac{x}{2}-\frac{1}{2}x\sqrt{4-x^2}+\frac{1}{4}x^3\sqrt{4-x^2}+C$. 149. $\int \frac{dx}{x^2\sqrt{1+x^2}}$. *Ans.* $-\frac{\sqrt{1+x^2}}{x}+C$. 150. $\int \frac{\sqrt{x^2-a^2}}{x}dx$. *Ans.* $\sqrt{x^2-a^2}-a\arccos\frac{a}{x}+C$. 151. $\int \frac{dx}{\sqrt{(a^2+x^2)^3}}$. *Ans.* $\frac{x}{a^2}\frac{1}{\sqrt{a^2+x^2}}+C$.

Integration of rational fractions:

152. $\int \frac{2x-1}{(x-1)(x-2)}dx$. *Ans.* $\ln\left|\frac{(x-2)^3}{x-1}\right|+C$. 153. $\int \frac{x\,dx}{(x+1)(x+3)(x+5)}$.

Ans. $\frac{1}{8}\ln\frac{(x+3)^6}{(x+5)^5(x+1)}$. **154.** $\int\frac{x^5+x^4-8}{x^3-4x}\,dx$. *Ans.* $\frac{x^3}{3}+\frac{x^2}{2}+4x+$ $+\ln\left|\frac{x^2(x-2)^5}{(x+2)^3}\right|+C$. **155.** $\int\frac{x^4\,dx}{(x^2-1)(x+2)}$. *Ans.* $\frac{x^2}{2}-2x+\frac{1}{6}\ln\times$ $\times\frac{(x-1)}{(x+1)^3}+\frac{16}{3}\ln(x+2)+C$. **156.** $\int\frac{dx}{(x-1)^2(x-2)}$. *Ans.* $\frac{1}{x-1}+$ $+\ln\frac{x-2}{x-1}+C$. **157.** $\int\frac{x-8}{x^3-4x^2+4x}\,dx$. *Ans.* $\frac{3}{x-2}+\ln\frac{(x-2)^2}{x^2}+C$.

158. $\int\frac{3x+2}{x(x+1)^3}\,dx$. *Ans.* $\frac{4x+3}{2(x+1)^2}+\ln\frac{x^2}{(x+1)^2}+C$. **159.** $\int\frac{x^2\,dx}{(x+2)^2(x+4)^2}$. *Ans.* $-\frac{5x+12}{x^2+6x+8}+\ln\left(\frac{x+4}{x+2}\right)^2+C$. **160.** $\int\frac{dx}{x(x^2+1)}$. *Ans.* $\ln\frac{x}{\sqrt{x^2+1}}+C$.

161. $\int\frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}\,dx$. *Ans.* $\ln\frac{(x^2-2x+5)^{\frac{3}{2}}}{x-1}+\frac{1}{2}\arctan\frac{x-1}{2}+C$.

162. $\frac{x^3-6}{x^4+6x^2+8}\,dx$. *Ans.* $\ln\frac{x^2+4}{\sqrt{x^2+2}}+\frac{3}{2}\arctan\frac{x}{2}-\frac{3}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+C$.

163. $\int\frac{dx}{x^3+1}$. *Ans.* $\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+C$.

164. $\int\frac{3x-7}{x^3+x^2+4x+4}$. *Ans.* $\ln\frac{x^2+4}{(x+1)^2}+\frac{1}{2}\arctan\frac{x}{2}+C$. **165.** $\int\frac{4dx}{x^4+1}$. *Ans.* $\frac{1}{\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\sqrt{2}\arctan\frac{x\sqrt{2}}{1-x^2}+C$. **166.** $\int\frac{x^5}{x^3-1}\,dx$. *Ans.* $\frac{1}{3}[x^3+\ln(x^3-1)]+C$. **167.** $\int\frac{x^3+x-1}{(x^2+2)^2}\,dx$. *Ans.* $\frac{2-x}{4(x^2+2)}+\ln(x^2+2)^{\frac{1}{2}}-$ $-\frac{1}{4\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+C$. **168.** $\int\frac{(4x^2-8x)\,dx}{(x-1)^2(x^2+1)^2}$. *Ans.* $\frac{3x^2-1}{(x-1)(x^2+1)}+$ $+\ln\frac{(x-1)^2}{x^2+1}+\arctan x+C$. **169.** $\int\frac{dx}{(x^2-x)(x^2-x+1)^2}$. *Ans.* $\ln\frac{x-1}{x}-$ $-\frac{10}{3\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}-\frac{2x-1}{3(x^2-x+1)}+C$.

Integration of irrational functions:

170. $\int\frac{\sqrt{x}}{\sqrt[4]{x^3}+1}\,dx$. *Ans.* $\frac{4}{3}\left[\sqrt[4]{x^3}-\ln(\sqrt[4]{x^3}+1)\right]+C$.

171. $\int\frac{\sqrt{x^3}-\sqrt[3]{x}}{6\sqrt[4]{x}}\,dx$. *Ans.* $\frac{2}{27}\sqrt[4]{x^9}-\frac{2}{13}\sqrt[12]{x^{13}}+C$. **172.** $\int\frac{\sqrt[6]{x}+1}{\sqrt[6]{x^7}+\sqrt[4]{x^5}}\,dx$. *Ans.* $-\frac{6}{\sqrt[6]{x}}+\frac{12}{\sqrt[12]{x}}+2\ln x-24\ln(\sqrt[12]{x}+1)+C$.

173. $\int \frac{2+\sqrt[3]{x}}{\sqrt[6]{x}+\sqrt[3]{x}+\sqrt{x}+1}\,dx$. *Ans.* $\frac{6}{5}\sqrt[6]{x^5}-\frac{3}{2}\sqrt[6]{x^4}+4\sqrt[6]{x^3}-$
$-6\sqrt[6]{x^2}+6\sqrt[6]{x}-9\ln\left(\sqrt[6]{x}+1\right)+\frac{3}{2}\ln\left(\sqrt[6]{x^2}+1\right)+3\arctan\sqrt[6]{x}+C.$

174. $\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{x^2}$. *Ans.* $\ln\left|\frac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1-x}-\sqrt{1+x}}\right|-\frac{\sqrt{1-x^2}}{x}+C.$

175. $\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{x}$. *Ans.* $2\arctan\sqrt{\frac{1-x}{1+x}}+\ln\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}+C.$

176. $\int \frac{\sqrt[7]{x}+\sqrt{x}}{\sqrt[7]{x^8}+\sqrt[14]{x^{15}}}\,dx$. *Ans.* $14\left[\sqrt[14]{x}-\frac{1}{2}\sqrt[7]{x}+\frac{1}{3}\sqrt[14]{x^3}-\frac{1}{4}\sqrt[7]{x^2}+\right.$
$\left.+\frac{1}{5}\sqrt[14]{x^5}\right]+C.$ **177.** $\int \sqrt{\frac{2+3x}{x-3}}\,dx$. *Ans.* $\sqrt{3x^2-7x-6}+\frac{11}{2\sqrt{3}}\times$
$\times\ln\left(x-\frac{7}{6}+\sqrt{x^2-\frac{7}{3}x-2}\right)+C.$

Integrals of the form $\int R(x,\ \sqrt{ax^2+bx+c})\,dx$:

178. $\int \frac{dx}{x\sqrt{x^2-x+3}}$. *Ans.* $\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2-x+3}-\sqrt{3}}{x}+\frac{1}{2\sqrt{3}}\right|+C.$

179. $\int \frac{dx}{x\sqrt{2+x-x^2}}$. *Ans.* $-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2+x-x^2}+\sqrt{2}}{x}+\frac{1}{2\sqrt{2}}\right|+C.$

180. $\int \frac{dx}{x\sqrt{x^2+4x-4}}$. *Ans.* $\frac{1}{2}\arcsin\frac{x-2}{x\sqrt{2}}+C.$ **181.** $\int \frac{\sqrt{x^2+2x}}{x}\,dx.$
Ans. $\sqrt{x^2+2x}+\ln|x+1+\sqrt{x^2+2x}|+C.$ **182.** $\int \frac{dx}{\sqrt{(2x-x^2)^3}}$.
Ans. $\frac{x-1}{\sqrt{2x-x^2}}+C.$ **183.** $\int \sqrt{2x-x^2}\,dx.$ *Ans.* $\frac{1}{2}[(x-1)\sqrt{2x-x^2}+$
$+\arcsin(x-1)]+C.$ **184.** $\int \frac{dx}{x-\sqrt{x^2-1}}$. *Ans.* $\frac{x^2}{2}+\frac{x}{2}\sqrt{x^2-1}-$
$-\frac{1}{2}\ln|x+\sqrt{x^2-1}|+C.$ **185.** $\int \frac{dx}{(1+x)\sqrt{1+x+x^2}}$.
Ans. $\ln\left|\frac{x+\sqrt{1+x+x^2}}{2+x+\sqrt{1+x+x^2}}\right|+C.$ **186.** $\int \frac{(x+1)}{(2x+x^2)\sqrt{2x+x^2}}\,dx.$
Ans. $-\frac{1}{\sqrt{2x+x^2}}+C.$ **187.** $\int \frac{1-\sqrt{1+x+x^2}}{x\sqrt{1+x+x^2}}\,dx.$ *Ans.* $\ln\left|\frac{2+x-2\sqrt{1+x+x^2}}{x^2}\right|+C.$

188. $\int \frac{\sqrt{x^2+4x}}{x^2}\,dx.$ *Ans.* $-\frac{8}{x+\sqrt{x^2+4x}}+\ln|x+2+\sqrt{x^2+4x}|+C.$

Integration of binomial differentials:

189. $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx$. *Ans.* $2\left(1+x^{\frac{1}{3}}\right)^{\frac{3}{2}}+C$. 190. $\int x^{\frac{1}{3}}\left(2+x^{\frac{2}{3}}\right)^{\frac{1}{4}}dx$.

Ans. $\frac{10x^{\frac{2}{3}}-16}{15}\left(2+x^{\frac{2}{3}}\right)^{\frac{5}{4}}+C$. 191. $\int \frac{dx}{(1+x^2)^{\frac{3}{2}}}$. *Ans.* $\frac{x}{\sqrt{1+x^2}}+C$.

192. $\int \frac{dx}{x^2(1+x^2)^{\frac{3}{2}}}$. *Ans.* $-(1+x^2)^{-\frac{1}{2}}\left(2x+\frac{1}{x}\right)+C$. 193. $\int \sqrt[4]{\left(1+x^{\frac{1}{2}}\right)^3}\,dx$.

Ans. $\frac{8}{77}(7\sqrt{x}-4)(1+\sqrt{x})^{\frac{7}{4}}+C$. 194. $\int \frac{\sqrt{2-\sqrt[3]{x}}}{\sqrt[3]{x}}\,dx$.

Ans. $\frac{2(4+3\sqrt[3]{x})(2-\sqrt[3]{x})^{\frac{3}{2}}}{5}$. 195. $\int x^5\sqrt[3]{(1+x^3)^2}\,dx$.

Ans. $\frac{5x^3-3}{40}(1+x^3)^{\frac{5}{3}}$.

Integration of trigonometric functions:

196. $\int \sin^3 x\,dx$. *Ans.* $\frac{1}{3}\cos^3 x-\cos x+C$. 197. $\int \sin^5 x\,dx$. *Ans.* $-\cos x+\frac{2}{3}\cos^3 x-\frac{\cos^5 x}{5}+C$. 198. $\int \cos^4 x\sin^3 x$. *Ans.* $-\frac{1}{5}\cos^5 x+\frac{1}{7}\cos^7 x+C$.

199. $\int \frac{\cos^3 x}{\sin^3 x}\,dx$. *Ans.* $\csc x-\frac{1}{3}\csc^3 x+C$. 200. $\int \cos^2 x\,dx$.

Ans. $\frac{x}{2}+\frac{1}{4}\sin 2x+C$. 201. $\int \sin^4 x\,dx$. *Ans.* $\frac{3}{8}x-\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+C$.

202. $\int \cos^6 x\,dx$. *Ans.* $\frac{1}{16}\left(5x+4\sin 2x-\frac{\sin^3 2x}{3}+\frac{3}{4}\sin 4x\right)+C$.

203. $\int \sin^4 x\cos^4 x\,dx$. *Ans.* $\frac{1}{128}\left(3x-\sin 4x+\frac{\sin 8x}{8}\right)+C$. 204. $\int \tan^3 x\,dx$.

Ans. $\frac{\tan^2 x}{2}+\ln|\cos x|+C$. 205. $\int \cot^5 x\,dx$. *Ans.* $-\frac{1}{4}\cot^4 x+\frac{1}{2}\cot^2 x+\ln|\sin x|+C$. 206. $\int \cot^3 x\,dx$. *Ans.* $-\frac{\cot^2 x}{2}-\ln|\sin x|+C$.

207. $\int \sec^8 x\,dx$. *Ans.* $\frac{\tan^7 x}{7}+\frac{3\tan^5 x}{5}+\tan^3 x+\tan x+C$.

208. $\int \tan^4 x \sec^4 x\, dx$. *Ans.* $\frac{\tan^7 x}{7}+\frac{\tan^5 x}{5}+C$. **209.** $\int \frac{dx}{\cos^4 x}$. *Ans.* $\tan x+\frac{1}{3}\tan^3 x+C$. **210.** $\int \frac{\cos x}{\sin^2 x}\, dx$. *Ans.* $C-\csc x$.

211. $\int \frac{\sin^3 x\, dx}{\sqrt[3]{\cos^4 x}}$. *Ans.* $\frac{3}{5}\cos^{\frac{5}{3}} x+3\cos^{-\frac{1}{3}} x+C$. **212.** $\int \sin x \sin 3x\, dx$. *Ans.* $-\frac{\sin 4x}{8}+\frac{\sin 2x}{4}+C$. **213.** $\int \cos 4x \cos 7x\, dx$. *Ans.* $\frac{\sin 11x}{22}+\frac{\sin 3x}{6}+C$.

214. $\int \cos 2x \sin 4x\, dx$. *Ans.* $-\frac{\cos 6x}{12}-\frac{\cos 2x}{4}+C$. **215.** $\int \sin\frac{1}{4}x\cos\frac{3}{4}x\, dx$. *Ans.* $-\frac{\cos x}{2}+\cos\frac{1}{2}x+C$. **216.** $\int \frac{dx}{4-5\sin x}$. *Ans.* $\frac{1}{3}\ln\left|\frac{\tan\frac{x}{2}-2}{2\tan\frac{x}{2}-1}\right|+C$.

217. $\int \frac{dx}{5-3\cos x}$. *Ans.* $\frac{1}{2}\arctan\left|2\tan\frac{x}{2}\right|+C$. **218.** $\int \frac{\sin x\, dx}{1+\sin x}$. *Ans.* $\frac{2}{1+\tan\frac{x}{2}}+x+C$. **219.** $\int \frac{\cos x\, dx}{1+\cos x}$. *Ans.* $x-\tan\frac{x}{2}+C$.

220. $\int \frac{\sin 2x}{\cos^4 x+\sin^4 x}\, dx$. *Ans.* $\arctan(2\sin^2 x-1)+C$. **221.** $\int \frac{dx}{(1+\cos x)^2}$. *Ans.* $\frac{1}{2}\tan\frac{x}{2}+\frac{1}{6}\tan^3\frac{x}{2}+C$. **222.** $\int \frac{dx}{\sin^2 x+\tan^2 x}$. *Ans.* $-\frac{1}{2}\left[\cot x+\frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)\right]+C$. **223.** $\int \frac{\sin^2 x}{1+\cos^2 x}\, dx$. *Ans.* $\sqrt{2}\arctan\times\left(\frac{\tan x}{\sqrt{2}}\right)-x+C$.

CHAPTER XI

THE DEFINITE INTEGRAL

SEC. 1. STATEMENT OF THE PROBLEM. THE LOWER AND UPPER INTEGRAL SUMS

The **definite integral** is one of the basic concepts of mathematical analysis and is a powerful research tool in mathematics, physics, mechanics, and other disciplines. Calculation of areas bounded by curves, of arc lengths, volumes, work, velocity, path length, moments of inertia, and so forth reduce to the evaluation of a definite integral.

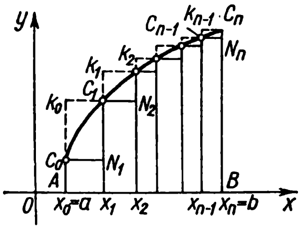


Fig. 206.

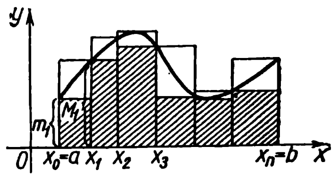


Fig. 207.

Let a **continuous** function $y=f(x)$ be given on the interval $[a, b]$ (Figs. 206 and 207). Denote by m and M its smallest and largest values on this interval. Divide the interval $[a, b]$ into n subintervals by points of division:

$$a=x_0,\ x_1,\ x_2,\ \ldots,\ x_{n-1},\ x_n=b,$$

so that

$$x_0<x_1<x_2<\ldots<x_n,$$

and put

$$x_1-x_0=\Delta x_1;\ x_2-x_1=\Delta x_2,\ \ldots,\ x_n-x_{n-1}=\Delta x_n.$$

Then denote the smallest and greatest values of the function $f(x)$

on the interval $[x_0, x_1]$ by m_1 and M_1
on the interval $[x_1, x_2]$ by m_2 and M_2
. . . .
on the interval $[x_{n-1}, x_n]$ by m_n and M_n

Form the sums

$$\underline{s_n}=m_1\Delta x_1+m_2\Delta x_2+\ldots+m_n\Delta x_n=\sum_{i=1}^{n} m_i\Delta x_i, \tag{1}$$

$$\bar{s}_n = M_1\Delta x_1 + M_2\Delta x_2 + \ldots + M_n\Delta x_n = \sum_{i=1}^{n} M_i\Delta x_i. \qquad (2)$$

The sum $\underline{s}_n$ is called the *lower (integral) sum*, and the sum $\bar{s}_n$ is called the *upper (integral) sum*.

If $f(x) \geqslant 0$, then the lower sum is numerically equal to the area of an "inscribed step-like figure" $AC_0N_1C_1N_2\ldots C_{n-1}N_nBA$ bounded by an "inscribed" broken line, the upper sum is equal numerically to the area of an "circumscribed step-like figure" $AK_0C_1K_1\ldots C_{n-1}K_{n-1}C_nBA$ bounded by an "circumscribed" broken line.

The following are some properties of upper and lower sums.

a) Since $m_i \leqslant M_i$ for any $i\,(i = 1, 2, \ldots, n)$, by formulas (1) and (2) we have

$$\underline{s}_n \leqslant \bar{s}_n.$$

(The equal sign occurs only when $f(x) = \text{const.}$)

b) Since

$$m_1 \geqslant m, \; m_2 \geqslant m, \; \ldots, \; m_n \geqslant m,$$

where m is the smallest value of $f(x)$ on $[a, b]$, we have

$$\underline{s}_n = m_1\Delta x_1 + m_2\Delta x_2 + \ldots + m_n\Delta x_n \geqslant m\Delta x_1 + m\Delta x_2 + \ldots + m\Delta x_n = \\ = m(\Delta x_1 + \Delta x_2 + \ldots + \Delta x_n) = m(b-a).$$

Thus,

$$\underline{s}_n \geqslant m(b-a).$$

c) Since

$$M_1 \leqslant M, \; M_2 \leqslant M, \; \ldots, \; M_n \leqslant M,$$

where M is the greatest value of $f(x)$ on $[a, b]$, we have

$$\bar{s}_n = M_1\Delta x_1 + M_2\Delta x_2 + \ldots + M_n\Delta x_n \leqslant M\Delta x_1 + M\Delta x_2 + \ldots \\ \ldots + M\Delta x_n = M(\Delta x_1 + \Delta x_2 + \ldots + \Delta x_n) = M(b-a).$$

Thus,

$$\bar{s}_n \leqslant M(b-a_n).$$

Combining the inequalities obtained, we have

$$m(b-a) \leqslant \underline{s}_n \leqslant \bar{s}_n \leqslant M(b-a).$$

If $f(x) \geqslant 0$, then the latter inequality has a simple geometric meaning (Fig. 208), because the products $m(b-a)$ and $M(b-a)$ are, respectively, numerically equal to the areas of the "inscribed" rectangle AL_1L_2B and the "circumscribed" rectangle $A\bar{L}_1\bar{L}_2B$.

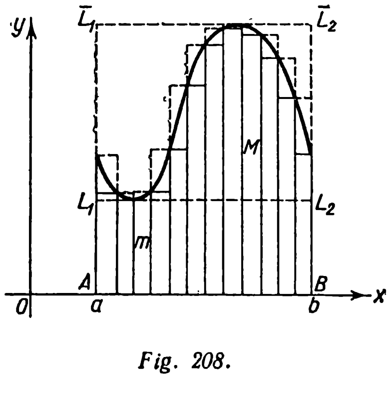


Fig. 208.

SEC. 2. THE DEFINITE INTEGRAL

Let us continue examining the question of the preceding section. In each of the intervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$ take a point and denote them by $\xi_1, \xi_2, \ldots, \xi_n$ (Fig. 209):

$$x_0<\xi_1<x_1,\ x_1<\xi_2<x_2,\ \ldots,\ x_{n-1}<\xi_n<x_n.$$

At each of these points find the value of the function $f(\xi_1)$, $f(\xi_2)$, $\ldots$, $f(\xi_n)$. Form a sum:

$$s_n=f(\xi_1)\,\Delta x_1+f(\xi_2)\,\Delta x_2+\ldots+f(\xi_n)\,\Delta x_n=\sum_{i=1}^{n} f(\xi_i)\,\Delta x_i. \quad (1)$$

This sum is called the *integral sum* of the function $f(x)$ on the interval $[a, b]$. Since for an arbitrary ξ_i belonging to the interval $[x_{i-1}, x_i]$ we will have

$$m_i \leqslant f(\xi_i) \leqslant M_i$$

and all $\Delta x_i > 0$, it follows that

$$m_i \Delta x_i \leqslant f(\xi_i)\,\Delta x_i \leqslant M_i\,\Delta x_i,$$

and consequently

$$\sum_{i=1}^{n} m_i\,\Delta x_i \leqslant \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i \leqslant \sum_{i=1}^{n} M_i \Delta x_i,$$

or

$$\underline{s_n} \leqslant s_n \leqslant \overline{s_n}. \quad (2)$$

The geometric meaning of the latter inequality for $f(x) \geqslant 0$ consists in the fact that the figure whose area is equal to s_n is bounded by a broken line lying between the "inscribed" broken line and the "circumscribed" broken line.

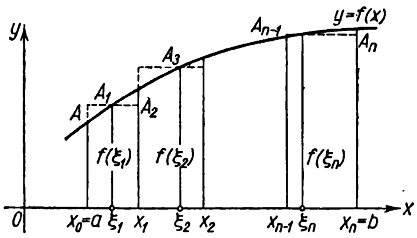


Fig. 209.

The sum s_n depends upon the way in which the interval $[a, b]$ is divided into the subintervals $[x_{i-1}, x_i]$ and also upon the choice of points ξ_i inside the resulting subintervals.

Let us now denote by $\max\,[x_{i-1}, x_i]$ the largest of the lengths of subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$. Let us consider different partitions of the interval $[a, b]$ into subintervals $[x_{i-1}, x_i]$ such that $\max\,[x_{i-1}, x_i] \to 0$. Obviously, the number of subintervals n approaches infinity here. Choosing the appro-

priate values of ξ_i, it is possible, for each partition, to form the integral sum

$$\sum_{i=1}^{n} f(\xi_i)\,\Delta x_i.$$

We can thus speak of a sequence of partitions and a corresponding sequence of integral sums. Let this sum*) approach the limit I for some chosen sequence of partitions when $\max \Delta x_i \to 0$.

If for any partitions of the interval $[a, b]$ such that $\max \Delta x_i \to 0$ and for any choice of points ξ_i the sum $\sum_{i=1}^{n} f(\xi_i)\,\Delta x_i$ approaches the same limit I, we say that the function $f(x)$ is *integrable* on the interval $[a, b]$; the limit I is called the *definite integral* of the function $f(x)$ on the interval $[a, b]$. It is denoted by $\int_a^b f(x)\,dx$ and we write

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The number a is called the *lower limit* of the integral, b is the *upper limit*. The interval $[a, b]$ is called the *interval of integration*, the letter x is the *variable of integration*.

Let it be stated without proof that *if a function $y = f(x)$ is continuous on the interval $[a, b]$, then it is integrable on this interval.*

It is obvious that if for some sequence of partitions such that $\max \Delta x_i \to 0$ we consider the sequence of lower integral sums $\underline{s_n}$ and of upper integral sums $\overline{s_n}$ for a continuous function $f(x)$, then these sums will tend towards the same limit I—the definite integral of the function $f(x)$:

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} m_i\,\Delta x_i = \int_a^b f(x)\,dx,$$

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} M_i\Delta x_i = \int_a^b f(x)\,dx.$$

Among discontinuous functions there are both integrable functions and nonintegrable ones.

*) In this case the sum is an ordered variable quantity.

If we construct the graph of the integrand $y=f(x)$, then in the case of $f(x)\geqslant 0$ the integral

$$\int_a^b f(x)\,dx$$

will be numerically equal to the area of a so-called curvilinear trapezoid bounded by the given curve, the straight lines $x=a$ and $x=b$, and the x-axis (Fig. 210).

For this reason, if it is required to compute the area of a curvilinear trapezoid bounded by the curve $y=f(x)$, the straight lines $x=a$ and $x=b$, and the x-axis, this area Q is computed by means of the integral

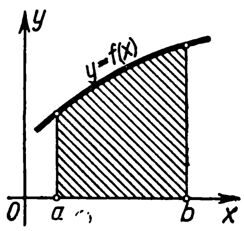


Fig. 210.

$$Q=\int_a^b f(x)\,dx. \tag{3}$$

Note 1. It will be noted that the definite integral depends only on the form of the function $f(x)$ and the limits of integration, and not on the variable of integration, which may be denoted by any letter. Thus, without changing the magnitude of a definite integral it is possible to replace the latter x by any other letter:

$$\int_a^b f(x)\,dx=\int_a^b f(t)\,dt=\ldots=\int_a^b f(z)\,dz.$$

When introducing the concept of the definite integral $\int_a^b f(x)\,dx$ we assumed that $a<b$. In the case where $b<a$ we will, **by definition,** have

$$\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx. \tag{4}$$

Thus, for instance,

$$\int_5^0 x^2\,dx=-\int_0^5 x^2\,dx.$$

Finally, in the case of $a=b$ we assume, **by definition,** that for

any function $f(x)$ we have

$$\int_a^a f(x)\,dx = 0. \quad (5)$$

This is natural also from the geometric standpoint. Indeed, the base of a curvilinear trapezoid has a length equal to zero; consequently, its area is zero too.

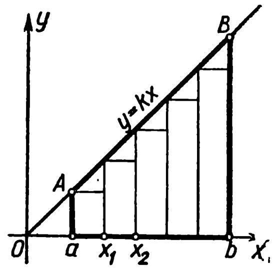


Fig. 211.

Example 1. Compute the integral $\int_a^b kx\,dx$ $(b>a)$.

Solution. Geometrically, the problem is equivalent to computing the area Q of a trapezoid bounded by the lines $y=kx$, $x=a$, $x=b$, $y=0$ (Fig. 211).

The function $y=kx$ under the integral sign is continuous. Therefore, in order to compute the definite integral we have the right, as was stated above, to divide the interval $[a, b]$ in any way and choose arbitrary intermediate points ξ_k. The result of computing a definite integral is independent of the way in which the integral sum is formed, provided that the subinterval approaches zero.

Divide the interval $[a, b]$ into n equal subintervals.

The length Δx of each subinterval is $\Delta x=\frac{b-a}{n}$; this number is the subinterval (partition unit). The division points have coordinates:

$$a=x_0, \quad x_1=a+\Delta x,$$
$$x_2=a+2\Delta x, \ \ldots, \ x_n=a+n\Delta x.$$

For the points ξ_k take the left end points of each subinterval:

$$\xi_1=a, \quad \xi_2=a+\Delta x, \ \xi_3=a+2\Delta x, \ \ldots, \ \xi_n=a+(n-1)\Delta x.$$

Form the integral sum (1). Since $f(\xi_i)=k\xi_i$, we have

$$s_n=k\xi_1\Delta x+k\xi_2\Delta x+\ldots+k\xi_n\Delta x=$$
$$=ka\,\Delta x+[k(a+\Delta x)]\,\Delta x+\ldots+\{k\,[a+(n-1)\,\Delta x]\}\,\Delta x=$$
$$=k\,\{a+(a+\Delta x)+(a+2\Delta x)+\ldots+[a+(n-1)\,\Delta x]\}\,\Delta x=$$
$$=k\,\{na+[\Delta x+2\Delta x+\ldots+(n-1)\,\Delta x]\}\,\Delta x=$$
$$=k\,\{na+[1+2+\ldots+(n-1)]\,\Delta x\}\,\Delta x,$$

where $\Delta x=\frac{b-a}{n}$. Taking into account that

$$1+2+\ldots+(n-1)=\frac{n(n-1)}{2}$$

(as the sum of an arithmetic progression),

$$s_n=k\left[na+\frac{n(n-1)}{2}\,\frac{b-a}{n}\right]\frac{b-a}{n}=k\left[a+\frac{n-1}{n}\,\frac{b-a}{2}\right](b-a).$$

Since $\lim\limits_{n\to\infty}\frac{n-1}{n}=1$, we have

$$\lim_{n\to\infty} s_n = Q = k\left[a+\frac{b-a}{2}\right](b-a) = k\frac{b^2-a^3}{2}.$$

Thus,

$$\int_a^b kx\,dx = k\frac{b^2-a^2}{2}.$$

The area of $ABba$ (Fig. 211) is readily computed by the methods of elementary geometry. The result will be the same.

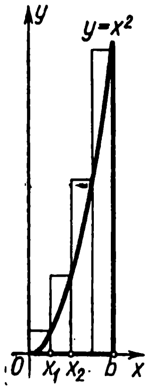


Fig. 212.

Example 2. Evaluate $\int_0^b x^2\,dx$.

Solution. The given integral is equal to the area Q of a curvilinear trapezoid bounded by a parabola $y=x^2$, the ordinate $x=b$, and the straight line $y=0$ (Fig. 212).

Divide the interval $[a, b]$ into n equal parts by the points

$$x_0=0,\ x_1=\Delta x,\ x_2=2\Delta x,\ \ldots,\ x_n=b=n\Delta x,\ \Delta x=\frac{b}{n}.$$

For the ξ_i points take the right extremities of each subinterval.

Form the integral sum

$$s_n = x_1^2\Delta x + x_2^2\,\Delta x + \ldots + x_n^2\,\Delta x = [(\Delta x)^2\,\Delta x + (2\Delta x)^2\,\Delta x + \ldots + (n\Delta x)^2\,\Delta x] = (\Delta x)^3\,[1^2+2^2+\ldots+n^2].$$

As we know,

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6},$$

therefore

$$s_n=\frac{b^3}{n^3}\,\frac{n(n+1)(2n+1)}{6}=\frac{b^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right);$$

$$\lim_{n\to\infty} s_n = Q = \int_0^b x^2\,dx = \frac{b^3}{3}.$$

Example 3. Evaluate $\int_a^b m\,dx$ ($m=$ const).

Solution.

$$\int_a^b m\,dx = \lim_{\max\Delta x_i\to 0}\sum_{i=1}^n m\,\Delta x_i = \lim_{\max\Delta x_i\to 0} m\sum_{i=1}^n \Delta x_i = m\lim_{\max\Delta x_i\to 0}\sum_{i=1}^n \Delta x_i = m(b-a).$$

Here, $\sum_{i=1}^{n} \Delta x_i$ is the sum of the lengths of the subintervals into which the interval $[a, b]$ was divided. No matter what the method of partition, the sum is equal to the length of the segment $b-a$.

Example 4. Evaluate $\int_a^b e^x dx$.

Solution. Again divide the interval $[a, b]$ into n equal parts:

$$x_0=a, \quad x_1=a+\Delta x, \ \ldots, \ x_n=a+n\Delta x; \quad \Delta x=\frac{b-a}{n}.$$

Take the left extremities as the points ξ_i. Then form the sum

$$s_n=e^a\Delta x+e^{a+\Delta x}\Delta x+\ldots+e^{a+(n-1)\Delta x}\Delta x=$$
$$=e^a(1+e^{\Delta x}+e^{2\Delta x}+\ldots+e^{(n-1)\Delta x})\Delta x.$$

The expression in the brackets is a geometric progression with common ratio $e^{\Delta x}$ and first term 1; therefore

$$s_n=e^a\frac{e^{n\Delta x}-1}{e^{\Delta x}-1}\Delta x=e^a(e^{n\Delta x}-1)\frac{\Delta x}{e^{\Delta x}-1}.$$

Then we have

$$n\Delta x=b-a; \ \lim_{\Delta x\to 0}\frac{\Delta x}{e^{\Delta x}-1}=1.$$

(By L'Hospital's rule $\lim_{z\to 0}\frac{z}{e^z-1}=\lim_{z\to 0}\frac{1}{e^z}=1.$) Thus,

$$\lim_{n\to\infty} s_n=Q=e^a(e^{b-a}-1)\cdot 1=e^b-e^a,$$

that is,

$$\int_a^b e^x\,dx=e^b-e^a.$$

Note 2. The foregoing examples show that the direct evaluation of definite integrals as the limits of integral sums involves great difficulties. Even when the integrands are very simple (kx, x^2, e^x), this method involves cumbersome computations. The finding of definite integrals of more complicated functions leads to still greater difficulties. The natural problem that arises is to find some practically convenient way of evaluating definite integrals. This method, which was discovered by Newton and Leibniz, utilises the profound relationship that exists between integration and differentiation. The following sections of this chapter are devoted to the exposition and substantiation of this method.

SEC. 3. BASIC PROPERTIES OF THE DEFINITE INTEGRAL

Property 1. *The constant factor may be taken outside the sign of the definite integral:* if $A=\text{const}$, then

$$\int_a^b Af(x)\,dx = A\int_a^b f(x)\,dx. \tag{1}$$

Proof.

$$\int_a^b Af(x)\,dx = \lim_{\max \Delta x \to 0} \sum_{i=1}^{n} Af(\xi_i)\,\Delta x_i =$$

$$= A \lim_{\max \Delta x \to 0} \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i = A\int_a^b f(x)\,dx.$$

Property 2. *The definite integral of an algebraic sum of several functions is equal to the algebraic sum of the integrals of the summands.* Thus, in the case of two terms

$$\int_a^b [f_1(x)+f_2(x)]\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx. \tag{2}$$

Proof.

$$\int_a^b [f_1(x)+f_2(x)]\,dx = \lim_{\max \Delta x \to 0} \sum_{i=1}^{n} [f_1(\xi_i)+f_2(\xi_i)]\,\Delta x_i =$$

$$= \lim_{\max \Delta x \to 0} \left[\sum_{i=1}^{n} f_1(\xi_1)\,\Delta x_i + \sum_{i=1}^{n} f_2(\xi_i)\,\Delta x_i\right] =$$

$$= \lim_{\max \Delta x \to 0} \sum_{i=1}^{n} f_1(\xi_i)\,\Delta x_i + \lim_{\max \Delta x \to 0} \sum_{i=1}^{n} f_2(\xi_i)\,\Delta x_i =$$

$$= \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx.$$

The proof is similar for any number of terms.

Properties 1 and 2, though proved only for the case $a<b$, hold also for $a \geqslant b$.

However, the following property holds only for $a<b$:

Property 3. *If on the interval* $[a, b]$ $(a<b)$, *the functions* $f(x)$ *and* $\varphi(x)$ *satisfy the condition* $f(x) \leqslant \varphi(x)$, *then*

$$\int_a^b f(x)\,dx \leqslant \int_a^b \varphi(x)\,dx. \tag{3}$$

Proof. Let us consider the difference

$$\int_a^b \varphi(x)\,dx-\int_a^b f(x)\,dx=\int_a^b [\varphi(x)-f(x)]\,dx=$$

$$=\lim_{\max\Delta x\to 0}\sum_{i=1}^{n}(\varphi(\xi_i)-f(\xi_i))\,\Delta x_i.$$

Here, each difference $\varphi(\xi_i)-f(\xi_i)\geqslant 0$, $\Delta x_i\geqslant 0$. Thus, each term of the sum is nonnegative, the entire sum is nonnegative, and its limit is nonnegative; that is,

$$\int_a^b [\varphi(x)-f(x)]\,dx\geqslant 0$$

or

$$\int_a^b \varphi(x)\,dx-\int_a^b f(x)\,dx\geqslant 0,$$

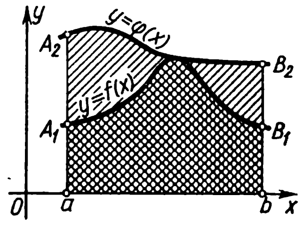


Fig. 213.

whence follows inequality (3).

If $f(x)>0$ and $\varphi(x)>0$, then this property is nicely illustrated geometrically (Fig. 213). Since $\varphi(x)\geqslant f(x)$, the area of the curvilinear trapezoid aA_1B_1b does not exceed the area of the curvilinear trapezoid aA_2B_2b.

Property 4. *If m and M are the smallest and greatest values of the function $f(x)$ on the interval $[a, b]$ and $a\leqslant b$, then*

$$m(b-a)\leqslant\int_a^b f(x)\,dx\leqslant M(b-a). \tag{4}$$

Proof. It is given that

$$m\leqslant f(x)\leqslant M.$$

On the basis of property (3) we have

$$\int_a^b m\,dx\leqslant\int_a^b f(x)\,dx\leqslant\int_a^b M\,dx. \tag{4'}$$

But

$$\int_a^b m\,dx=m(b-a),\quad \int_a^b M\,dx=M(b-a)$$

(see Example 3, Sec. 2, Ch. XI). Putting these expressions into inequality (4'), we get inequality (4).

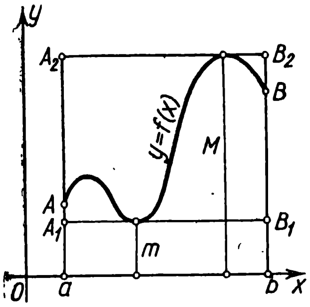


Fig. 214.

If $f(x) \geqslant 0$, this property is clearly illustrated geometrically (Fig. 214). The area of the curvilinear trapezoid $aABb$ lies between the areas of the rectangles aA_1B_1b and aA_2B_2b.

Property 5. (Mean-value theorem). *If a function $f(x)$ is continuous on the interval $[a, b]$, then there is a point ξ on this interval such that the following equality holds:*

$$\int_a^b f(x)\,dx = (b-a) f(\xi). \qquad (5)$$

Proof. For definiteness let $a < b$. If m and M are, respectively, the smallest and greatest values of $f(x)$ on $[a, b]$, then by virtue of (4)

$$m \leqslant \frac{1}{b-a} \int_a^b f(x)\,dx \leqslant M.$$

Whence

$$\frac{1}{b-a} \int_a^b f(x)\,dx = \mu, \text{ where } m \leqslant \mu \leqslant M.$$

Since $f(x)$ is continuous, it takes on all intermediate values between m and M. Therefore, for some value $\xi\,(a \leqslant \xi \leqslant b)$ we will have $\mu = f(\xi)$, or

$$\int_a^b f(x)\,dx = f(\xi)(b-a).$$

Property 6. *For any three numbers a, b, c the equality*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \qquad (6)$$

is true, provided all these three integrals exist.

Proof. First suppose that $a < c < b$, and form the integral sum of the function $f(x)$ on the interval $[a, b]$.

Since the limit of the integral sum is independent of the way in which the interval $[a, b]$ is divided into subintervals, we shall divide $[a, b]$ into subintervals such that the point c is the division point. Then we partition the sum $\sum\limits_a^b$, which corresponds to the

interval $[a, b]$, into two sums: $\sum_a^c$, which corresponds to $[a, c]$, and $\sum_c^b$, which corresponds to $[c, b]$. Then

$$\sum_a^b f(\xi_i)\,\Delta x_i = \sum_a^c f(\xi_i)\,\Delta x_i + \sum_c^b f(\xi_i)\,\Delta x_i.$$

Now, passing to the limit as $\max \Delta x_i \to 0$, we get relation (6).

If $a < b < c$, then on the basis of what has been proved we can write

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \quad \text{or} \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx - \int_b^c f(x)\,dx;$$

but by formula (4), Sec. 2, we have

$$\int_b^c f(x)\,dx = -\int_c^b f(x)\,dx.$$

Therefore,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

This property is similarly proved for any other arrangement of points a, b, and c.

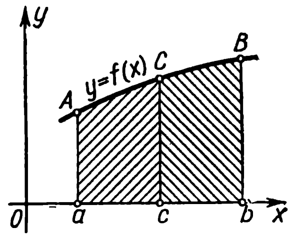


Fig. 215.

Fig. 215 illustrates, geometrically, Property 6 for the case when $f(x) > 0$ and $a < c < b$: the area of the trapezoid $aABb$ is equal to the sum of the areas of the trapezoids $aACc$ and $cCBb$.

SEC. 4. EVALUATING A DEFINITE INTEGRAL. NEWTON-LEIBNIZ FORMULA

In a definite integral

$$\int_a^b f(x)\,dx$$

let the lower limit a be fixed, and let the upper limit b vary. Then the value of the integral will vary as well; that is, the integral is a **function of the upper limit**.

So as to retain customary notations, we shall denote the upper limit by x, and to avoid confusion we shall denote the variable

of integration by t. (This change in notation does not change the value of the integral.) We get the integral $\int_a^x f(t)\,dt$. For constant a, this integral will be a function of the upper limit x. We denote this function by $\Phi(x)$:

$$\Phi(x)=\int_a^x f(t)\,dt. \tag{1}$$

If $f(t)$ is a nonnegative function, the quantity $\Phi(x)$ is numerically equal to the area of the curvilinear trapezoid $aAXx$ (Fig. 216). It is obvious that this area varies with x.

Let us find the derivative of $\Phi(x)$ with respect to x, or the derivative of the definite integral (1) with respect to the upper limit.

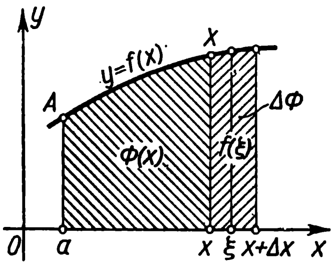


Fig. 216.

Theorem 1. *If $f(x)$ is a continuous function and $\Phi(x)=\int_a^x f(t)\,dt$, then we have the equality*

$$\Phi'(x)=f(x).$$

In other words, *the derivative of a definite integral with respect to the upper limit is equal to the integrand, in which the value of the upper limit replaces the variable of integration* (provided that the integrand is continuous).

Proof. Let us give the argument x a positive or negative increment Δx; then (taking into account Property 6 of a definite integral) we get

$$\Phi(x+\Delta x)=\int_a^{x+\Delta x} f(t)\,dt=\int_a^x f(t)\,dt+\int_x^{x+\Delta x} f(t)\,dt.$$

The increment of the function $\Phi(x)$ is equal to

$$\Delta\Phi=\Phi(x+\Delta x)-\Phi(x)=\int_a^x f(t)\,dt+\int_x^{x+\Delta x} f(t)\,dt-\int_a^x f(t)\,dt;$$

that is,

$$\Delta\Phi=\int_x^{x+\Delta x} f(t)\,dt.$$

Apply to the latter integral the mean-value theorem (Property 5 of a definite integral):

$$\Delta\Phi = f(\xi)(x+\Delta x - x) = f(\xi)\,\Delta x,$$

where ξ lies between x and $x+\Delta x$.

Find the ratio of the increment of the function to the increment of the argument:

$$\frac{\Delta\Phi}{\Delta x} = \frac{f(\xi)\,\Delta x}{\Delta x} = f(\xi).$$

Hence,

$$\Phi'(x) = \lim_{\Delta x \to 0} \frac{\Delta\Phi}{\Delta x} = \lim_{\Delta x \to 0} f(\xi).$$

But since $\xi \to x$ as $\Delta x \to 0$, we have

$$\lim_{\Delta x \to 0} f(\xi) = \lim_{\xi \to x} f(\xi),$$

and due to the continuity of the function $f(x)$,

$$\lim_{\xi \to x} f(\xi) = f(x).$$

Thus, $\Phi'(x) = f(x)$, and the theorem is proved.

The geometric illustration of this theorem (Fig. 216) is simple: the increment $\Delta\Phi = f(\xi)\,\Delta x$ is equal to the area of a curvilinear trapezoid with base Δx, and the derivative $\Phi'(x) = f(x)$ is equal to the length of the interval xX.

Note. One consequence of the theorem that has been proved is that *every continuous function has an antiderivative*. Indeed, if the function $f(t)$ is continuous on the interval $[a, x]$, then as was pointed out in Sec. 2, Ch. XI, in this case the definite integral $\int_a^x f(t)\,dt$ exists, which is to say that the following function exists:

$$\Phi(x) = \int_a^x f(t)\,dt.$$

But from what has already been proved, it is the antiderivative of $f(x)$.

Theorem 2 *If $F(x)$ is some antiderivative of the continuous function $f(x)$, then the formula*

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad (2)$$

holds.

This formula is known as the *Newton-Leibniz formula.* *)

Proof. Let $F(x)$ be some antiderivative of the function $f(x)$. By Theorem 1, the function $\int_a^x f(t)\,dt$ is also an antiderivative of $f(x)$. But any two antiderivatives of a given function differ by the constant C^*. And so we can write

$$\int_a^x f(t)\,dt = F(x) + C^*. \tag{3}$$

Within an appropriate choice of C^* this equality holds for all values of x, that is, it is an identity. To determine the constant C^* put $x=a$ in the identity; then

$$\int_a^a f(t)\,dt = F(a) + C^*,$$

or

$$0 = F(a) + C^*,$$

whence

$$C^* = -F(a).$$

Hence,

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

Putting $x=b$, we obtain the Newton-Leibniz formula:

$$\int_a^b f(t)\,dt = F(b) - F(a),$$

or, replacing the notation of the variable of integration by x,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

It will be noted that the difference $F(b)-F(a)$ is independent of the choice of antiderivative F, since all antiderivatives differ by a constant quantity, which disappears upon subtraction anyway.

*) It is necessary to point out that the name of formula (2) is not exact, since neither Newton nor Leibniz had any such formula. The important thing, however, is that namely Leibniz and Newton were the first to establish a relationship between integration and differentiation, thus making possible the rule for evaluating definite integrals.

If we introduce the notation *)

$$F(b)-F(a)=F(x)\big|_a^b,$$

then formula (2) may be rewritten as follows:

$$\int_a^b f(x)\,dx=F(x)\big|_a^b=F(b)-F(a).$$

The Newton-Leibniz formula yields a practical and convenient method for computing definite integrals in cases where the antiderivative of the integrand is known. Only when this formula was discovered did the definite integral acquire its present significance in mathematics. Although the ancients (Archimedes) were familiar with a process similar to the computation of a definite integral as the limit of an integral sum, the applications of this method were confined to the very simple cases when the limit of the sum could be computed directly. The Newton-Leibniz formula greatly expanded the field of application of the definite integral, because mathemat cs obtained a **general method** for solving various problems of a particular type and so could considerably extend the range of applications of the definite integral to technology, mechanics, astronomy, and so on.

Example 1.

$$\int_a^b x\,dx=\frac{x^2}{2}\Big|_a^b=\frac{b^2-a^2}{2}.$$

Example 2.

$$\int_a^b x^2\,dx=\frac{x^3}{3}\Big|_a^b=\frac{b^3-a^3}{3}.$$

Example 3.

$$\int_a^b x^n\,dx=\frac{x^{n+1}}{n+1}\Big|_a^b=\frac{b^{n+1}-a^{n+1}}{n+1}\ (n\neq-1).$$

*) The expression $\big|_a^b$ is called the sign of double substitution. In the literature we find two notations:

$$F(b)-F(a)=[F(x)]_a^b$$

or

$$F(b)-F(a)=F(x)\big|_a^b.$$

We shall use both notations.

Example 4.

$$\int_a^b e^x\,dx = e^x\Big|_a^b = e^b - e^a.$$

Example 5.

$$\int_0^{2\pi} \sin x\,dx = -\cos x\Big|_0^{2\pi} = -(\cos 2\pi - \cos 0) = 0.$$

Example 6.

$$\int_0^1 \frac{x\,dx}{\sqrt{1+x^2}} = \sqrt{1+x^2}\Big|_0^1 = \sqrt{2} - 1.$$

SEC. 5. CHANGING THE VARIABLE IN THE DEFINITE INTEGRAL

Theorem. *Let there be an integral*

$$\int_a^b f(x)\,dx,$$

where the function $f(x)$ *is continuous on the interval* $[a, b]$.
Introduce a new variable t *using the formula*

$$x = \varphi(t).$$

If
1) $\varphi(\alpha) = a$, $\varphi(\beta) = b$,
2) $\varphi(t)$ and $\varphi'(t)$ *are continuous on* $[\alpha, \beta]$,
3) $f[\varphi(t)]$ *is defined and is continuous on* $[\alpha, \beta]$, *then*

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\varphi(t)]\,\varphi'(t)\,dt. \tag{1}$$

Proof. If $F(x)$ is an antiderivative of the function $f(x)$, we can write the following equations:

$$\int f(x)\,dx = F(x) + C, \tag{2}$$

$$\int f[\varphi(t)]\,\varphi'(t)\,dt = F[\varphi(t)] + C. \tag{3}$$

The truth of the latter equation is checked by differentiation of both sides with respect to t. [It likewise follows from formula (2), Sec. 4, Ch. X.] From (2) we have

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

From (3) we have

$$\int_{\alpha}^{\beta} f[\varphi(t)]\,\varphi'(t)\,dt = F[\varphi(t)]\Big|_{\alpha}^{\beta} =$$

$$= F[\varphi(\beta)] - F[\varphi(\alpha)] =$$

$$= F(b) - F(a).$$

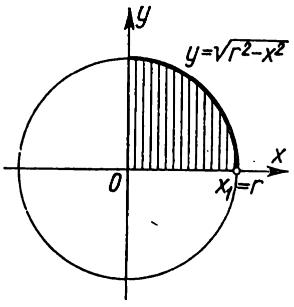


Fig. 217.

The right sides of the latter expressions are equal, and so the left sides are equal as well, thus proving the theorem.

Note. It will be noted that when computing the definite integral from formula (1) we do not return to the old variable. If we compute the second of the definite integrals of (1), we get a certain number; the first integral is also equal to this number.

Example. Compute the integral

$$\int_0^r \sqrt{r^2-x^2}\,dx.$$

Solution. Change the variable:

$$x = r\sin t, \quad dx = r\cos t\,dt.$$

Determine the new limits:

$$x=0 \text{ for } t=0,$$

$$x=r \text{ for } t=\frac{\pi}{2}.$$

Consequently,

$$\int_0^r \sqrt{r^2-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{r^2-r^2\sin^2 t}\; r\cos t\,dt = r^2\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\,\cos t\,dt =$$

$$= r^2\int_0^{\frac{\pi}{2}} \cos^2 t\,dt = r^2\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}+\frac{1}{2}\cos 2t\right)dt = r^2\left[\frac{t}{2}+\frac{\sin 2t}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi r^2}{4}.$$

Geometrically, the computed integral is the area of $\frac{1}{4}$ of the circle bounded by the circle $x^2+y^2=r^2$ (Fig. 217).

SEC. 6. INTEGRATION BY PARTS

Let u and v be differentiable functions of x. Then

$$(uv)' = u'v + uv'.$$

Integrating both sides of the identity from a to b, we have

$$\int_a^b (uv)'\,dx = \int_a^b u'v\,dx + \int_a^b uv'\,dx. \tag{1}$$

Since $\int (uv)'\,dx = uv + C$, we have $\int_a^b (uv)'\,dx = uv\Big|_a^b$; for this reason, the equation can be written in the form

$$uv\Big|_a^b = \int_a^b v\,du + \int_a^b u\,dv,$$

or, finally,

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Example. Evaluate the integral $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$.

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x\,dx = -\int_0^{\frac{\pi}{2}} \underbrace{\sin^{n-1} x}_{u}\, \underbrace{d\cos x}_{dv} =$$

$$= -\sin^{n-1} x \cos x\Big|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos x \cos x\,dx =$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\,dx =$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,(1-\sin^2 x)\,dx =$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\,dx.$$

In the notation chosen we can write the latter equation as

$$I_n = (n-1)\,I_{n-2} - (n-1)\,I_n,$$

whence we find

$$I_n=\frac{n-1}{n}I_{n-2}. \qquad (2)$$

Using the same technique we find

$$I_{n-2}=\frac{n-3}{n-2}I_{n-4},$$

and so

$$I_n=\frac{n-1}{n}\,\frac{n-3}{n-2}I_{n-4}.$$

Continuing in the same way, we arrive at I_0 or I_1 depending on whether the number n is even or odd.

Let us consider two cases:

1) n is even, $n=2m$:

$$I_{2m}=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\frac{3}{4}\cdot\frac{1}{2}I_0;$$

2) n is odd, $n=2m+1$:

$$I_{2m+1}=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\frac{4}{5}\cdot\frac{2}{3}I_1,$$

but since

$$I_0=\int_0^{\frac{\pi}{2}}\sin^0 x\,dx=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2},$$

$$I_1=\int_0^{\frac{\pi}{2}}\sin x\,dx=1,$$

we have

$$I_{2m}=\int_0^{\frac{\pi}{2}}\sin^{2m}x\,dx=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},$$

$$I_{2m+1}=\int_0^{\frac{\pi}{2}}\sin^{2m+1}x\,dx=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}.$$

From these formulas there follows the Wallis formula, which expresses the number $\frac{\pi}{2}$ in the form of an infinite product.

Indeed, from the latter two equations we find, by means of termwise division,

$$\frac{\pi}{2}=\left(\frac{2\cdot4\cdot6\ldots2m}{3\cdot5\ldots(2m-1)}\right)^2\frac{1}{2m+1}\,\frac{I_{2m}}{I_{2m+1}}. \qquad (3)$$

We shall now prove that

$$\lim_{m \to \infty} \frac{I_{2m}}{I_{2m+1}} = 1.$$

For all x of the interval $\left(0, \frac{\pi}{2}\right)$ the inequalities

$$\sin^{2m-1} x > \sin^{2m} x > \sin^{2m+1} x$$

hold.

Integrating from 0 to $\frac{\pi}{2}$, we get

$$I_{2m-1} \geqslant I_{2m} \geqslant I_{2m+1},$$

whence

$$\frac{I_{2m-1}}{I_{2m+1}} \geqslant \frac{I_{2m}}{I_{2m+1}} \geqslant 1. \tag{4}$$

From (2) it follows that

$$\frac{I_{2m-1}}{I_{2m+1}} = \frac{2m+1}{2m}.$$

Hence,

$$\lim_{m \to \infty} \frac{I_{2m-1}}{I_{2m+1}} = \lim_{m \to \infty} \frac{2m+1}{2m} = 1.$$

From inequality (4) we have

$$\lim_{m \to \infty} \frac{I_{2m}}{I_{2m+1}} = 1.$$

Passing to the limit in formula (3), we get *Wallis' formula (Wallis' product)* for

$$\frac{\pi}{2} = \lim_{m \to \infty} \left[\left(\frac{2 \cdot 4 \cdot 6 \ldots 2m}{3 \cdot 5 \ldots (2m-1)}\right)^2 \frac{1}{2m+1}\right].$$

This formula may be written in the form

$$\frac{\pi}{2} = \lim_{m \to \infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdots \frac{2m-2}{2m-1} \cdot \frac{2m}{2m-1} \cdot \frac{2m}{2m+1}\right).$$

SEC. 7. IMPROPER INTEGRALS

1. Integrals with infinite limits. Let the function $f(x)$ be defined and continuous for all values of x such that $a \leqslant x < +\infty$. Consider the integral

$$I(b) = \int_a^b f(x)\,dx.$$

This integral is meaningful for any $b > a$. The integral varies with b and is a continuous function of b (see Sec. 4, Ch. XI). Let us consider the behaviour of this integral when $b \to +\infty$ (Fig. 218).

Definition. If there exists a finite limit

$$\lim_{b \to +\infty} \int_a^b f(x)\,dx,$$

then this limit is called the *improper integral* of the function $f(x)$ in the interval $[a, +\infty]$ and is denoted by the symbol

$$\int_a^{+\infty} f(x)\,dx.$$

Thus, by definition, we have

$$\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx.$$

In this case it is said that the improper integral $\int_a^{+\infty} f(x)\,dx$ *exists* or *converges*. If $\int_a^b f(x)\,dx$ as $b \to +\infty$ does not have a finite limit, one says that $\int_a^{+\infty} f(x)\,dx$ *does not exist* or *diverges*.

It is easy to see the geometric meaning of an improper integral for the case when $f(x) \geqslant 0$: if the integral $\int_a^b f(x)\,dx$ expresses the area of a region bounded by the curve $y=f(x)$, the x-axis and the ordinates $x=a$, $x=b$, it is natural to consider that the improper integral $\int_a^{+\infty} f(x)\,dx$ expresses the area of an unbounded (infinite) region lying between the lines $y=f(x)$, $x=a$, and the axis of abscissas.

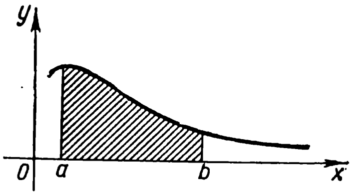


Fig. 218.

We similarly define the improper integrals of other infinite intervals:

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{\alpha \to -\infty} \int_{\alpha}^{a} f(x)\,dx,$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx.$$

The latter equation should be understood as follows: if each of the improper integrals on the right exists, then, by definition, the integral on the left also exists (converges).

Example 1. Evaluate the integral $\int\limits_0^{+\infty} \frac{dx}{1+x^2}$ (see Figs. 219 and 220).

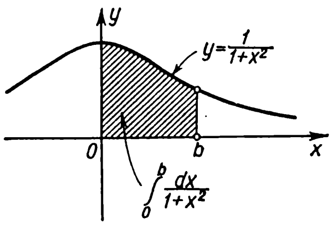


Fig. 219.

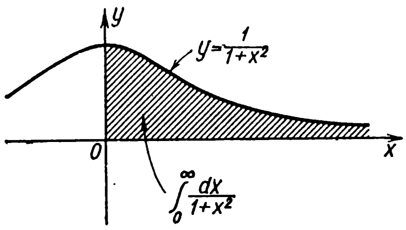


Fig. 220.

Solution. By the definition of an improper integral we find

$$\int\limits_0^{+\infty} \frac{dx}{1+x^2} = \lim_{b \to +\infty} \int\limits_0^b \frac{dx}{1+x^2} = \lim_{b \to +\infty} \arctan x \Big|_0^b = \lim_{b \to +\infty} \arctan b = \frac{\pi}{2}.$$

This integral expresses the area of an infinite curvilinear trapezoid cross-hatched in Fig. 220.

Example 2. Find out at which values of α (Fig. 221) the integral

$$\int\limits_1^{+\infty} \frac{dx}{x^\alpha}$$

converges and at which it diverges.

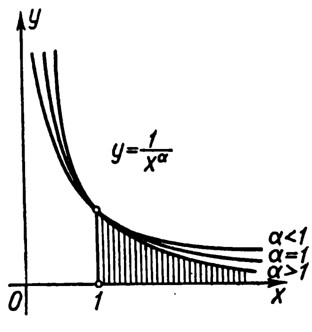


Fig. 221.

Solution. Since (when $\alpha \neq 1$)

$$\int\limits_1^b \frac{dx}{x^\alpha} = \frac{1}{1-\alpha} x^{1-\alpha} \Big|_1^b = \frac{1}{1-\alpha} [b^{1-\alpha} - 1],$$

we have

$$\int\limits_1^{+\infty} \frac{dx}{x^\alpha} = \lim_{b \to +\infty} \frac{1}{1-\alpha} (b^{1-\alpha} - 1).$$

Consequently,

if $\alpha > 1$, then $\int\limits_1^{+\infty} \frac{dx}{x^\alpha} = \frac{1}{\alpha - 1}$, and the integral converges;

if $\alpha < 1$, then $\int\limits_1^{+\infty} \frac{dx}{x^\alpha} = \infty$, and the integral diverges.

When $\alpha = 1$, $\int\limits_1^{+\infty} \frac{dx}{x} = \ln x \Big|_1^{+\infty} = \infty$, the integral diverges.

Example 3. Evaluate $\int\limits_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

Solution.

$$\int\limits_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int\limits_{-\infty}^{0} \frac{dx}{1+x^2} + \int\limits_{0}^{+\infty} \frac{dx}{1+x^2}.$$

The second integral is equal to $\frac{\pi}{2}$ (see Example 1). Compute the first integral:

$$\int\limits_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{\alpha \to -\infty} \int\limits_{\alpha}^{0} \frac{dx}{1+x^2} = \lim_{\alpha \to -\infty} \arctan x \Big|_{\alpha}^{0} =$$

$$= \lim_{\alpha \to -\infty} (\arctan 0 - \arctan \alpha) = \frac{\pi}{2}.$$

Therefore,

$$\int\limits_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

In many cases it is sufficient to determine whether the given integral converges or diverges, and to estimate its value. The following theorems, which we give without proof, may be useful in this respect. We shall illustrate their application in a few cases.

Theorem 1. *If for all $x\,(x \geqslant a)$ the inequality*

$$0 \leqslant f(x) \leqslant \varphi(x)$$

is fulfilled and if $\int\limits_a^{+\infty} \varphi(x)\,dx$ *converges, then* $\int\limits_a^{+\infty} f(x)\,dx$ *also converges, and*

$$\int\limits_a^{+\infty} f(x)\,dx \leqslant \int\limits_a^{+\infty} \varphi(x)\,dx.$$

Example 4. Investigate the integral

$$\int\limits_1^{+\infty} \frac{dx}{x^2(1+e^x)}$$

for convergence.

Solution. It will be noted that when $1 \leqslant x$,

$$\frac{1}{x^2(1+e^x)} < \frac{1}{x^2}.$$

And

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^{+\infty} = 1.$$

Consequently,

$$\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$$

converges, and its value is less than 1.

Theorem 2. *If for all $x\,(x \geqslant a)$ the inequality $0 \leqslant \varphi(x) \leqslant f(x)$ is fulfilled, and $\int_a^{+\infty} \varphi(x)\,dx$ diverges, then the integral $\int_a^{+\infty} f(x)\,dx$ also diverges.*

Example 5. Find out whether the following integral converges:

$$\int_1^{+\infty} \frac{x+1}{\sqrt{x^3}}.$$

We notice that

$$\frac{x+1}{\sqrt{x^3}} > \frac{x}{\sqrt{x^3}} = \frac{1}{\sqrt{x}}.$$

But

$$\int_1^{+\infty} \frac{dx}{\sqrt{x}} = \lim_{b \to +\infty} 2\sqrt{x}\,\bigg|_1^b = +\infty.$$

Consequently, the given integral also converges.

In the last two theorems we considered improper integrals of nonnegative functions. For the case of a function $f(x)$ which changes its sign in an infinite interval we have the following theorem.

Theorem 3. *If the integral $\int_a^{+\infty} |f(x)|\,dx$ converges, then the integral $\int_a^{+\infty} f(x)\,dx$ also converges.*

In this case, the latter integral is called an *absolutely convergent integral*.

Example 6. Investigate the convergence of the integral

$$\int_{1}^{+\infty} \frac{\sin x}{x^3}\,dx.$$

Solution. Here, the integrand is an alternating function. We note that

$$\left|\frac{\sin x}{x^3}\right| \leqslant \left|\frac{1}{x^3}\right|. \quad \text{But} \quad \int_{1}^{+\infty} \frac{dx}{x^3} = -\frac{1}{2x^2}\Bigg|_{1}^{+\infty} = \frac{1}{2}.$$

Therefore, the integral $\int_{1}^{+\infty} \left|\frac{\sin x}{x^3}\right| dx$ converges. Whence it follows that the given integral also converges.

2. The integral of a discontinuous function. Let the function $f(x)$ be defined and continuous when $a \leqslant x < c$, and for $x = c$ let the function be either not defined or let it be discontinuous. In this case, one cannot speak of the integral $\int_{a}^{c} f(x)\,dx$ as of the limit of integral sums, because $f(x)$ is not continuous on the interval $[a, c]$, and for this reason the limit may not exist.

The integral $\int_{a}^{c} f(x)\,dx$ of the function $f(x)$ **discontinuous at the point** c is defined as follows:

$$\int_{a}^{c} f(x)\,dx = \lim_{b \to c-0} \int_{a}^{b} f(x)\,dx.$$

If the limit on the right exists, the integral is called an improper *convergent* integral, otherwise it is *divergent*.

If the function $f(x)$ is discontinuous at the left extremity of the interval $[a, c]$ (that is, for $x = a$), then by **definition**

$$\int_{a}^{c} f(x)\,dx = \lim_{b \to a+0} \int_{b}^{c} f(x)\,dx.$$

If the function $f(x)$ is discontinuous at some point $x = x_0$ **inside** the interval $[a, c]$, we put

$$\int_{a}^{c} f(x)\,dx = \int_{a}^{x_0} f(x)\,dx + \int_{x_0}^{c} f(x)\,dx,$$

if both improper integrals on the right side of the equation exist.

Example 7. Evaluate

$$\int_0^1 \frac{dx}{\sqrt{1-x}}.$$

Solution.

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{b \to 1-0} \int_0^b \frac{dx}{\sqrt{1-x}} = -\lim_{b \to 1-0} 2\sqrt{1-x}\,\Big|_0^b =$$

$$= -\lim_{b \to 1-0} 2\,[\sqrt{1-b}-1] = 2.$$

Example 8. Evaluate the integral $\int_{-1}^{1} \frac{dx}{x^2}$.

Solution. Since inside the interval of integration there exists a point $x=0$ where the integrand is discontinuous, the integral must be represented as the sum of two terms:

$$\int_{-1}^{1} \frac{dx}{x^2} = \lim_{\varepsilon_1 \to -0} \int_{-1}^{\varepsilon_1} \frac{dx}{x^2} + \lim_{\varepsilon_2 \to +0} \int_{\varepsilon_2}^{1} \frac{dx}{x^2}.$$

Calculate each limit separately:

$$\lim_{\varepsilon_1 \to -0} \int_{-1}^{\varepsilon_1} \frac{dx}{x^2} = -\lim_{\varepsilon_1 \to -0} \frac{1}{x}\Big|_{-1}^{\varepsilon_1} = -\lim_{\varepsilon_1 \to -0} \left(\frac{1}{\varepsilon_1} - \frac{1}{-1}\right) = \infty.$$

Thus, the integral diverges on the interval $[-1, 0]$:

$$\lim_{\varepsilon_2 \to +0} \int_{\varepsilon_2}^{1} \frac{dx}{x^2} = -\lim_{\varepsilon_2 \to +0} \left(1 - \frac{1}{\varepsilon_2}\right) = \infty.$$

And this means that the integral also diverges on the interval $[0, 1]$.

Hence, the given integral diverges on the entire interval $[-1, 1]$.

It should be noted that if we had begun to evaluate the given integral without paying attention to the discontinuity of the integrand at the point $x=0$, the result would have been wrong. Indeed,

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -\left(\frac{1}{1} - \frac{1}{-1}\right) = -2,$$

which is impossible (Fig. 222).

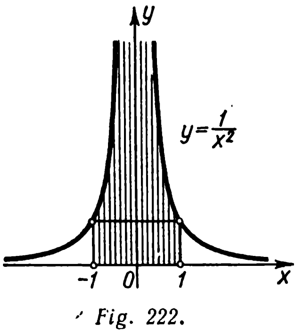


Fig. 222.

Note. If the function $f(x)$, defined on the interval $[a, b]$, has, within this interval, a finite number of points of discontinuity $a_1, a_2, \ldots, a_n$, then the integral of the function $f(x)$ on the

interval $[a, b]$ is defined as follows:

$$\int_a^b f(x)\,dx = \int_a^{a_1} f(x)\,dx + \int_{a_1}^{a_2} f(x)\,dx + \ldots + \int_{a_n}^{b} f(x)\,dx,$$

if each of the improper integrals on the right side of the equation converges. But if even one of these integrals diverges, then $\int_a^b f(x)\,dx$ is called divergent as well.

For determining the convergence of improper integrals of discontinuous functions and for estimating their values, one can frequently make use of theorems similar to those used to estimate integrals with infinite limits.

Theorem 1′. *If on the interval* $[a, c]$ *the functions* $f(x)$ *and* $\varphi(x)$ *are discontinuous at the point* c, *and at all points of this interval the inequalities* $\varphi(x) \geqslant f(x) \geqslant 0$ *are fulfilled and* $\int_a^c \varphi(x)\,dx$ *converges, then* $\int_a^c f(x)\,dx$ *also converges.*

Theorem 2′. *If on the interval* $[a, c]$ *the functions* $f(x)$ *and* $\varphi(x)$ *are discontinuous at the point* c, *and at all points of this interval the inequalities* $f(x) \geqslant \varphi(x) \geqslant 0$ *are fulfilled and* $\int_a^c \varphi(x)\,dx$ *diverges, then* $\int_a^c f(x)\,dx$ *also diverges.*

Theorem 3′. *If* $f(x)$ *is an alternating function on the interval* $[a, c]$ *and discontinuous only at the point* c, *and the improper integral* $\int_a^c |f(x)|\,dx$ *of the absolute value of this function converges, then the integral* $\int_a^c f(x)\,dx$ *of the function itself also converges.*

Use is frequently made of $\frac{1}{(c-x)^\alpha}$ as functions with which it is convenient to compare the functions under the sign of the improper integral. It is easy to verify that $\int_a^c \frac{1}{(c-x)^\alpha}\,dx$ converges for $\alpha < 1$, and diverges for $\alpha \geqslant 1$.

The same applies also to the integrals $\int_a^c \frac{1}{(x-a)^\alpha}\,dx$.

Example 9. Does the integral $\int_0^1 \frac{1}{\sqrt{x}+4x^3}\,dx$ converge?

Solution. The integrand is discontinuous at the left extremity of the interval [0, 1]. Comparing it with the function $\frac{1}{\sqrt{x}}$, we have

$$\frac{1}{\sqrt{x}+4x^3} < \frac{1}{\sqrt{x}}.$$

The improper integral $\int_0^1 \frac{dx}{x^{1/2}}$ exists. Consequently, the improper integral of a lesser function, that is, $\int_0^1 \frac{1}{\sqrt{x}+4x^3}\,dx$, also exists.

SEC. 8. APPROXIMATING DEFINITE INTEGRALS

At the end of Chapter X it was pointed out that not for every continuous function is its antiderivative expressible in terms of elementary functions. In these cases, computation of definite integrals by the Newton-Leibniz formula is involved, and various methods of **approximation** are used to evaluate the definite integrals. The following are several methods of approximate integration based on the concept of a definite integral as the limit of a sum.

1. Rectangular formula. Let a continuous function $y=f(x)$ be given on an interval $[a, b]$. It is required to evaluate the definite integral

$$\int_a^b f(x)\,dx.$$

Divide the interval $[a, b]$ by the points $a=x_0, x_1, x_2, \ldots, x_n=b$ into n equal parts of length Δx:

$$\Delta x = \frac{b-a}{n}.$$

Then denote by $y_0, y_1, y_2, \ldots, y_{n-1}, y_n$ the values of the function $f(x)$ at the points $x_0, x_1, x_2, \ldots, x_n$; that is,

$$y_0=f(x_0);\ y_1=f(x_1);\ \ldots;\ y_n=f(x_n).$$

Form the sums:

$$y_0\Delta x + y_1\Delta x + \ldots + y_{n-1}\Delta x,$$
$$y_1\Delta x + y_2\Delta x + \ldots + y_n\Delta x.$$

Each of these sums is an integral sum for $f(x)$ on the interval $[a, b]$ and for this reason approximately expresses the integral

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}(y_0 + y_1 + y_2 + \ldots + y_{n-1}), \tag{1}$$

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}(y_1 + y_2 + \ldots + y_n). \tag{1'}$$

This is the *rectangular formula*. From Fig. 223 it is evident that if $f(x)$ is a positive and increasing function, then formula (1) expresses the area of the step-like figure composed of "inside" rectangles, while formula (1′) yields the area of the step-like figure composed of "outside" rectangles.

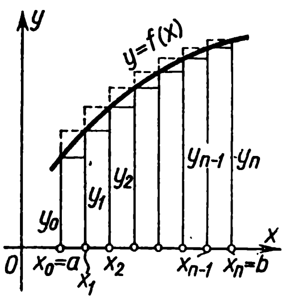


Fig. 223.

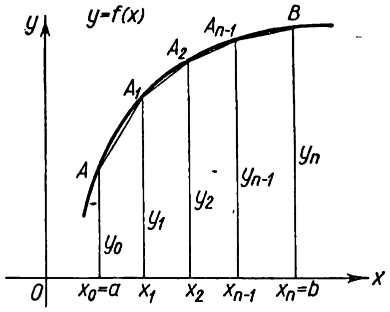


Fig. 224.

The error made when calculating integrals by the rectangular formula diminishes with increasing n $\left(\text{that is, the smaller the divisions } \Delta x = \frac{b-a}{n}\right)$.

II. The trapezoidal rule. It is natural to expect that we will obtain a more exact value of the definite integral if we replace the curve $y = f(x)$ not by a stepped line, as in the rectangular formula, but by an **inscribed** broken line (Fig. 224). Then the area of the curvilinear trapezoid $aABb$ will be replaced by the sum of the areas of the rectilinear trapezoids bounded from above by the chords AA_1, A_1A_2, ..., $A_{n-1}B$. Since the area of the first of

these trapezoids is $\frac{y_0+y_1}{2}\Delta x$, the area of the second is $\frac{y_1+y_2}{2}\Delta x$, and so forth, so

$$\int_a^b f(x)\,dx \approx \left(\frac{y_0+y_1}{2}\Delta x+\frac{x_1+y_2}{2}\Delta x+\ldots+\frac{y_{n-1}+y_n}{2}\Delta x\right)$$

or

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left(\frac{y_0+y_n}{2}+y_1+y_2+\ldots+y_{n-1}\right). \qquad (2)$$

This is the *trapezoidal formula (trapezoidal rule)*.

The choice of n is arbitrary. The greater this number, the smaller will be the division (subinterval) $\Delta x=\frac{b-a}{n}$ and the greater will be the accuracy with which the sum, written on the right side of the approximate equality (2), yields the value of the integral.

III. Parabolic formula (Simpson's rule). Divide the interval $[a, b]$ into an **even** number of parts $n=2m$. Replace the area of the curvilinear trapezoid, corresponding to the first two subintervals $[x_0, x_1]$ and $[x_1, x_2]$ and bounded by the given curve $y=f(x)$, by the area of a curvilinear trapezoid such that is bounded by a **quadratic parabola** passing through three points:

$$M(x_0, y_0); \quad M_1(x_1, y_1); \quad M_2(x_2, y_2),$$

and with an axis parallel to the y-axis (Fig. 225). We shall call this kind of curvilinear trapezoid a *parabolic* trapezoid.

The equation of a parabola with an axis parallel to the y-axis is of the form

$$y=Ax^2+Bx+C.$$

The coefficients A, B and C are uniquely determined from the condition that the parabola passes through three specified points. Analogous parabolas are constructed for other pairs of intervals as well. The sum of the areas of the parabolic trapezoids will yield the approximate value of the integral.

Let us first compute the areas of one parabolic trapezoid.

Lemma. *If a curvilinear trapezoid is bounded by the parabola*

$$y=Ax^2+Bx+C,$$

the x-axis and two ordinates separated by a distance 2h, then its area is

$$S=\frac{h}{3}(y_0+4y_1+y_2), \qquad (3)$$

where y_0 and y_2 are the extreme ordinates and y_1 is the ordinate of the curve at the midpoint of the interval.

Proof. Arrange an auxiliary coordinate system as shown in Fig. 226.

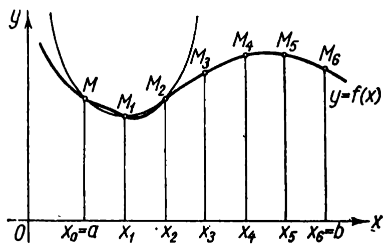


Fig. 225.

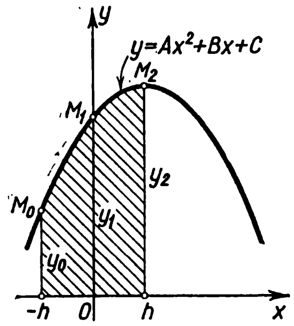


Fig. 226.

The coefficients in the equation of the parabola $y = Ax^2 + Bx + C$ are determined from the following equations:

$$\left.\begin{array}{lll} \text{if } x_0 = -h, & \text{then } y_0 = Ah^2 - Bh + C; \\ \text{if } x_1 = 0, & \text{then } y_1 = \qquad\qquad C; \\ \text{if } x_2 = h, & \text{then } y_2 = Ah^2 + Bh + C. \end{array}\right\} \qquad (4)$$

Considering the coefficients A, B, C known, we determine the area of the parabolic trapezoid with the aid of a definite integral:

$$S = \int_{-h}^{h} (Ax^2 + Bx + C)\,dx = \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_{-h}^{h} = \frac{h}{3}(2Ah^2 + 6C).$$

But from equalities (4) it follows that

$$y_0 + 4y_1 + y_2 = 2Ah^2 + 6C.$$

Hence,

$$S = \frac{h}{3}(y_0 + 4y_1 + y_2),$$

which is what had to be proved.

Let us come back to our basic problem (see Fig. 225). Using formula (3) we can write the following approximate equalities ($h = \Delta x$):

$$\int_{a=x_0}^{x_2} f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + y_2),$$

$$\int_{x_2}^{x_4} f(x)\,dx \approx \frac{\Delta x}{3}(y_2+4y_3+y_4),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\int_{x_{2m-2}}^{x_{2m}=b} f(x)\,dx \approx \frac{\Delta x}{3}(y_{2m-2}+4y_{2m-1}+y_{2m}).$$

Adding the left and right sides, we get (on the left) the sought-for integral and (on the right) its approximate value:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0+4y_1+2y_2+4y_3+\ldots$$
$$\ldots+2y_{2m-2}+4y_{2m-1}+y_{2m}), \qquad (5)$$

or

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6m}[y_0+y_{2m}+2(y_2+y_4+\ldots+y_{2m-2})+$$
$$+4(y_1+y_3+\ldots+y_{2m-1})].$$

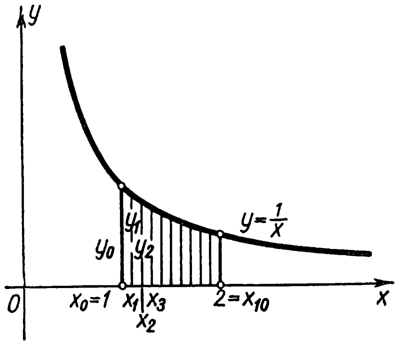


Fig. 227.

This is *Simpson's formula (rule)*. Here, the number of division points $2m$ is arbitrary; but the more of them there are, the more accurately the sum on the right side of (5) yields the value of the integral. *)

Example. Evaluate approximately

$$\ln 2 = \int_1^2 \frac{dx}{x}.$$

Solution. Divide the interval [1, 2] into 10 equal parts (Fig. 227). Assuming

$$\Delta x = \frac{2-1}{10} = 0.1,$$

*) To find out how many division points are needed to compute an integral to the desired number of decimal places, one can make use of formulas for estimating the error resulting from approximating the integral. We do not give these estimates here. The reader will find them in more advanced courses of analysis; see, for example, Fikhtengolts, "Course of Differential and Integral Calculus", 1959, Vol. II, Ch. IX, Sec. 5. (Russian edition).

we make a table of the values of the integrand:

x	$y=\frac{1}{x}$	x	$y=\frac{1}{x}$
$x_0=1.0$	$y_0=1\ 00000$	$x_6=1.6$	$y_6=0.62500$
$x_1=1.1$	$y_1=0.90909$	$x_7=1.7$	$y_7=0.58824$
$x_2=1.2$	$y_2=0.83333$	$x_8=1.8$	$y_8=0.55556$
$x_3=1.3$	$y_3=0.76923$	$x_9=1.9$	$y_9=0.52632$
$x_4=1.4$	$y_4=0.71429$	$x_{10}=2.0$	$y_{10}=0.50000$
$x_5=1.5$	$y_5=0.66667$		

I By the first rectangular formula (1) we get

$$\int_1^2 \frac{dx}{x} \approx 0.1\,(y_0+y_1+\ldots+y_9)=0.1\cdot 7.18773=0.71877.$$

By the second rectangular formula (1′) we get

$$\int_1^2 \frac{dx}{x} \approx 0.1\,(y_1+y_2+\ldots+y_{10})=0.1\cdot 6.68773=0.66877.$$

It follows directly from Fig. 227 that in this case the first formula yields the value of the integral with an **excess**, the second, with a **defect**.

II. By the trapezoidal rule (2), we have

$$\int_1^2 \frac{dx}{x} \approx 0.1\left(\frac{1+0.5}{2}+6.18773\right)=0.69377.$$

III. By Simpson's rule (5), we have

$$\int_1^2 \frac{dx}{x} \approx \frac{0.1}{3}\,[y_0+y_{10}+2\,(y_2+y_4+y_6+y_8)+4\,(y_1+y_3+y_5+y_7+y_9)]=$$

$$=\frac{0.1}{3}\,(1+0.5+2\cdot 2.72818+4\cdot 3.45955)=0.69315.$$

Actually, $\ln 2=\int_1^2 \frac{dx}{x}=0.6931472$ (to seven places of decimals).

Thus, when dividing the interval [0, 1] into 10 parts by Simpson's rule, we get five significant decimals; by the trapezoidal rule, only three; and by the rectangular formula, we are sure only of the first decimal.

SEC. 9. CHEBYSHEV'S FORMULA

In engineering computations, use is frequently made of Chebyshev's formula of approximate integration.

Once again, let it be required to compute $\int_a^b f(x)\,dx$.

Replace the integrand by the Lagrange interpolation polynomial $P(x)$ (Sec. 9, Ch. VII) and take certain n values of the function on the interval $[a, b]: f(x_1), f(x_2), \ldots, f(x_n)$ where $x_1, x_2, \ldots, x_n$ are any points of the interval $[a, b]$:

$$P(x)=\frac{(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_1-x_3)\ldots(x_1-x_n)}f(x_1)+$$
$$+\frac{(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}f(x_2)+$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$+\frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}f(x_n). \quad (1)$$

We get the following approximate formula of integration:

$$\int_a^b f(x)\,dx\approx\int_a^b P(x)\,dx; \quad (2)$$

after some computation it takes the form

$$\int_a^b f(x)\,dx\approx C_1 f(x_1)+C_2 f(x_2)+\ldots+C_n f(x_n), \quad (3)$$

where the coefficients C_i are calculated by the formulas

$$C_i=\int_a^b \frac{(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}\,dx. \quad (4)$$

Formula (3) is cumbersome and inconvenient for computation because the coefficients C_i are expressed by complex fractions.

Chebyshev posed the inverse problem: specify not the abscissas $x_1, x_2, \ldots, x_n$ but the coefficients $C_1, C_2, \ldots, C_n$ and determine the abscissas $x_1, x_2, \ldots, x_n$.

The coefficients C_i are specified so that formula (3) should be as simple as possible for computation. This will obviously occur when all the coefficients C_i are equal:

$$C_1 = C_2 = \dots = C_n.$$

If we denote the total value of the coefficients $C_1, C_2 \dots, C_n$ by C_n, formula (3) will take the form

$$\int_a^b f(x)\,dx \approx C_n[f(x_1) + f(x_2) + \dots + f(x_n)]. \quad (5)$$

Formula (5) is, generally speaking, an **approximate** equality, but if $f(x)$ is a polynomial of degree not higher than $n-1$, then the equality will be **exact**. This circumstance is what permits determining the quantities $C_n, x_1, x_2, \dots, x_n$.

To obtain a formula that is convenient for any interval of integration, let us transform the interval of integration $[a, b]$ into the interval $[-1, 1]$. To do this, put

$$x = \frac{a+b}{2} + \frac{b-a}{2} t;$$

then for $t = -1$ we will have $x = a$, for $t = 1$, $x = b$.

Hence,

$$\int_a^b f(x)\,dx = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2} t\right) dt = \frac{b-a}{2} \int_{-1}^{1} \varphi(t)\,dt,$$

where $\varphi(t)$ denotes the function of t under the integral sign. Thus, the problem of integrating the given function $f(x)$ on the interval $[a, b]$ can always be reduced to integrating some other function $\varphi(x)$ on the interval $[-1, 1]$.

To summarise, then, the problem has reduced to choosing, in the formula

$$\int_{-1}^{1} f(x)\,dx = C_n[f(x_1) + f(x_2) + \dots + f(x_n)], \quad (6)$$

the numbers $C_n, x_1, x_2, \dots, x_n$ so that this formula will be exact for any function $f(x)$ of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}. \quad (7)$$

It will be noted that

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (a_0 + a_1x + a_2x^2 \dots + a_{n-1}x^{n-1})\,dx =$$

$$= \begin{cases} 2\left(a_0 + \frac{a_2}{3} + \frac{a_4}{5} + \frac{a_6}{7} + \dots + \frac{a_{n-1}}{n}\right), \text{ if } n \text{ is odd;} \\ 2\left(a_0 + \frac{a_2}{3} + \dots + \frac{a_{n-2}}{n-1}\right), \text{ if } n \text{ is even.} \end{cases} \quad (8)$$

On the other hand, the sum on the right side of (6) will, on the basis of (7), be equal to

$$C_n [na_0 + a_1 (x_1 + x_2 + \dots + x_n) + a_2 (x_1^2 + x_2^2 + \dots + x_n^2) + \dots$$
$$\dots + a_{n-1} (x_1^{n-1} + x_2^{n-1} + \dots + x_n^{n-1})]. \quad (9)$$

Equating expressions (8) and (9), we get an equation that should hold for all $a_0, a_1, a_2, \dots, a_{n-1}$:

$$2\left(a_0 + \frac{a_2}{3} + \frac{a_4}{5} + \frac{a_6}{7} + \dots\right) =$$
$$= C_n [na_0 + a_1 (x_1 + x_2 + \dots + x_n) +$$
$$+ a_2 (x_1^2 + x_2^2 + \dots + x_n^2) + \dots a_{n-1} (x_1^{n-1} + x_2^{n-1} + \dots + x_n^{n-1})].$$

Equate the coefficients of $a_0, a_1, a_2, a_3, \dots, a_{n-1}$ on the left and right sides of the equation:

$$\left.\begin{array}{l} 2 = C_n n \text{ or } C_n = \frac{2}{n}; \\ x_1 + x_2 + \dots + x_n = 0; \\ x_1^2 + x_2^2 + \dots + x_n^2 = \frac{2}{3C_n} = \frac{n}{3}; \\ x_1^3 + x_2^3 + \dots + x_n^3 = 0; \\ x_1^4 + x_2^4 + \dots + x_n^4 = \frac{2}{5C_n} = \frac{n}{5}; \\ \dots\dots\dots\dots\dots\dots \end{array}\right\} \quad (10)$$

From the latter n equations we find the abscissas $x_1, x_2, \dots, x_n$. These solutions were found by Chebyshev for various values of n.

The following solutions are those that he found for cases when the number of intermediate points n is equal to 3, 4, 5, 6, 7, 9:

Number of ordinates n	Coefficient C_n	Values of abscissas $x_1, x_2, \ldots, x_n$
3	$\frac{2}{3}$	$x_1 = -x_3 = 0.707107$ $x_2 = 0$
4	$\frac{1}{2}$	$x_1 = -x_4 = 0.794654$ $x_2 = -x_3 = 0.187592$
5	$\frac{2}{5}$	$x_1 = -x_5 = 0.832498$ $x_2 = -x_4 = 0.374541$ $x_3 = 0$
6	$\frac{1}{3}$	$x_1 = -x_6 = 0.866247$ $x_2 = -x_5 = 0.422519$ $x_3 = -x_4 = 0.266635$
7	$\frac{2}{7}$	$x_1 = -x_7 = 0.883862$ $x_2 = -x_6 = 0.529657$ $x_3 = -x_5 = 0.323912$ $x_4 = 0$
9	$\frac{2}{9}$	$x_1 = -x_9 = 0.911589$ $x_2 = -x_8 = 0.601019$ $x_3 = -x_7 = 0.528762$ $x_4 = -x_6 = 0.167906$ $x_5 = 0$

Thus, on the interval $[-1, 1]$, an integral can be approximated by the following Chebyshev formula:

$$\int_{-1}^{1} f(x)\,dx = \frac{2}{n}[f(x_1) + f(x_2) + \ldots + f(x_n)],$$

where n is one of the numbers 3, 4, 5, 6, 7 or 9, and $x_1, \ldots, x_n$ are the numbers given in the table. Here, n cannot be 8 or any number exceeding 9; for then the system of equations (10) yields imaginary roots.

When the given integral has limits of integration a and b, the Chebyshev formula takes on the form

$$\int_a^b f(x)\,dx = \frac{b-a}{n}\,[f(X_1)+f(X_2)+\ldots+f(X_n)],$$

where $X_i = \frac{b+a}{2} + \frac{b-a}{2} x_i$ $(i = 1, 2, \ldots, n)$ and x_i have the values given in the table.

The following example illustrates the use of Chebyshev's approximation formula for calculating an integral.

Example. Evaluate $\int_1^2 \frac{dx}{x}\ (=\ln 2)$.

Solution. First, by changing variables, transform this integral into a new one with limits of integration -1 and 1:

$$x = \frac{1+2}{2} + \frac{2-1}{2}\,t = \frac{3}{2} + \frac{t}{2} = \frac{3+t}{2},$$

$$dx = \frac{dt}{2}.$$

Then

$$\int_1^2 \frac{dx}{x} = \int_{-1}^{1} \frac{dt}{3+t}.$$

Compute the latter integral, taking $n=3$, by Chebyshev's formula:

$$\int_{-1}^{1} f(t)\,dt = \frac{2}{3}\,[f(0.707107)+f(0)+f(-0.707107)].$$

Since

$$f(0.707107) = \frac{1}{3+0.707107} = \frac{1}{3.707107} = 0.269752,$$

$$f(0) = \frac{1}{3+0} = 0.333333,$$

$$f(-0.707107) = \frac{1}{3-0.707107} = \frac{1}{2.292893} = 0.436130,$$

we have

$$\int_{-1}^{1} \frac{dt}{3+t} = \frac{2}{3}\,(0.269752+0.333333+0.436130) =$$

$$= \frac{2}{3}\cdot 1.039215 = 0.692810 \approx 0.693.$$

Comparing this result with the results of computation using the rectangular formulas, the trapezoidal rule, and Simpson's rule (see the example in the preceding section), we note that the result given by Chebyshev's formula (with three intermediate points) is in better agreement with the true value of the integral than the result obtained by the trapezoidal rule (with nine intermediate points).

The theory of approximating integrals was further developed in the works of Academician A. N. Krylov (1863-1945).

SEC. 10. INTEGRALS DEPENDENT ON A PARAMETER

Differentiating integrals dependent on a parameter. Let there be an integral

$$I(\alpha)=\int_a^b f(x, \alpha)\,dx, \tag{1}$$

in which the integrand is dependent upon some parameter α. If the parameter α varies, then the value of the definite integral will also vary. And the definite integral is a **function** of α; we can therefore denote it by $I(\alpha)$.

1. Suppose that $f(x, \alpha)$ and $f'_\alpha(x, \alpha)$ are continuous functions when

$$c \leqslant \alpha \leqslant d \text{ and } a \leqslant x \leqslant b. \tag{2}$$

Find the derivative of the integral with respect to the parameter α:

$$\lim_{\Delta\alpha \to 0} \frac{I(\alpha+\Delta\alpha)-I(\alpha)}{\Delta\alpha}=I'_\alpha(\alpha).$$

In finding this derivative we note that

$$I(\alpha+\Delta\alpha)=\int_a^b f(x, \alpha+\Delta\alpha)\,dx$$

and, consequently,

$$I(\alpha+\Delta\alpha)-I(\alpha)=\int_a^b f(x, \alpha+\Delta\alpha)\,dx-\int_a^b f(x, \alpha)\,dx=$$

$$=\int_a^b [f(x, \alpha+\Delta\alpha)-f(x, \alpha)]\,dx;$$

$$\frac{I(\alpha+\Delta\alpha)-I(\alpha)}{\Delta\alpha}=\int_a^b \frac{f(x, \alpha+\Delta\alpha)-f(x, \alpha)}{\Delta\alpha}\,dx.$$

Applying the Lagrange theorem to the integrand we have

$$\frac{f(x, \alpha+\Delta\alpha)-f(x, \alpha)}{\Delta\alpha}=f'_\alpha(x, \alpha+\theta\Delta\alpha),$$

where $0<\theta<1$.

Since $f'_\alpha(x, \alpha)$ is continuous in the closed domain (2), we have

$$f'_\alpha(x, \alpha+\theta\Delta\alpha)=f'_\alpha(x, \alpha)+\varepsilon,$$

where the quantity ε, which depends on x, α, $\Delta\alpha$, approaches zero as $\Delta\alpha \to 0$.

Thus,

$$\frac{I(\alpha+\Delta\alpha)-I(\alpha)}{\Delta\alpha}=\int_a^b [f'_\alpha(x, \alpha)+\varepsilon]\,dx=\int_a^b f'_\alpha(x, \alpha)\,dx+\int_a^b \varepsilon\,dx.$$

Passing to the limit as $\Delta\alpha \to 0$, we have*)

$$\lim_{\Delta\alpha\to 0}\frac{I(\alpha+\Delta\alpha)-I(\alpha)}{\Delta\alpha}=I'_\alpha(\alpha)=\int_a^b f'_\alpha(x, \alpha)\,dx$$

or

$$\left[\int_a^b f(x, \alpha)\,dx\right]'_\alpha=\int_a^b f'_\alpha(x, \alpha)\,dx.$$

This formula is called the *Leibniz formula*.

2. Now suppose that in the integral (1) the *limits of integration a and b are functions of* α:

$$I(\alpha)=\Phi[\alpha, a(\alpha), b(\alpha)]=\int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)\,dx. \qquad (1')$$

$\Phi[\alpha, a(\alpha), b(\alpha)]$ is a composite function of α, and a and b are intermediate arguments. To find the derivative of $I(\alpha)$, apply the rule for differentiating a composite function of several variables (see Sec. 10, Ch. VIII):

$$I'(\alpha)=\frac{\partial\Phi}{\partial\alpha}+\frac{\partial\Phi}{\partial a}\frac{da}{d\alpha}+\frac{\partial\Phi}{\partial b}\frac{db}{d\alpha}. \qquad (3)$$

---

*) The integrand in the integral $\int_a^b \varepsilon\,d\alpha$ approaches zero as $\Delta\alpha \to 0$. From the fact that the integrand approaches zero it does not always follow that the integral also approaches zero. However, in the given case, $\int_a^b \varepsilon\,dx$ approaches zero as $\Delta\alpha \to 0$. We accept this fact without proof.

By the theorem for the differentiation of a definite integral with respect to the variable upper limit [see formula (1), Sec. 5] we get

$$\frac{\partial \Phi}{\partial b}=\frac{\partial}{\partial b}\int_a^b f(x, \alpha)\,dx=f[b(\alpha), \alpha],$$

$$\frac{\partial \Phi}{\partial a}=\frac{\partial}{\partial a}\int_b^b f(x, \alpha)\,dx=-\frac{\partial}{\partial a}\int_b^a f(x, \alpha)\,dx=-f[a(\alpha), \alpha].$$

Finally, to evaluate $\frac{\partial \Phi}{\partial \alpha}$ use the above-derived Leibniz formula:

$$\frac{\partial \Phi}{\partial \alpha}=\int_a^b f'_\alpha(x, \alpha)\,dx.$$

Substituting into (3) the expressions obtained for the derivatives, we have

$$I'_\alpha(\alpha)=\int_{a(\alpha)}^{b(\alpha)} f'_\alpha(x, a)\,dx+f[b(\alpha), \alpha]\frac{db}{d\alpha}-f[a(\alpha), \alpha]\frac{da}{d\alpha}. \tag{4}$$

Using the Leibniz formula it is possible to compute some definite integrals.

Example. Evaluate the integral

$$\int_0^\infty e^{-x}\frac{\sin \alpha x}{x}\,dx.$$

Solution. First note that it is impossible to compute the integral directly, because the antiderivative of the function $e^{-x}\frac{\sin \alpha x}{x}$ is not expressible in terms of elementary functions. To compute this integral we shall consider it as a function of the parameter α:

$$I(\alpha)=\int_0^\infty e^{-x}\frac{\sin \alpha x}{x}\,dx.$$

Then its derivative with respect to α is found from the above-derived Leibniz formula *):

$$I'(\alpha)=\int_0^\infty \left[e^{-x}\frac{\sin \alpha x}{x}\right]'_\alpha dx=\int_0^\infty e^{-x}\cos \alpha x\,dx.$$

*) Leibniz' formula was derived on the assumption that the limits of integration a and b are finite. However, in this case Leibniz' formula also holds, even though one of the limits of integration is equal to infinity.

But the latter integral is readily evaluated by means of elementary functions; it is equal to $\frac{1}{1+\alpha^2}$. Therefore,

$$I'(\alpha)=\frac{1}{1+\alpha^2}.$$

Integrating the identity obtained, we find $I(\alpha)$:

$$I(\alpha)=\operatorname{arc}\,\tan\alpha+C. \qquad (5)$$

We have C to determine now. To do this, we note that

$$I(0)=\int_0^{\infty} e^{-x}\frac{\sin 0\cdot x}{x}\,dx=\int_0^{\infty} 0\,dx=0.$$

What is more, $\operatorname{arc}\,\tan 0=0$.

Substituting into (5) $\alpha=0$, we get

$$I(0)=\operatorname{arc}\,\tan 0+C,$$

whence $C=0$. Hence, for any value of α we have the equality

$$I(\alpha)=\operatorname{arc}\,\tan\alpha;$$

that is,

$$\int_0^{\infty} e^{-x}\frac{\sin\alpha x}{x}\,dx=\operatorname{arc}\,\tan\alpha.$$

Exercises on Chapter XI

1. Forming the integral sum s_n and passing to the limit, compute the definite integrals

$$\int_0^b x^2\,dx.$$

Hint. Divide the interval $[a, b]$ into n parts by the points $x_i=aq^i$ ($i=0, 1, 2, \ldots, n$), where $q=\sqrt[n]{\frac{b}{a}}$. *Ans.* $\frac{b^3-a^3}{3}$.

2. $\int_a^b \frac{dx}{x}$ where $0<a<b$. *Ans.* $\ln\frac{b}{a}$.

Hint. Divide the interval $[a, b]$ in the same way as in the preceding example.

3. $\int_a^b \sqrt{x}\,dx$. *Ans.* $\frac{2}{3}(b^{3/2}-a^{3/2})$.

Hint. See Example 2.

4. $\int_a^b \sin x\,dx$. *Ans.* $\cos a-\cos b$.

Hint. First establish the following identity:

$$\sin a+\sin(a+h)+\sin(a+2h)+\ldots+\sin[a+(n-1)h]=$$
$$=\frac{\cos\left(a+\frac{h}{2}\right)-\cos\left[a+nh-\frac{h}{2}\right]}{2\sin\frac{h}{2}}.$$

To do this, multiply and divide all the terms of the left side by $\sin\frac{h}{2}$ and replace the product of sines by the difference of cosines.

5. $\int_a^b \cos x\,dx$. *Ans.* $\sin b-\sin a$.

Using the Newton-Leibniz formula, compute the definite integrals:

6. $\int_0^1 x^4\,dx$. *Ans.* $\frac{1}{5}$. 7. $\int_0^1 e^x\,dx$. *Ans.* $e-1$. 8. $\int_0^{\frac{\pi}{2}} \sin x\,dx$. *Ans.* 1.

9. $\int_0^1 \frac{dx}{1+x^2}$. *ns.* $\frac{\pi}{4}$. 10. $\int_0^{\frac{\sqrt{2}}{2}} \frac{dx}{\sqrt{1-x^2}}$. *Ans.* $\frac{\pi}{4}$. 11. $\int_0^{\frac{\pi}{3}} \tan x\,dx$. *Ans.* $\ln 2$.

12. $\int_1^e \frac{dx}{x}$. *Ans.* 1. 13. $\int_1^x \frac{dx}{x}$. *Ans.* $\ln x$. 14. $\int_0^x \sin x\,dx$. *Ans.* $2\sin^2\frac{x}{2}$. 15. $\int_{\sqrt[3]{a}}^x x^2dx$. *Ans.* $\frac{x^3-a}{3}$. 16. $\int_1^z \frac{dx}{2x-1}$. *Ans.* $\ln(2z-1)$. 17. $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$. *Ans.* $\frac{\pi}{4}$.

18. $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$. *Ans.* $\frac{\pi}{4}$.

Evaluate the following integrals applying the indicated substitutions:

19. $\int_0^{\frac{\pi}{2}} \sin x\cos^2 x\,dx$, $\cos x=t$. *Ans.* $\frac{1}{3}$. 20. $\int_0^{\pi} \frac{dx}{3+2\cos x}$, $\tan\frac{x}{2}=t$. *Ans.* $\frac{\pi}{\sqrt{5}}$.

21. $\int_1^4 \frac{x\,dx}{\sqrt{2+4x}}$, $2+4x=t^2$. *Ans.* $\frac{3\sqrt{2}}{2}$. 22. $\int_{-1}^1 \frac{dx}{(1+x^2)^2}$, $x=\tan t$.

Ans. $\frac{\pi}{4}+\frac{1}{2}$. 23. $\int_1^5 \frac{\sqrt{x-1}}{x}\,dx$, $x-1=t^2$. *Ans.* $2(2-\arctan 2)$.

24. $\int_{\frac{3}{4}}^{\frac{4}{3}} \frac{dz}{z\sqrt{z^2+1}}$, $z=\frac{1}{x}$. *Ans.* $\ln\frac{3}{2}$. 25. $\int_0^{\frac{\pi}{2}} \frac{\cos\varphi\,d\varphi}{6-5\sin\varphi+\sin^2\varphi}$, $\sin\varphi=t$. Ans $\ln\frac{4}{3}$.

Prove that 26. $\int_0^1 x^m(1-x)^n\,dx=\int_0^1 x^n(1-x)^m\,dx\ (m>0,\quad n>0)$.

27. $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$. 28. $\int_0^a f(x^2)\,dx=\frac{1}{2}\int_{-a}^a f(x^2)\,dx$.

Evaluate the following improper integrals: 29. $\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}}$. *Ans.* 1.

30. $\int_0^\infty e^{-x}\,dx$. *Ans.* 1. 31. $\int_0^\infty \frac{dx}{a^2+x^2}$. *Ans.* $\frac{\pi}{2a}\,(a>0)$. 32. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$. *Ans.* $\frac{\pi}{2}$

33. $\int_1^\infty \frac{dx}{x^5}$. *Ans.* $\frac{1}{4}$. 34. $\int_0^1 \ln x\,dx$. *Ans.* -1. 35. $\int_0^\infty x\sin x\,dx$. Ans. The integral diverges. 36. $\int_1^\infty \frac{dx}{\sqrt{x}}$ Ans. The integral diverges. 37. $\int_{-\infty}^{+\infty} \frac{dx}{x^2+2x+2}$. *Ans.* π.

38. $\int_0^1 \frac{dx}{\sqrt[3]{x}}$. *Ans.* $\frac{3}{2}$. 39. $\int_0^2 \frac{dx}{x^3}$. *Ans.* The integral diverges. 40. $\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$.

Ans. $\frac{\pi}{2}$. 41. $\int_{-1}^1 \frac{dx}{x^4}$. *Ans.* The integral diverges. 42. $\int_0^\infty e^{-ax}\sin bx\,dx\ (a>0)$.

Ans. $\frac{b}{a^2-b^2}$. 43. $\int_0^\infty e^{-ax}\cos bx\,dx\ (a>0)$. *Ans.* $\frac{a}{a^2+b^2}$.

Evaluate the following integrals approximately: 44. $\ln 5=\int_1^5 \frac{dx}{x}$ by the trapezoidal rule and by Simpson's rule ($n=12$). Ans. 1.6182 (by the trapezoidal rule); 1.6098 (by Simpson's rule). 45. $\int_1^{11} x^3\,dx$ by the trapezoidal rule and by

Simpson's rule ($n=10$). *Ans.* 3690; 3660. **46.** $\int\limits_0^1 \sqrt{1-x^3}\,dx$ by the trapezoidal rule ($n=6$). *Ans.* 0.8109. **47.** $\int\limits_0^3 \frac{dx}{2x-1}$ by Simpson's rule ($n=4$). *Ans.* 0.8111. **48.** $\int\limits_1^{10} \log_{10} x\,dx$ by the trapezoidal rule and by Simpson's rule ($n=10$). *Ans.* 6.0656; 6.0896. **49.** Evaluate π from the relation $\frac{\pi}{4}=\int\limits_0^1 \frac{dx}{1+x^2}$ applying Simpson's rule ($n=10$). *Ans.* 3.14159. **50.** $\int\limits_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx$ by Simpson's rule ($n=10$). *Ans.* 1.371. **51.** Evaluate $\int\limits_0^{\infty} e^{-x} x^n\,dx$ for integral $n>0$ by proceeding from the equality $\int\limits_0^{\infty} e^{-\alpha x}\,dx=\frac{1}{\alpha}$ where $\alpha>0$. *Ans.* $n!$ **52.** Proceeding from the equality $\int\limits_0^{\infty} \frac{dx}{x^2+a}=\frac{\pi}{2\sqrt{a}}$, evaluate the integral $\int\limits_0^{\infty} \frac{dx}{(x^2+1)^{n+1}}$. *Ans.* $\frac{\pi}{2}\,\frac{1\cdot3\cdot5\ldots(2n-1)}{2^n\,n!}$. **53.** Evaluate the integral $\int\limits_0^{\infty} \frac{1-e^{-\alpha x}}{xe^x}\,dx$. *Ans.* $\ln(1+\alpha)$ $(\alpha>-1)$. **54.** Utilising the equality $\int\limits_0^1 x^{n-1}\,dx=\frac{1}{n}$, compute the integral $\int\limits_0^1 x^{n-1}(\ln x)^k dx$. *Ans.* $(-1)^k\frac{k!}{n^{k+1}}$.

CHAPTER XII

GEOMETRIC AND MECHANICAL APPLICATIONS OF THE DEFINITE INTEGRAL

SEC. 1. COMPUTING AREAS IN RECTANGULAR COORDINATES

If on the interval $[a, b]$ the function $f(x) \geqslant 0$, then, as we know from Sec. 2, Ch. XI, the area of a curvilinear trapezoid bounded by the curve $y = f(x)$, the x-axis, and the straight lines $x = a$ and $x = b$ (Fig. 210) is

$$Q = \int_a^b f(x)\,dx. \tag{1}$$

If $f(x) \leqslant 0$ on $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ is also $\leqslant 0$. It is equal, in absolute value, to the area Q corresponding to the curvilinear trapezoid:

$$-Q = \int_a^b f(x)\,dx.$$

If $f(x)$ changes sign on the interval $[a, b]$ a finite number of times, then we break up the integral throughout $[a, b]$ into the sum of integrals of the subintervals. The integral will be positive on those subintervals where $f(x) \geqslant 0$, and negative where $f(x) \leqslant 0$. The integral over the entire interval will yield the difference of the areas above and below the x-axis (Fig. 228). To find the sum of the areas in the ordinary sense, one has to find the sum of the absolute values of the integrals over the above-indicated subintervals or compute the integral

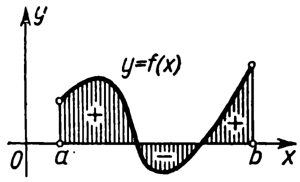


Fig. 228.

$$Q = \int_a^b |f(x)|\,dx.$$

Example 1. Compute the area Q bounded by the sine curve $y = \sin x$ and the x-axis, for $0 \leqslant x \leqslant 2\pi$ (Fig. 229).

Solution. Since $\sin x \geqslant 0$ when $0 \leqslant x \leqslant \pi$ and $\sin x \leqslant 0$ when $\pi < x \leqslant 2\pi$, we have

$$Q = \int_0^{\pi} \sin x\, dx + \left| \int_{\pi}^{2\pi} \sin x\, dx \right| = \int_0^{2\pi} |\sin x|\, dx,$$

$$\int_0^{\pi} \sin x\, dx = -\cos x \Big|_0^{\pi} = -(\cos \pi - \cos 0) = -(-1-1) = 2,$$

$$\int_{\pi}^{2\pi} \sin x\, dx = -\cos x \Big|_{\pi}^{2\pi} = -(\cos 2\pi - \cos \pi) = -2.$$

Consequently, $Q = 2 + |-2| = 4$.

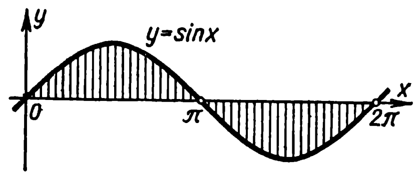


Fig. 229.

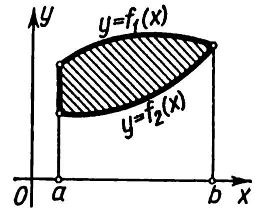


Fig. 230.

If one needs to compute the area bounded by the curves $y = f_1(x)$, $y = f_2(x)$ and the ordinates $x = a$, $x = b$, then provided $f_1(x) \geqslant f_2(x)$ we will obviously have (Fig. 230)

$$Q = \int_a^b f_1(x)\, dx - \int_a^b f_2(x)\, dx = \int_a^b [f_1(x) - f_2(x)]\, dx. \tag{2}$$

Example 2. Compute the area bounded by the curves (Fig. 231)

$$y = \sqrt{x} \quad \text{and} \quad y = x^2.$$

Solution. Find the points of intersection of the curves: $\sqrt{x} = x^2$, $x = x^4$, whence $x_1 = 0$, $x_2 = 1$.
Therefore,

$$Q = \int_0^1 \sqrt{x}\, dx - \int_0^1 x^2\, dx = \int_0^1 (\sqrt{x} - x^2)\, dx = \frac{2}{3} x^{3/2} \Big|_0^1 - \frac{x^3}{3} \Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

Now let us compute the area of the curvilinear trapezoid bounded by a curve represented by equations in parametric form (Fig. 232):

$$x = \varphi(t) \quad y = \psi(t), \tag{3}$$

where

$$\alpha \leqslant t \leqslant \beta$$

and

$$\varphi(\alpha)=a, \quad \varphi(\beta)=b.$$

Let equations (3) define some function $y=f(x)$ on the interval $[a, b]$ and, consequently, the area of the curvilinear trapezoid may be computed from the formula

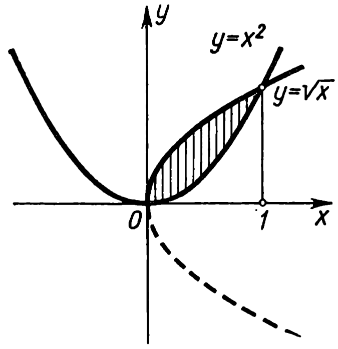


Fig. 231.

$$Q=\int_a^b f(x)\,dx=\int_a^b y\,dx.$$

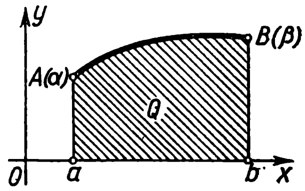


Fig. 232.

Change the variable in this integral:

$$x=\varphi(t); \quad dx=\varphi'(t)\,dt.$$

From (3) we have

$$y=f(x)=f[\varphi(t)]=\psi(t).$$

Consequently,

$$Q=\int_\alpha^\beta \psi(t)\,\varphi'(t)\,dt. \tag{4}$$

This is the formula for computing the area of a curvilinear trapezoid bounded by a curve represented parametrically.

Example 3. Compute the area of a region bounded by the ellipse

$$x=a\cos t, \quad y=b\sin t.$$

Solution. Compute the area of the upper half of the ellipse and double it. Here, x varies from $-a$ to $+a$, and so t varies between π and 0.

$$Q=2\int_\pi^0 (b\sin t)(-a\sin t\,dt)=-2ab\int_\pi^0 \sin^2 t\,dt=2ab\int_0^\pi \sin^2 t\,dt=$$

$$=2ab\int_0^\pi \frac{1-\cos 2t}{2}\,dt=2ab\left[\frac{t}{2}-\frac{\sin 2t}{4}\right]_0^\pi=\pi ab.$$

Example 4. Compute the area bounded by the x-axis and an arc of the cycloid

$$x=a(t-\sin t), \quad y=a(1-\cos t)$$

Solution. The variation of x from 0 to $2\pi a$ corresponds to the variation of t from 0 to 2π.

From (4) we have

$$Q=\int_0^{2\pi} a(1-\cos t)\,a(1-\cos t)\,dt=a^2\int_0^{2\pi}(1-\cos t)^2\,dt=$$

$$=a^2\left[\int_0^{2\pi} dt-2\int_0^{2\pi}\cos t\,dt+\int_0^{2\pi}\cos^2 t\,dt\right];$$

$$\int_0^{2\pi} dt=2\pi;\quad \int_0^{2\pi}\cos t\,dt=0;\quad \int_0^{2\pi}\cos^2 t\,dt=\int_0^{2\pi}\frac{1+\cos 2t}{2}\,dt=\pi.$$

We finally get

$$Q=a^2(2\pi+\pi)=3\pi a^2.$$

SEC. 2. THE AREA OF A CURVILINEAR SECTOR IN POLAR COORDINATES

Suppose in a polar coordinate system we have a curve given by the equation

$$\varrho=f(\theta),$$

where $f(\theta)$ is a continuous function when $\alpha\leqslant\theta\leqslant\beta$.

Let us determine the area of the sector OAB bounded by the curve $\varrho=f(\theta)$ and by the radius vectors $\theta=\alpha$ and $\theta=\beta$.

Divide the given area by radius vectors $\theta_0=\alpha$, $\theta=\theta_1, \ldots, \theta_n=\beta$ into n parts. Denote by $\Delta\theta_1, \Delta\theta_2, \ldots, \Delta\theta_n$ the angles between the radius vectors that we have drawn (Fig. 233).

Denote by $\bar{\varrho}_i$ the length of a radius vector corresponding to some angle $\bar{\theta}_i$ between θ_{i-1} and θ_i.

Let us consider the circular sector with radius $\bar{\varrho}_i$ and central angle $\Delta\theta_i$. Its area will be

$$\Delta Q_i=\frac{1}{2}\bar{\varrho}_i^2\,\Delta\theta_i.$$

The sum

$$Q_n=\frac{1}{2}\sum_{i=1}^{n}\bar{\varrho}_i^2\,\Delta\theta_i=\frac{1}{2}\sum_{i=1}^{n}[f(\bar{\theta}_i)]^2\,\Delta\theta_i$$

will yield the area of the "step-like" sector.

Since this sum is an integral sum of the function $\varrho^2=[f(\theta)]^2$ on the interval $\alpha\leqslant\theta\leqslant\beta$, its limit, as max $\Delta\theta_i\to 0$, is the definite integral

$$\frac{1}{2}\int_{\alpha}^{\beta}\varrho^2\,d\theta.$$

It is not dependent on which radius vector $\bar{\varrho}_i$ we take inside the

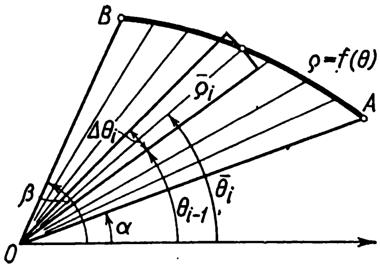


Fig. 233.

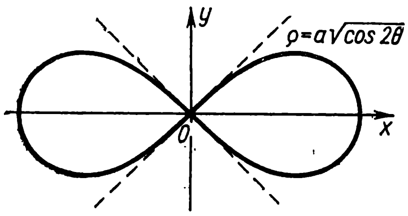


Fig. 234.

angle $\Delta\theta_i$. It is natural to consider this limit the sought-for area of the figure*).

Thus, the area of the sector OAB is

$$Q=\frac{1}{2}\int_{\alpha}^{\beta}\varrho^2 d\theta \qquad (1)$$

or

$$Q=\frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta. \qquad (1')$$

Example. Compute the area bounded by the lemniscate

$$\varrho=a\sqrt{\cos 2\theta}.$$

(Fig. 234).

Solution. The radius vector will describe a fourth of the sought-for area if θ varies between 0 and $\frac{\pi}{4}$:

$$\frac{1}{4}Q=\frac{1}{2}\int_{0}^{\frac{\pi}{4}}\varrho^2\,d\theta=\frac{1}{2}a^2\int_{0}^{\frac{\pi}{4}}\cos 2\theta\,d\theta=\frac{a^2}{2}\,\frac{\sin 2\theta}{2}\Big|_0^{\frac{\pi}{4}}=\frac{a^2}{4}.$$

Hence

$$Q=a^2.$$

*) It might be shown that this determination of the area does not contradict that given earlier. In other words, if one computes the area of a curvilinear sector by means of curvilinear trapezoids, the result will be the same.

SEC. 3. THE ARC LENGTH OF A CURVE

1. The arc length of a curve in rectangular coordinates. Let a curve be given by the equation $y=f(x)$ in rectangular coordinates in a plane.

Let us find the length of the arc AB of this curve between the vertical straight lines $x=a$ and $x=b$ (Fig. 235).

The definition of the length of an arc was given in Chapter VI, Sec. 1. Let us recall that definition. On an arc AB take points A, M_1, M_2, ..., M_i ..., B with abscissas $x_0=a$, x_1, x_2, ..., x_i, ..., $b=x_n$ and draw the chords AM_1, M_1M_2, ..., $M_{n-1}B$ whose lengths we shall denote by ΔS_1, ΔS_2, ..., ΔS_n, respectively. This gives the broken line $AM_1M_2 \ldots M_{n-1}B$ inscribed in the arc $\widehat{AB}$. The length of the broken line is

$$s_n=\sum_{i=1}^{n} \Delta s_i.$$

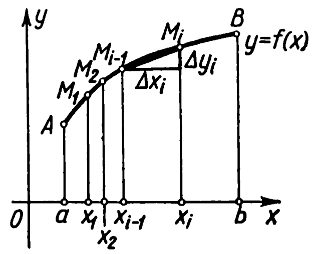


Fig. 235.

The *length, s, of the arc AB* is the limit which the length of the inscribed broken line approaches when the length of its greatest segment approaches zero:

$$s=\lim_{\max \Delta s_i \to 0} \sum_{i=1}^{n} \Delta s_i. \qquad (1)$$

We shall now prove that if on the interval $a \leqslant x \leqslant b$ the function $f(x)$ and its derivative $f'(x)$ are continuous, then this limit exists. At the same time we shall specify a technique for computing the length of the arc.

Let us introduce the notation

$$\Delta y_i=f(x_i)-f(x_{i-1}).$$

Then

$$\Delta s_i=\sqrt{(\Delta x_i)^2+(\Delta y_i)^2}=\sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i.$$

By Lagrange's theorem we have

$$\frac{\Delta y_i}{\Delta x_i}=\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}=f'(\xi_i),$$

where

$$x_{i-1}<\xi_i<x_i.$$

Hence,

$$\Delta s_i = \sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i.$$

Thus, the length of an inscribed broken line is

$$s_n = \sum_{i=1}^{n} \sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i.$$

It is given that $f'(x)$ is continuous; hence, the function $\sqrt{1+[f'(x)]^2}$ is also continuous. Therefore, this integral sum has a limit that is equal to a definite integral:

$$s = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1+[f'(\xi_i)]^2}\,\Delta x_i = \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

We thus have a formula for computing the arc length:

$$s = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx. \tag{2}$$

Note. Using this formula, it is possible to obtain the derivative of the arc length with respect to the abscissa. If we consider the upper limit of integration as variable and denote it by x (we shall not change the variable of integration), then the arc length s will be a function of x:

$$s(x) = \int_a^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Differentiating this integral with respect to the upper limit, we obtain

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}. \tag{3}$$

This formula was derived in Sec. 1, Ch. VI, on certain other assumptions.

Example 1. Determine the circumference of the circle

$$x^2+y^2=r^2.$$

Solution. First compute the length of a fourth part of the circumference lying in the first quadrant. Then the equation of the arc AB will be

$$y=\sqrt{r^2-x^2},$$

whence

$$\frac{dy}{dx} = -\frac{x}{\sqrt{r^2-x^2}}.$$

Consequently,

$$\frac{1}{4}s=\int_0^r\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx=\int_0^r\frac{r}{\sqrt{r^2-x^2}}\,dx=r\arcsin\frac{x}{r}\Big|_0^r=r\frac{\pi}{2}.$$

The length of the circumference is $s=2\pi r$.

Let us now find the arc length of a curve when the equation of the curve is represented in parametric form:

$$x=\varphi(t),\quad y=\psi(t)\quad(\alpha\leqslant t\leqslant\beta),\tag{4}$$

where $\varphi(t)$ and $\psi(t)$ are continuous functions with continuous derivatives, and $\varphi(t)$ does not vanish in the given interval. In this case, equations (4) define a function $y=f(x)$ which is continuous and has a continuous derivative:

$$\frac{dy}{dx}=\frac{\psi'(t)}{\varphi'(t)}.$$

Let $a=\varphi(\alpha)$, $b=\varphi(\beta)$. Then substituting in the integral (2)

$$x=\varphi(t),$$
$$dx=\varphi'(t)\,dt,$$

we have

$$s=\int_\alpha^\beta\sqrt{1+\left[\frac{\psi'(t)}{\varphi'(t)}\right]^2}\,\varphi'(t)\,dt,$$

or, finally,

$$s=\int_\alpha^\beta\sqrt{\varphi'(t)^2+\psi'(t)^2}\,dt.\tag{5}$$

Note 2. It may be proved that formula (5) holds also for curves that are crossed by vertical lines in more than one point (in particular, for closed curves), provided that both derivatives $\varphi'(t)$ and $\psi'(t)$ are continuous at all points of the curve.

Example 2. Compute the length of the hypocycloid (astroid):

$$x=a\cos^3 t,\quad y=a\sin^3 t.$$

Solution. Since the curve is symmetric about both coordinate axes, we shall first compute the length of a fourth part of it located in the first quadrant. We find

$$\frac{dx}{dt}=-3a\cos^2 t\sin t,$$

$$\frac{dy}{dt}=3a\sin^2 t\cos t.$$

The parameter t will vary from 0 to $\frac{\pi}{2}$. Hence

$$\frac{1}{4}s=\int_0^{\frac{\pi}{2}}\sqrt{9a^2\cos^4 t\sin^2 t+9a^2\sin^4 t\cos^2 t}\,dt=3a\int_0^{\frac{\pi}{2}}\sqrt{\cos^2 t\sin^2 t}\,dt=$$

$$=3a\int_0^{\frac{\pi}{2}}\sin t\cos t\,dt=3a\frac{\sin^2 t}{2}\Big|_0^{\frac{\pi}{2}}=\frac{3a}{2};\quad s=6a.$$

Note 3. If a space curve is represented by the parametric equations

$$x=\varphi(t),\quad y=\psi(t),\quad z=\chi(t) \tag{6}$$

where $\alpha\leqslant t\leqslant\beta$ (see Sec. 1, Ch. IX), then the length of its arc is defined (in the same way as for a plane arc) as the limit which the length of an inscribed broken line approaches when the length of the greatest segment approaches zero. If the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are continuous and have continuous derivatives on the interval $[\alpha, \beta]$, then the curve has a definite length (that is, it has the above-mentioned limit) which is computed from the formula

$$s=\int_\alpha^\beta\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2+[\chi'(t)]^2}\,dt. \tag{7}$$

This result we accept without proof.

Example 3. Compute the arc length of the helix

$$x=a\cos t,\quad y=a\sin t,\quad z=amt$$

as t varies from 0 to 2π.

Solution. From the given equations we have

$$dx=-a\sin t\,dt,\quad dy=a\cos t\,dt,\quad dz=am\,dt.$$

Substituting into formula (7), we have

$$s=\int_0^{2\pi}\sqrt{a^2\sin^2 t+a^2\cos^2 t+a^2m^2}\,dt=a\int_0^{2\pi}\sqrt{1+m^2}\,dt=2\pi a\sqrt{1+m^2}.$$

2. The arc length of a curve in polar coordinates. Given (in polar coordinates) the equation of the curve

$$\rho=f(\theta) \tag{8}$$

where ρ is the radius vector and θ is the vectorial (polar) angle.

Let us write the formulas for passing from polar coordinates to Cartesian coordinates:

$$x=\varrho\cos\theta, \quad y=\varrho\sin\theta.$$

If in place of ϱ we put its expression (8) in terms of θ, we get the equations

$$x=f(\theta)\cos\theta, \quad y=f(\theta)\sin\theta.$$

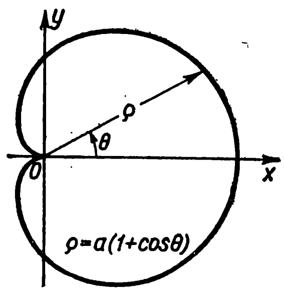


Fig. 236.

These equations may be regarded as the parametric equations of the curve and we can apply formula (5) for computing the arc length. To do this, find the derivatives of x and y with respect to the parameter θ:

$$\frac{dx}{d\theta}=f'(\theta)\cos\theta-f(\theta)\sin\theta;$$

$$\frac{dy}{d\theta}=f'(\theta)\sin\theta+f(\theta)\cos\theta.$$

Then

$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=[f'(\theta)]^2+[f(\theta)]^2=\varrho'^2+\varrho^2.$$

Hence,

$$s=\int_{\theta_0}^{\theta}\sqrt{\varrho'^2+\varrho^2}\,d\theta.$$

Example 4. Find the length of the cardioid

$$\varrho=a(1+\cos\theta)$$

(Fig. 236).

Varying the vectorial angle θ from 0 to π, we get half the sought-for length. Here, $\varrho'=-a\sin\theta$. Hence,

$$s=2\int_0^{\pi}\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,d\theta=$$

$$=2a\int_0^{\pi}\sqrt{2+2\cos\theta}\,d\theta=$$

$$=4a\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta=8a\sin\frac{\theta}{2}\Big|_0^{\pi}=8a.$$

Example 5. Compute the length of the ellipse

$$\left.\begin{aligned} x &= a\cos t, \\ y &= b\sin t, \end{aligned}\right\} \quad 0 \leqslant t \leqslant 2\pi,$$

assuming that $a > b$.

Solution. We take advantage of formula (5), first computing $\frac{1}{4}$ the arc length; that is, the length of the arc that corresponds to a variation of the parameter from $t=0$ to $t=\frac{\pi}{2}$:

$$\frac{s}{4} = \int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt =$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2(1-\cos^2 t) + b^2\cos^2 t}\, dt = \int_0^{\frac{\pi}{2}} \sqrt{a^2-(a^2-b^2)\cos^2 t}\, dt =$$

$$= a\int_0^{\frac{\pi}{2}} \sqrt{1-\frac{a^2-b^2}{a^2}\cos^2 t}\, dt = a\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\cos^2 t}\, dt,$$

where $k = \frac{\sqrt{a^2-b^2}}{a} < 1$. Hence,

$$s = 4a\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\cos^2 t}\, dt.$$

The only thing that remains is to compute the last integral. But we know that it is not expressible by elementary functions (see Sec. 16, Ch. X). This integral can be computed only by approximation methods (by Simpson's rule, for example).

For instance, if the semi-major axis of an ellipse is equal to 5 and the semi-minor axis is 4, then $k=\frac{3}{5}$, and the circumference of the ellipse is

$$s = 4\cdot 5\int_0^{\frac{\pi}{2}} \sqrt{1-\left(\frac{3}{5}\right)^2\cos^2 t}\, dt.$$

Computing this integral by Simpson's rule (by dividing the interval $\left[0, \frac{\pi}{2}\right]$ into four parts) we get an approximate value of the integral:

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\frac{3}{5}\cos^2 t}\, dt \approx 1.298,$$

and so the length of the arc of the entire ellipse, is approximately equal to

$$s \approx 25.96 \text{ units of length.}$$

SEC. 4. COMPUTING THE VOLUME OF A SOLID FROM THE AREAS OF PARALLEL SECTIONS (VOLUMES BY SLICING)

Suppose we have some solid T. Let us assume that we know the area of any section of this solid made by a plane perpendicular to the x-axis (Fig. 237). This area will depend on the position of the cutting plane; that is, it will be a **function of** x:

$$Q = Q(x).$$

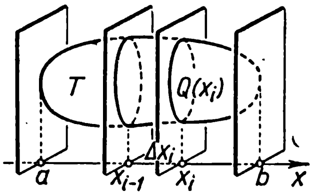


Fig. 237.

We assume that $Q(x)$ is a continuous function of x and calculate the volume of the body.

Draw the planes $x=a$, $x=x_1$, $x=x_2$, $\ldots$, $x=x_n=b$.

These planes will cut the solid up into layers (slices).

In each subinterval $x_{i-1} \leqslant x \leqslant x_i$ we choose an arbitrary point ξ_i and for each value $i=1, 2, \ldots, n$ we construct a cylindrical body, the generatrix of which is parallel to the x-axis, while the directrix is the boundary of the slice of the solid T made by the plane $x=\xi_i$.

The volume of such an elementary cylinder, the area of the base of which is

$$Q(\xi_i) \ (x_{i-1} \leqslant \xi_i \leqslant x_i)$$

and the altitude Δx_i, is

$$Q(\xi_i)\,\Delta x_i.$$

The volume of all the cylinders will be

$$v_n = \sum_{i=1}^{n} Q(\xi_i)\,\Delta x_i.$$

The limit of this sum as $\max \Delta x_i \to 0$ (if it exists) is the volume of the given solid:

$$v = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} Q(\xi_i)\,\Delta x_i.$$

Since v_n is obviously the integral sum of the continuous function $Q(x)$ on the interval $a \leqslant x \leqslant b$, the indicated limit exists and is expressed by the definite integral

$$v = \int_a^b Q(x)\,dx. \tag{1}$$

Example. Compute the volume of the triaxial ellipsoid (Fig. 238).

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

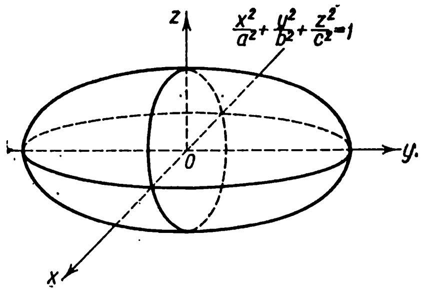


Fig. 238.

Solution. In a section of the ellipsoid made by a plane parallel to the yz-plane and at a distance x from it, we have the ellipse

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{x^2}{a^2}$$

or

$$\frac{y^2}{\left[b\sqrt{1-\frac{x^2}{a^2}}\right]^2}+\frac{z^2}{\left[c\sqrt{1-\frac{x^2}{a^2}}\right]^2}=1$$

with semi-axes

$$b_1=b\sqrt{1-\frac{x^2}{a^2}};\quad c_1=c\sqrt{1-\frac{x^2}{a^2}}.$$

But the area of such an ellipse is $\pi b_1 c_1$ (see Example 3, Sec. 1).

Therefore,

$$Q(x)=\pi bc\left(1-\frac{x^2}{a^2}\right).$$

The volume of the ellipsoid will be

$$v=\pi bc\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)dx=\pi bc\left(x-\frac{x^3}{3a^2}\right)\Big|_{-a}^{a}=\frac{4}{3}\pi abc.$$

In the particular case, $a=b=c$, the ellipsoid turns into a sphere, and we have

$$v=\frac{4}{3}\pi a^3.$$

SEC. 5. THE VOLUME OF A SOLID OF REVOLUTION

Let us consider a solid generated by the revolution, about the x-axis, of a curvilinear trapezoid $aABb$ bounded by the curve $y=f(x)$, the x-axis, and the lines $x=a$, $x=b$.

In this case, an arbitrary section of the solid made by a plane perpendicular to the x-axis is a circle of area

$$Q=\pi y^2=\pi [f(x)]^2.$$

Applying the general formula for computing volume [(1), Sec. 4], we get a formula for calculating the volume of a **solid of revolution:**

$$v=\pi \int_a^b y^2\,dx=\pi \int_a^b [f(x)]^2\,dx.$$

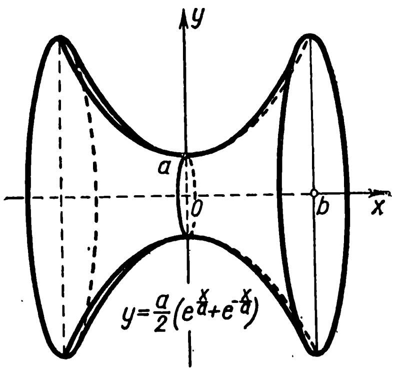


Fig. 239.

Example. Find the volume of a solid generated by the revolution of the catenary

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$$

about the x-axis on the interval from $x=0$ to $x=b$ (Fig. 239).

Solution.

$$v=\pi\frac{a^2}{4}\int_0^b\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2dx=\frac{\pi a^2}{4}\int_0^b\left(e^{\frac{2x}{a}}+2+e^{-\frac{2x}{a}}\right)dx=$$

$$=\frac{\pi a^2}{4}\left[\frac{a}{2}e^{\frac{2x}{a}}+2x-\frac{a}{2}e^{-\frac{2x}{a}}\right]_0^b=\frac{\pi a^3}{8}\left(e^{\frac{2b}{a}}-e^{-\frac{2b}{a}}\right)+\frac{\pi a^2 b}{2}.$$

SEC. 6. THE SURFACE OF A SOLID OF REVOLUTION

Suppose we have a surface generated by the revolution of a curve $y=f(x)$ about the x-axis. Let us determine the area of this surface on the interval $a\leqslant x\leqslant b$. We take the function $f(x)$ to be continuous and to have a continuous derivative at all points of the interval $[a, b]$.

As in Sec. 3, draw the chords AM_1, M_1M_2, ..., $M_{n-1}B$, whose lengths are denoted by Δs_1, Δs_2, ..., Δs_n (Fig. 240).

Each chord of length Δs_i $(i=1, 2, \ldots, n)$ describes (in the process of revolution) a truncated cone whose surface ΔP_i is

$$\Delta P_i = 2\pi \frac{y_{i-1}+y_i}{2} \Delta s_i.$$

But

$$\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x_i.$$

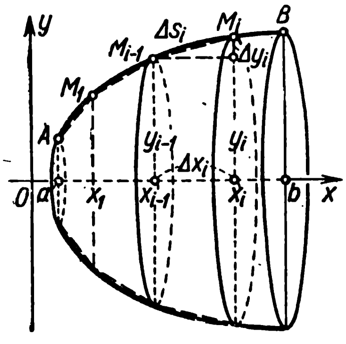


Fig. 240.

Applying Lagrange's theorem, we get

$$\frac{\Delta y_i}{\Delta x_i} = \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} = f'(\xi_i)$$

where

$$x_{i-1} < \xi_i < x_i;$$

hence,

$$\Delta s_i = \sqrt{1+f'^2(\xi_i)}\, \Delta x_i,$$

$$\Delta P_i = 2\pi \frac{y_{i-1}+y_i}{2} \sqrt{1+f'^2(\xi_i)}\, \Delta x_i.$$

The surface described by the broken line will be equal to the sum

$$P_n = 2\pi \sum_{i=1}^{n} \frac{y_{i-1}+y_i}{2} \sqrt{1+f'^2(\xi_i)}\, \Delta x_i$$

or the sum

$$P_n = \pi \sum_{i=1}^{n} [f(x_{i-1}) + \\ + f(x_i)] \sqrt{1+f'^2(\xi_i)}\, \Delta x_i, \qquad (1)$$

extended to all segments of the broken line. The limit of this sum, when the largest segment Δs_i approaches zero is called the area of the surface of revolution under consideration. The sum (1) is not the integral sum of the function

$$2\pi f(x) \sqrt{1+f'(x)^2}, \qquad (2)$$

because the term corresponding to the interval $[x_{i-1}, x_i]$ involves several points of this interval x_{i-1}, x_i, ξ_i. But it is possible to prove that the limit of the sum (1) is equal to the limit of the

integral sum of function (2); that is,

$$P=\lim_{\max \Delta x_i \to 0} \pi \sum_{i=1}^{n} [f(x_{i-1})+f(x_i)] \sqrt{1+f'(\xi_i)^2}\, \Delta x_i =$$

$$=\lim_{\max \Delta x_i \to 0} \pi \sum_{i=1}^{n} 2f(\xi_i) \sqrt{1+f'(\xi_i)^2}\, \Delta x_i$$

or

$$P=2\pi \int_a^b f(x) \sqrt{1+f'^2(x)}\, dx. \tag{3}$$

Example. Determine the surface of a paraboloid generated by revolution about the x-axis of an arc of the parabola $y^2=2px$, which corresponds to the variation of x from $x=0$ to $x=a$:

$$y=\sqrt{2px}, \quad y'=\frac{\sqrt{2p}}{2\sqrt{x}}, \quad \sqrt{1+y'^2}=\sqrt{1+\frac{2p}{4x}}=\sqrt{\frac{2x+p}{2x}}.$$

Solution. By (3) we have

$$P=2\pi \int_0^a \sqrt{2px} \sqrt{\frac{2x+p}{2x}}\, dx=2\pi \sqrt{p} \int_0^a \sqrt{2x+p}\, dx=$$

$$=2\pi \sqrt{p}\, \frac{2}{3}(2x+p)^{3/2} \frac{1}{2} \Big|_0^a = \frac{2\pi \sqrt{p}}{3}[(2a+p)^{3/2}-p^{3/2}].$$

SEC. 7. COMPUTING WORK BY THE DEFINITE INTEGRAL

Suppose a material point M is moving in a straight line Os under a force F, and the direction of the force coincides with the direction of motion. It is required to find the work performed by the force F as the point M is moved from $s=a$ to $s=b$.

1) If the force F is constant, then the work A is expressed by the product of the force F by the path length:

$$A=F(b-a).$$

2) Let us assume that the force F is constantly varying, depending on the position of the material point; that is to say, it is a function $F(s)$ continuous on the interval $a \leqslant s \leqslant b$.

Divide the interval $[a, b]$ into n arbitrary parts of length

$$\Delta s_1, \Delta s_2, \ldots, \Delta s_n,$$

then in each subinterval $[s_{i-1}, s_i]$ choose an arbitrary point ξ_i and replace the work of the force $F(s)$ along the path

Δs_i $(i = 1, 2, \ldots, n)$ by the product

$$F(\xi_i)\,\Delta s_i.$$

This means that within the limits of each subinterval we take the force F to be constant: we assume $F = F(\xi_i)$. Here, the expression $F(\xi_i)\,\Delta s_i$ will yield an approximate value of the work done by the force F over the path Δs_i (for a sufficiently small Δs_i), and the sum

$$A_n = \sum_{i=1}^{n} F(\xi_i)\,\Delta s_i$$

will be the approximate expression of the work of the force F over the interval $[a, b]$.

Obviously, A_n is an integral sum of the function $F = F(s)$ on the interval $[a, b]$. The limit of this sum as $\max(\Delta s_i) \to 0$ exists and expresses the work of the force $F(s)$ over the path from $s = a$ to $s = b$:

$$A = \int_a^b F(s)\,ds. \qquad (1)$$

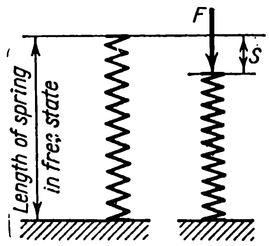


Fig. 241.

Example 1. The compression S of a helical spring is proportional to the applied force F. Compute the work of the force F when the spring is compressed 5 cm, if a force of one kilogram is required to compress it 1 cm (Fig. 241).

Solution. It is given that the force F and the distance covered S are connected by the relation $F = kS$, where k is a constant.

Let us express S in metres and F in kilograms. When $S = 0.01$, $F = 1$, that is, $1 = k\cdot 0.01$, whence $k = 100$, $F = 100S$.

By (1) we have

$$A = \int_0^{0.05} 100S\,dS = 100\frac{S^2}{2}\Big|_0^{0.05} = 0.125 \text{ kilogram-metre.}$$

Example 2. The force F with which an electric charge e_1 repulses another charge e_2 (of the same sign) at a distance of r is expressed by the formula

$$F = k\frac{e_1e_2}{r_2},$$

where k is a constant.

Determine the work done by a force F in moving the charge e_2 from the point A_1 (at a distance of r_1 from e_1) to A_2 (at a distance of r_2 from e_1) assuming that e_1 is located at the point A_0 as the origin.

Solution. From formula (1) we have

$$A = \int_{r_1}^{r_2} k\frac{e_1e_2}{r^2}\,dr = -ke_1e_2\frac{1}{r}\Big|_{r_1}^{r_2} = ke_1e_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

When $r_2=\infty$, we have

$$A=\int\limits_{r_1}^{\infty}\frac{ke_1e_2}{r^2}\,dr=\frac{ke_1e_2}{r_1}.$$

When $e_2=1$, $A=k\frac{e_1}{r}$. This quantity is called the *potential of the field* generated by the charge e_1.

SEC. 8. COORDINATES OF THE CENTRE OF GRAVITY

Suppose on an xy-plane we have a system of material points

$$P_1(x_1,\ y_1);\quad P_2(x_2,\ y_2),\ \ldots,P_n(x_n,\ y_n)$$

with masses $m_1,\ m_2,\ \ldots,\ m_n$.

The products x_im_i and y_im_i are called the *static moments* of the mass m_i relative to the y- and x-axes.

We denote by x_c and y_c the coordinates of the centre of gravity of the given system. Then, as we know from mechanics, the coordinates of the centre of gravity of this material system will be defined by the formulas

$$x_c=\frac{x_1m_1+x_2m_2+\ldots+x_nm_n}{m_1+m_2+\ldots+m_n}=\frac{\sum\limits_{i=1}^{n}x_im_i}{\sum\limits_{i=1}^{n}m_i},\tag{1}$$

$$y_c=\frac{y_1m_1+y_2m_2+\ldots+y_nm_n}{m_1+m_2+\ldots+m_n}=\frac{\sum\limits_{i=1}^{n}y_im_i}{\sum\limits_{i=1}^{n}m_i}.\tag{2}$$

We shall use these formulas in finding the centres of gravity of various figures and solids.

1. The centre of gravity of a plane line. Let there be a curve AB given by the equation $y=f(x)$, $a\leqslant x\leqslant b$, and let this curve be a **material** line.

Let the linear density*) of such a material curve be γ. Divide the line into n parts of length $\Delta s_1,\ \Delta s_2,\ \ldots,\ \Delta s_n$. The masses of these parts will be equal to the product of their lengths by the (constant) density: $\Delta m_i=\gamma\Delta s_i$. On each part of the arc Δs_i take

*) Linear density is the mass of unit length of a given line. We assume that the linear density is the same in all portions of the curve.

an arbitrary point with abscissa ξ_i. Now representing each part of the arc Δs_i by the material point $p_i[\xi_i, f(\xi_i)]$ with mass $\gamma\Delta s_i$ and substituting into (1) and (2) ξ_i in place of x_i, $f(\xi_i)$ in place of y_i, and the value of $\gamma\Delta s_i$ (the mass of the parts Δs_i) in place of m_i, we obtain approximate formulas for determining the centre of gravity of the arc:

$$x_c \approx \frac{\sum \xi_i \gamma \Delta s_i}{\sum \gamma \Delta s_i}, \quad y_c \approx \frac{\sum f(\xi_i)\gamma\Delta s_i}{\sum \gamma\Delta s_i}.$$

If the function $y=f(x)$ is continuous and has a continuous derivative, the sums in the numerator and denominator of each fraction have, as $\max \Delta s_i \to 0$, limits equal to the limits of the corresponding integral sums. Thus, the coordinates of the centre of gravity of the arc are expressed by definite integrals:

$$x_c = \frac{\int_a^b x\,ds}{\int_a^b ds} = \frac{\int_a^b x\sqrt{1+f'^2(x)}\,dx}{\int_a^b \sqrt{1+f'^2(x)}\,dx}, \tag{1'}$$

$$y_c = \frac{\int_a^b f(x)\,ds}{\int_a^b ds} = \frac{\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx}{\int_a^b \sqrt{1+f'^2(x)}\,dx}. \tag{2'}$$

Example 1. Find the coordinates of the centre of gravity of the semi-circle $x^2+y^2=a^2$ situated above the x-axis.

Solution. Determine the abscissa of the centre of gravity:

$$y=\sqrt{a^2-x^2}, \quad \frac{dy}{dx}=-\frac{x}{\sqrt{a^2-x^2}}, \quad ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\frac{a}{\sqrt{a^2-x^2}}\,dx,$$

$$x_c=\frac{a\int_{-a}^{a}\frac{x\,dx}{\sqrt{a^2-x^2}}}{a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}}=\frac{-a\sqrt{a^2-x^2}\,\Big|_{-a}^{a}}{a\arcsin\frac{x}{a}\Big|_{-a}^{a}}=\frac{0}{\pi a}=0.$$

Find the ordinate of the centre of gravity:

$$y_c = \frac{\int\limits_{-a}^{a} \sqrt{a^2-x^2}\,\frac{a}{\sqrt{a^2-x^2}}\,dx}{\pi a} = \frac{a\int\limits_{-a}^{a} dx}{\pi a} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}.$$

2. The centre of gravity of a plane figure. Given a figure bounded by the lines $y=f_1(x)$, $y=f_2(x)$, $x=a$, $x=b$, which is a **material** plane figure. We consider constant the surface density, which is the mass of unit area of the surface. It is equal to δ for all parts of the figure.

Divide the given figure by straight lines $x=a, x=x_1, \ldots, x=x_n=b$ into strips of width $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$. The mass of each strip will be equal to the product of its area by the density δ. If each strip is replaced by a rectangle (Fig. 242) with base Δx_i

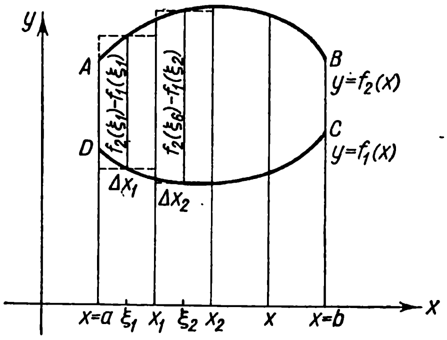


Fig. 242.

and altitude $f_2(\xi_i)-f_1(\xi_i)$, where $\xi_i=\frac{x_{i-1}+x_i}{2}$, then the mass of a strip will be approximately equal to

$$\Delta m_i = \delta\,[f_2(\xi_i)-f_1(\xi_i)]\,\Delta x_i \quad (i=1, 2, \ldots, n).$$

The centre of gravity of this strip will be situated approximately in the centre of the appropriate rectangle:

$$(x_i)_c=\xi_i; \quad (y_i)_c=\frac{f_2(\xi_i)+f_1(\xi_i)}{2}.$$

Now replacing each strip by a material point, whose mass is equal to the mass of the corresponding strip and is concentrated at the centre of gravity of this strip, we find the approximate value of the coordinates of the centre of gravity of the entire figure [by formulas (1) and (2)]:

$$x_c \approx \frac{\sum \xi_i \delta\,[f_2(\xi_i)-f_1(\xi_i)]\,\Delta x_i}{\sum \delta\,[f_2(\xi_i)-f_1(\xi_i)]\,\Delta x_i},$$

$$y_c \approx \frac{\frac{1}{2}\sum [f_2(\xi_i)+f_1(\xi_i)]\,\delta\,[f_2(\xi_i)-f_1(\xi_i)]\,\Delta x_i}{\sum \delta\,[f_2(\varepsilon_i)-f_1(\varepsilon_i)]\,\Delta x_i}.$$

Passing to the limit as $\Delta x_i \to 0$, we obtain the exact coordinates of the centre of gravity of the given figure:

$$x_c = \frac{\int_a^b x\,[f_2(x) - f_1(x)]\,dx}{\int_a^b [f_2(x) - f_1(x)]\,dx}; \quad y_c = \frac{\frac{1}{2}\int_a^b [f_2(x) + f_1(x)]\,[f_2(x) - f_1(x)]\,dx}{\int_a^b [f_2(x) - f_1(x)]\,dx}.$$

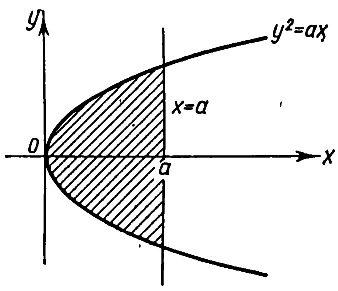


Fig. 243.

These formulas hold for any homogeneous (that is, having constant density at all points) plane figure. We see that the coordinates of the centre of gravity are independent of the density δ of the figure (δ was cancelled out in the process of computation).

Example 2. Determine the coordinates of the centre of gravity of a segment of the parabola $y^2 = ax$ cut off by the straight line $x = a$ (Fig. 243).

Solution. In this case $f_2(x) = \sqrt{ax}$, $f_1(x) = -\sqrt{ax}$; therefore

$$x_c = \frac{2\int_0^a x\sqrt{ax}\,dx}{2\int_0^a \sqrt{ax}\,dx} = \frac{\frac{2}{5}\,2\sqrt{a}x^{5/2}\Big|_0^a}{2\sqrt{a}\,\frac{2}{3}x^{3/2}\Big|_0^a} = \frac{\frac{4}{5}a^3}{\frac{4}{3}a^2} = \frac{3}{5}a,$$

$y_c = 0$ (since the segment is symmetric about the x-axis).

Exercises on Chapter XII
Computing Areas

1. Find the area of a figure bounded by the lines $y^2 = 9x$, $y = 3x$. *Ans.* $\frac{1}{2}$.

2. Find the area of a figure bounded by the equilateral hyperbola $xy = a^2$, the x-axis, and the lines $x = a$, $b = 2a$. *Ans.* $a^2 \ln 2$.

3. Find the area of a figure lying between the curve $y = 4 - x^2$ and the x-axis. *Ans.* $10\frac{2}{3}$.

4. Find the area of a figure bounded by the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\frac{3}{8}\pi a^2$.

5. Find the area of a figure bounded by the catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$, the x-axis, the y-axis, and the straight line $x=a$. *Ans.* $\frac{a^2}{2e}(e^2-1)$.

6. Find the area of a figure bounded by the curve $y=x^3$, the line $y=8$, and the y-axis. *Ans.* 12.

7. Find the area of a region bounded by one loop of a sine wave and the x-axis. *Ans.* 2.

8. Find the area of a region lying between the parabolas $y^2=2px$, $x^2=2py$. *Ans.* $\frac{4}{3}p^2$.

9. Find the total area of a figure bounded by the lines $y=x^3$, $y=2x$, $y=x$. *Ans.* $\frac{3}{2}$.

10. Find the area of a region bounded by one arc of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$ and the x-axis. *Ans.* $3\pi a^2$.

11. Find the area of a figure bounded by the hypocycloid $x=a\cos^3 t$, $y=$ $=a\sin^3 t$. *Ans.* $\frac{3}{8}\pi a^2$.

12. Find the area of the entire region bounded by the lemniscate $\varrho^2=a^2\cos 2\varphi$. *Ans.* a^2.

13. Compute the area of a region bounded by one loop of the curve $\varrho=a\sin 2\varphi$. *Ans.* $\frac{1}{8}\pi a^2$.

14. Compute the total area of a region bounded by the cardioid $\varrho=a(1-\cos\varphi)$. *Ans.* $\frac{3}{2}\pi a^2$.

15. Find the area of the region bounded by the curve $\varrho=a\cos\varphi$. *Ans.* $\frac{\pi a^2}{4}$.

16. Find the area of the region bounded by the curve $\varrho=a\cos 2\varphi$. *Ans.* $\frac{\pi a^2}{2}$.

17. Find the area of the region bounded by the curve $\varrho=\cos 3\varphi$. *Ans.* $\frac{\pi}{4}$.

18. Find the area of the region bounded by the curve $\varrho=a\cos 4\varphi$. *Ans.* $\frac{\pi a^2}{2}$.

Computing Volumes

19. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves about the x-axis. Find the volume of the solid of revolution. *Ans.* $\frac{4}{3}\pi ab^2$.

20. The segment of a line connecting the origin with the point (a, b) revolves about the y-axis. Find the volume of the resulting cone. *Ans.* $\frac{1}{3}\pi a^2 b$.

21. Find the volume of a torus generated by the revolution of the circle $x^2+(y-b)^2=a^2$ about the x-axis (it is assumed that $b\geqslant a$). *Ans.* $2\pi^2a^2b$.

22. The area bounded by the lines $y^2=2px$ and $x=a$ revolves about the x-axis. Find the volume of the solid of revolution. *Ans.* πpa^2.

23. A figure bounded by the hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ is revolved about the x-axis. Find the volume of the solid of revolution. *Ans.* $\frac{32\pi a^3}{105}$.

24. A figure bounded by one arc of the sine wave $y=\sin x$ and the x-axis is revolved about the x-axis, Find the volume of the solid of revolution. *Ans.* $\frac{\pi^2}{2}$.

25. A figure bounded by the parabola $y^2=4x$ and the straight line $x=4$ is revolved about the x-axis. Find the volume of the solid of revolution. *Ans.* 32π.

26. A figure bounded by the curve $y=xe^x$ and the straight lines $y=0, x=1$, is revolved about the x-axis. Find the volume of the solid of revolution. *Ans.* $\frac{\pi}{4}(e^2-1)$.

27. A figure bounded by one arc of a cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$ and the x-axis is revolved about the x-axis. Find the volume of the solid of revolution. *Ans.* $5\pi^2a^3$.

28. The same figure as in Problem 27 is revolved about the y-axis. Find the volume of the solid of revolution. *Ans.* $6\pi^3a^3$.

29. The same figure as in Problem 27 is revolved about a straight line that is parallel to the y-axis and passes through the vertex of a cycloid. Find the volume of the solid of revolution. *Ans.* $\frac{\pi a^3}{6}(9\pi^2-16)$.

30. The same figure as in Problem 27 is revolved about a straight line parallel to the x-axis and passing through the vertex of a cycloid. Find the volume of the solid of revolution. *Ans.* $7\pi^2a^3$.

31. A cylinder of radius R is cut by a plane that passes through the diameter of the base at an angle α to the plane of the base. Find the volume of the cut-off part. *Ans.* $\frac{2}{3}R^3\tan\alpha$.

32. Find a volume that is common to the two cylinders: $x^2+y^2=R^2$, $y^2+z^2=R^2$. *Ans.* $\frac{16}{3}R^3$.

33. The point of intersection of the diagonals of a square is in motion along the diameter of a circle of radius a; the plane in which the square lies remains perpendicular to the plane of the circle, while the two opposite vertices of the square move along the circle (as a result of this motion, the size of the square obviously varies). Find the volume of the solid generated by this moving square. *Ans.* $\frac{8}{3}a^3$.

34. Compute the volume of a segment cut off the elliptical paraboloid $\frac{y^2}{2p}+\frac{z^2}{2q}=x$ by the plane $x=a$. *Ans.* $\pi a^2\sqrt{pq}$.

35. Compute the volume of a solid bounded by the planes $z=0$, $y=0$, the cylindrical surfaces $x^2=2py$ and $z^2=2px$ and the plane $x=a$. *Ans.* $\frac{a^3\sqrt{2a}}{7\sqrt{p}}$ (in first octant).

36. A straight line is in motion parallel to the yz-plane, and cuts two ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$ lying in the xy- and xz-planes. Compute the volume of the solid thus obtained. *Ans.* $\frac{8}{3}abc$.

Computing Arc Lengths

37. Find the entire length of the hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. *Ans.* $6a$.

38. Compute the arc length of the semicubical parabola $ay^2=x^3$ from the origin to a point with abscissa $x=5a$. *Ans.* $\frac{335}{27}a$.

39. Find the arc length of the catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ from the origin to the point (x, y). *Ans.* $\frac{a}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)=\sqrt{y^2-a^2}$.

40. Find the length of one arc of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$. *Ans.* $8a$.

41. Find the length of an arc of the curve $y=\ln x$ within the limits from $x=\sqrt{3}$ to $x=\sqrt{8}$. *Ans.* $1+\frac{1}{2}\ln\frac{3}{2}$.

42. Find the arc length of the curve $y=1-\ln\cos x$ between $x=0$ and $x=\frac{\pi}{4}$. *Ans.* $\ln\tan\frac{3\pi}{8}$.

43. Find the length of the spiral of Archimedes $\varrho=a\varphi$ from the pole to the end of the first loop. *Ans.* $\pi a\sqrt{1+4\pi^2}+\frac{a}{2}\ln(2\pi+\sqrt{1+4\pi^2})$.

44. Find the length of the spiral $\varrho=e^{a\varphi}$ from the pole to the point (ϱ, φ). *Ans.* $\frac{\sqrt{1+a^2}}{a}e^{a\varphi}=\frac{\varrho}{a}\sqrt{1+a^2}$.

45. Find the entire length of the curve $\varrho=a\sin^3\frac{\varphi}{3}$. *Ans.* $\frac{3}{2}\pi a$.

46. Find the length of the evolute of the ellipse $x=\frac{c^2}{a}\cos^3 t$, $y=\frac{c^2}{b}\sin^3 t$. *Ans.* $\frac{4(a^3-b^3)}{ab}$.

47. Find the length of the cardioid $\varrho=a(1+\cos\varphi)$. *Ans.* $8a$.

48. Find the arc length of the involute of the circle $x=a(\cos\varphi+\varphi\sin\varphi)$, $y=a(\sin\varphi-\varphi\cos\varphi)$ from $\varphi=0$ to $\varphi=\varphi_1$. *Ans.* $\frac{1}{2}a\varphi_1^2$.

Computing Areas of Surfaces of Solids of Revolution

49. Find the area of a surface obtained by revolving the parabola $y^2=4ax$ about the x-axis, from the origin O to a point with abscissa $x=3a$. *Ans.* $\frac{56}{3}\pi a^2$.

50. Find the area of the surface of a cone generated by the revolution of a line segment $y=2x$ from $x=0$ to $x=2$: a) About the x-axis. *Ans.* $8\pi\sqrt{5}$. b) About the y-axis. *Ans.* $4\pi\sqrt{5}$.

51. Find the area of the surface of a torus obtained by revolving the circle $x^2+(y-b)^2=a^2$ about the x-axis. *Ans.* $4\pi^2ab$.

52. Find the area of the surface of a solid generated by revolving a cardioid about the x-axis. The cardioid is represented by the parametric equations $x=a(2\cos\varphi-\cos 2\varphi)$, $y=a(2\sin\varphi-\sin 2\varphi)$. *Ans.* $\frac{128}{5}\pi a^2$.

53. Find the area of the surface of a solid obtained by revolving one arc of a cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$ about the x-axis. *Ans.* $\frac{64\pi a^2}{3}$.

54. The arc of a cycloid (see Problem 53) is revolved about the y-axis. Find the surface of the solid of revolution. *Ans.* $16\pi^2a^2$.

55. The arc of a cycloid (see Problem 53) is revolved about a tangent line parallel to the x-axis and passing through the vertex. Find the surface of the solid of revolution. *Ans.* $\frac{32\pi a^2}{3}$.

56. The astroid $x=a\sin^3 t$, $y=a\cos^3 t$ is revolved about the x-axis. Find the surface of the solid of revolution. *Ans.* $\frac{12\pi a^2}{5}$.

57. An arc of the sine wave $y=\sin x$ from $x=0$ to $x=2\pi$ is revolved about the x-axis. Find the surface of the solid of revolution. *Ans.* $4\pi[\sqrt{2}+$ $+\ln(\sqrt{2}+1)]$.

58. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ revolves about the x-axis. Find the surface of the solid of revolution. *Ans.* $2\pi b^2+2\pi ab\frac{\arcsin e}{c}$, where $e=\frac{\sqrt{a^2-b^2}}{a}$.

Various Applications of the Definite Integral

59. Find the centre of gravity of the area of one-fourth of the ellipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(x\geqslant 0,\ y\geqslant 0)$. *Ans.* $\frac{4a}{3\pi}$, $\frac{4b}{3\pi}$.

60. Find the centre of gravity of the area of a figure bounded by the parabola $x^2+4y-16=0$ and the x-axis. *Ans.* $\left(0,\ \frac{8}{5}\right)$.

61. Find the centre of gravity of the volume of a hemisphere. *Ans.* On the axis of symmetry at a distance $\frac{3}{8}R$ from the base.

62. Find the centre of gravity of the surface of a hemisphere. *Ans.* On the axis of symmetry at a distance $\frac{R}{2}$ from the base.

63. Find the centre of gravity of the surface of a circular right cone, the radius of the base of which is R and the altitude h. *Ans.* On the axis of symmetry at a distance $\frac{h}{3}$ from the base.

64. The figure is bounded by the lines $y=\sin x$ $(0\leqslant x\leqslant\pi)$, $y=0$. Find the centre of gravity of the area of this figure. *Ans.* $\left(\frac{\pi}{2},\ \frac{\pi}{8}\right)$.

65. Find the centre of gravity of the area of a figure bounded by the parabolas $y^2=20x$, $x^2=20y$. *Ans.* (9, 9).

66. Find the centre of gravity of the area of a circular sector with central angle 2α and radius R. *Ans.* On the axis of symmetry at a distance $\frac{2}{3}R\frac{\sin\alpha}{\alpha}$ from the vertex of the sector.

67. Find the pressure of water on a rectangle vertically submerged in water at a depth of $5m$ if it is known that the base is 8 metres, the altitude, 12 metres, and the upper base is parallel to the free surface of the water. *Ans.* 1,056 m.

68. The upper edge of a canal lock has the shape of a square with a side of 8 m lying on the surface of the water. Determine the pressure on each part of the lock formed by dividing the square by one of its diagonals. *Ans.* 85,333.33 kg, 170,666.67 kg.

69. Compute the work needed to pump the water out of a hemispherical vessel of diameter 20 metres. *Ans.* $2.5\times10^6\pi$ kg-m.

70. A body is in rectilinear motion according to the law $x=ct^3$, where x is the path length traversed in time t, $c=$const. The resistance of the medium is proportional to the square of the velocity, and k is the constant of proportionality. Find the work done by the resistance when the body moves from the point $x=0$ to the point $x=a$. *Ans.* $\frac{27}{7}k\sqrt[3]{c^2a^7}$.

71. Compute the work that has to be done in order to pump a liquid of density γ from a reservoir having the shape of a cone with vertex pointing down, altitude H and radius of base R. *Ans.* $\frac{\pi\gamma R^2H^2}{12}$.

72. A wooden float of cylindrical shape whose basal area $S=4{,}000$ cm^2 and altitude $H=50$ cm is floating on the surface of the water. What work must be done to pull the float up to the surface? (Specific weight of the wood, 0.8). *Ans.* $\frac{\gamma^2H^2S}{2}=32$ kg-m.

73. Compute the force with which the water presses on a dam in the form of an equilateral trapezoid (upper base $a=6.4$ m, lower base $b=4.2$ m, altitude $H=3$ m). *Ans.* 22.2 m.

74. Find the axial component P kg of total pressure of steam on the spherical bottom of a boiler. The diameter of the cylindrical part of the boiler is D mm, the pressure of the steam in the boiler is P kg/cm^2. *Ans.* $P=\frac{\pi PD^2}{400}$.

75. The end of a vertical shaft of radius r is supported by a flat thrust bearing. The weight of the shaft P is distributed equally over the entire surface of the support. Compute the total work of friction in one rotation of the shaft. Coefficient of friction is μ. *Ans.* $\frac{4}{3}\pi\mu Pr$.

76. A vertical shaft ends in a thrust pin having the shape of a truncated cone. The specific pressure of the pin on the thrust bearing is constant and equal to P. The upper diameter of the pin is D, the lower, d, and the angle at the vertex of the cone is 2α. Coefficient of friction, μ.

Find the work of friction for one rotation of the shaft. *Ans.* $\frac{\pi^2P\mu}{6\sin\alpha}(D^3-d^3)$.

77. A prismatic rod of length l is slowly extended by a force increasing from 0 to P so that at each moment the tensile force is balanced by the forces of elasticity of the rod. Compute the work A expended by the force on tension, assuming that the tension occurred within the limits of elasticity. F is the cross-sectional area of the rod, and E is the modulus of elasticity of the material.

Hint. If x is the elongation of the rod and f is the corresponding force, then $f=\frac{FE}{l}x$. The elongation due to the force P is equal to $\Delta l=\frac{Pl}{EF}$.

Ans. $A=\frac{P\Delta l}{2}=\frac{P^2l}{2EF}$.

78. A prismatic beam is suspended vertically and a tensile force P is applied to its lower end. Compute the elongation of the beam due to the force of its weight and to the force P if it is given that the original length of the

beam is l, the cross-sectional area F, the weight Q and the modulus of elasticity of the material E. *Ans.* $\Delta l=\frac{(Q+2P)\,l}{2EF}$.

79. Determine the time during which a liquid will flow out of a prismatic vessel filled to a height H. The cross-sectional area of the vessel is F, the area of the aperture f, the exit velocity is computed from the formula $v=\mu\sqrt{2gh}$, where μ is the coefficient of viscosity, g is the acceleration of gravity, and h is the distance from the aperture to the level of the liquid. *Ans.* $T=\frac{2FH}{\mu f\sqrt{2gH}}=\frac{F}{\mu f}\sqrt{\frac{2H}{g}}$.

80. Determine the discharge Q (the quantity of water flowing in unit time) over a spillway of rectangular cross section. Height of spillway, h, width, b. *Ans.* $Q=\frac{2}{3}\mu bh\sqrt{2gh}$.

81. Determine the discharge of water Q flowing from a side rectangular opening of height a and width b, if the height of the open surface of the water above the lower side of the opening is H. *Ans.* $Q=\frac{2b\mu\sqrt{2g}}{3}\left[H^{\frac{3}{2}}-(H-a)^{\frac{3}{2}}\right]$.

CHAPTER XIII

DIFFERENTIAL EQUATIONS

SEC. 1. STATEMENT OF THE PROBLEM. THE EQUATION OF MOTION OF A BODY WITH RESISTANC OF THE MEDIUM PROPORTIONAL TO THE VELOCITY. THE EQUATION OF A CATENARY

Let the function $y=f(x)$ reflect the quantitative aspect of some phenomenon. Frequently, it is not possible to establish directly the type of dependence of y on x, but it is possible to give the relationship between x and y and the derivatives of y with respect to x: y', y'', ..., $y^{(n)}$. That is, we are able to write a **differential equation.**

From the relationship established between the variable x, y and the derivatives it is required to determine the direct dependence of y on x; that is, to find $y=f(x)$ or, as we say, **to integrate the differential equation.**

Let us consider two examples.

Example 1. A body of mass m is dropped from some height. It is required to establish that law according to which the velocity v will vary as the body falls, if, in addition to the force of gravity, the body is acted upon by the decelerating force of the air, which is proportional to the velocity (with constant of proportionality k); in other words, it is required to find $v=f(t)$.

Solution. By Newton's second law

$$m\frac{dv}{dt}=F$$

where $\frac{dv}{dt}$ is the acceleration of a moving body (the derivative of the velocity with respect to time) and F is the force acting on the body in the direction of motion. This force is the resultant of two forces: the force of gravity mg and the force of air resistance, $-kv$, which has the minus sign because it is in the opposite direction to that of the velocity. And so we have

$$m\frac{dv}{dt}=mg-kv. \tag{1}$$

This relation connects the unknown function v and its derivative $\frac{dv}{dt}$, *which is a differential equation in the unknown function* v. To solve the differential equation is to find a function $v=f(t)$ such that identically satisfies the given differential equation. There is an infinitude of such functions. The student can easily verify that any function of the form

$$v=Ce^{-\frac{k}{m}t}+\frac{mg}{k} \tag{2}$$

satisfies equation (1) no matter what the constant C is. Which one of these

functions yields the sought-for dependence of v on t? To find it we take advantage of a supplementary condition: when the body was dropped it was imparted an initial velocity v_0 (which may be zero as a particular case); we assume this initial velocity to be known. But then the unknown function $v=f(t)$ must be such that when $t=0$ (when motion begins) the condition $v=v_0$ is fulfilled. Substituting $t=0$, $v=v_0$ into formula (2), we find

$$v_0=C+\frac{mg}{k},$$

whence

$$C=v_0-\frac{mg}{k}.$$

Thus, the constant C is found, and the sought-for dependence of v on t is

$$v=\left(v_0-\frac{mg}{k}\right)e^{-\frac{kt}{m}}+\frac{mg}{k}. \qquad (2')$$

It will be noted that if $k=0$ (the air resistance is absent or negligibly small so that we can disregard it), then we have a result familiar from physics *):

$$v=v_0+gt. \qquad (2'')$$

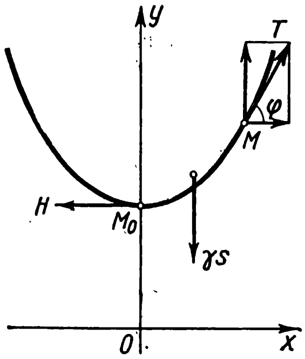


Fig. 244.

This function satisfies the differential equation (1) and the initial condition: $v=v_0$ when $t=0$.

Example 2. A flexible homogeneous thread is suspended at two ends. Find the equation of the curve that it describes under its own weight (it is the same as any suspended ropes, wires, chains, as for instance the caterpillar track of a tank between two supporting rollers).

Solution. Let $M_0(0, b)$ be the lowest point of the thread, and M an arbitrary point (Fig. 244). Let us consider a part of the thread, M_0M. This part is in equilibrium, the resultant of three forces:

1) the tension T, acting along the tangent to the point M and forming an angle φ with the x-axis;

2) the tension H at M_0 acting horizontally;

3) the weight of the thread γs acting vertically downwards, where s is the length of the arc M_0M and γ is the linear specific weight of the thread.

Breaking up the tension T into horizontal and vertical components, we get the equations of equilibrium:

$$T\cos\varphi=H, \qquad T\sin\varphi=\gamma s.$$

Dividing the terms of the second equation by the corresponding terms of the first, we obtain

$$\tan\varphi=\frac{\gamma}{H}s. \qquad (3)$$

---

*) Formula (2″) can be obtained from (2′) by passing to the limit:

$$\lim_{k\to 0}\left[\left(v_0-\frac{mg}{k}\right)e^{-\frac{kt}{m}}+\frac{mg}{k}\right]=v_0+gt.$$

Now suppose that the equation of the sought-for curve may be written in the form $y=f(x)$. Here, $f(x)$ is an unknown function that has to be found. It will be noted that

$$\tan\varphi = f'(x) = \frac{dy}{dx}.$$

Hence,

$$\frac{dy}{dx} = \frac{1}{a} s \tag{4}$$

where the ratio $\frac{H}{\gamma}$ is denoted in terms of a.

Differentiate both sides of (4) with respect to x:

$$\frac{d^2y}{dx^2} = \frac{1}{a}\frac{ds}{dx}. \tag{5}$$

But, as we know (see Sec. 1, Ch. VI),

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Substituting this expression into equation (5), we get the differential equation of the sought-for curve:

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}. \tag{6}$$

It expresses the relationship between the first and second derivatives of the unknown function y.

Without going into the methods of solving the equations, we shall note that any function of the form

$$y = \frac{a}{2}\left[e^{+\left(\frac{x}{a}+C_1\right)} + e^{-\left(\frac{x}{a}+C_1\right)}\right] + C_2 \tag{7}$$

satisfies equation (6) for any values that C_1 and C_2 may assume. This is evident if we put the first and second derivatives of the given function into (6). We shall indicate, without proof, that these functions (for different C_1 and C_2) exhaust all possible solutions of equation (6).

The graphs of all the functions thus obtained are called *catenaries*.

Let us now find out how one should choose the constants C_1 and C_2 so as to obtain precisely that catenary whose lowest point M has coordinates $(0, b)$. Since for $x=0$ the point of the catenary occupies the lowest possible position, the tangent here is horizontal, $\frac{dy}{dx}=0$. Also, it is given that at this point the ordinate is equal to b, $y=b$.

From (7) we find

$$y' = \frac{1}{2}\left(e^{\frac{x}{a}+C_1} - e^{-\left(\frac{x}{a}+C_1\right)}\right).$$

Putting $x=0$ here, we obtain $0=\frac{1}{2}(e^{C_1}-e^{-C_1})$. Hence, $C_1=0$.

If the ordinate of the point M_0 is b, then $y=b$ when $x=0$.

From equation (7) we get $b=\frac{a}{2}(1+1)+C_2$, assuming $x=0$ and $C_1=0$, whence $C_2=b-a$. Finally we have

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)+b-a.$$

Equation (7) assumes a very simple form if we take the ordinate of M_0 equal to a. Then the equation of the catenary is

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right).$$

SEC. 2. DEFINITIONS

Definition 1. A *differential equation* is one which connects an independent variable, x, an unknown function, $y=f(x)$, and its derivatives y', y'', ..., y^n.

Symbolically, a differential equation may be written as follows:

$$F(x, y, y', y'', \ldots, y^{(n)})=0$$

or

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right)=0.$$

If the sought-for function $y=f(x)$ is a function of **one** independent variable, then the differential equation is called *ordinary*. We shall deal only with ordinary differential equations *).

Definition 2. The *order* of a differential equation is the order of the highest derivative which appears.

For example, the equation

$$y'-2xy^2+5=0$$

is an equation of the first order.

*) In addition to ordinary differential equations, mathematical analysis makes a study of *partial differential equations*. Such an equation is a relation between an unknown function z (that is, dependent upon two or several variables x, y, ...), these variables x, y, ..., and the partial derivatives of z: $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x^2}$, etc.

The following is an example of a partial differential equation with unknown function $z(x, y)$:

$$x\frac{\partial z}{\partial x}=y\frac{\partial z}{\partial y}.$$

It is easy to verify that this equation is satisfied by the function $z=x^2y^2$ (and also by a multitude of other functions).

In this course we shall have little to do with partial differential equations.

The equation

$$y'' + ky' - by - \sin x = 0$$

is an equation of the second order, etc.

The equation considered in the preceding section in Example 1 is an equation of the first order, in Example 2, one of the second order.

Definition 3. The *solution* or *integral* of a differential equation is any function $y = f(x)$, which, when put into the equation, converts it into an identity.

Example 1. Let there be an equation

$$\frac{d^2y}{dx^2} + y = 0.$$

The functions $y = \sin x$, $y = 2\cos x$, $y = 3\sin x - \cos x$ and, in general, functions of the form $y = C_1 \sin x$, $y = C_2 \cos x$ or

$$y = C_1 \sin x + C_2 \cos x$$

are solutions of the given equation for any choice of constants C_1 and C_2; this is evident if we put these functions into the equation.

Example 2. Let us consider the equation

$$y'x - x^2 - y = 0.$$

Its solutions are all functions of the form

$$y = x^2 + Cx$$

where C is any constant. Indeed, differentiating the functions $y = x^2 + Cx$, we find

$$y' = 2x + C.$$

Putting the expressions for y and y' into the initial equation, we get the identity

$$(2x + C)\,x - x^2 - x^2 - Cx = 0.$$

Each of the equations considered in Examples 1 and 2 has an infinitude of solutions.

SEC. 3. FIRST-ORDER DIFFERENTIAL EQUATIONS (GENERAL NOTIONS)

1. A differential equation of the **first** order is of the form

$$F(x,\ y,\ y') = 0. \tag{1}$$

If this equation can be solved for y', it can be written in the form

$$y' = f(x,\ y). \tag{1'}$$

In this case we say that the differential equation is solved for the derivative. For such an equation the following theorem, called the theorem of the unique existence of solution of a differential equation, holds.

Theorem. *If in the equation*

$$y'=f(x, y)$$

the function $f(x, y)$ and its partial derivative with respect to y, $\frac{\partial f}{\partial y}$, are continuous in some region D in an xy-plane containing some point (x_0, y_0), then there is only one solution to this equation $y=\varphi(x)$ which satisfies the condition $x=x_0$, $y=y_0$. The geometric meaning of the theorem consists in the fact that there exists one and only one such function $y=\varphi(x)$, the graph of which passes through the point (x_0, y_0).

It follows from this theorem that equation (1′) has an infinitude of various solutions [for example, a solution the graph of which passes through (x_0, y_0); another solution whose graph passes through (x_0, y_1); through (x_0, y_2), etc., provided these points lie in the region D].

The condition that for $x=x_0$ the function y must be equal to the given number y_0 is called the *initial condition*. It is frequently written in the form

$$y|_{x=x_0}=y_0.$$

Definition 1. The *general solution* of a first-order differential equation is the function

$$y=\varphi(x, C), \tag{2}$$

which depends on a single arbitrary constant C and satisfies the following conditions:

a) It satisfies the differential equation for any specific value of the constant C.

b) No matter what the initial condition $y=y_0$ for $x=x_0$, that is, $(y)_{x=x_0}=y_0$, it is possible to find a value $C=C_0$ such that the function $y=\varphi(x, C_0)$ satisfies the given initial condition. It is assumed here that the values x_0 and y_0 belong to the range of the variables x and y in which the conditions of the existence theorem are fulfilled.

2. In searching for the general solution of a differential equation we often arrive at a relation like

$$\Phi(x, y, C)=0, \tag{2′}$$

which is not solved for y. Solving this relationship for y, we get the general solution. However, it is not always possible to express y from (2′) in terms of elementary functions; in such cases, the general solution is left in implicit form.

An equation of the form $\Phi(x, y, C)=0$ which gives an implicit general solution is called the *complete integral* of the differential equation.

Definition 2. A *particular solution* is any function $y=\varphi(x, C_0)$ which is obtained from the general solution $y=\varphi(x, C)$, if in the latter we assign to the arbitrary constant C a definite value $C=C_0$. In this case, the relation $\Phi(x, y, C_0)=0$ is called a *particular integral* of the equation.

Example 1. For the first-order equation

$$\frac{dy}{dx}=-\frac{y}{x}$$

the general solution is a family of functions $y=\frac{C}{x}$; this can be checked by simple substitution in the equation.

Let us find a particular solution that will satisfy the following initial condition: $y_0=1$ when $x_0=2$.

Putting these values into the formula $y=\frac{C}{x}$, we have $1=\frac{C}{2}$ or $C=2$.

Consequently, the function $y=\frac{2}{x}$ will be the particular solution we are seeking.

From the geometric viewpoint, the **general solution (complete integral)** is a **family of curves** in a coordinate plane, which family depends on a single arbitrary constant C (or, as it is common to say, on a single parameter C). These curves are called *integral curves* of the given differential equation. A **particular** integral is associated with **one curve** of this family that passes through a certain given point of the plane.

Thus, in the latter example, the complete integral is geometrically depicted by a family of hyperbolas $y=\frac{C}{x}$ while the particular integral defined by the given initial condition is depicted by one of these hyperbolas passing through the point $M_0(2, 1)$. Fig. 245 shows the curves of a family that are associated with certain values of the parameter: $C=\frac{1}{2}$, $C=1$, $C=2$, $C=-1$, etc.

To make the reasoning still more pictorial, we shall from now on say that not only the function $y=\varphi(x, C_0)$ that satisfies the equation but also the associated **integral curve** is a **solution of the equation.** We will therefore speak of **a solution passing through the point** (x_0, y_0).

Note. The equation $\frac{dy}{dx}=-\frac{y}{x}$ has no solution passing through a point lying on the y-axis (see Fig. 245). This is because the right

side of the equation is not defined for $x=0$ and consequently is not continuous.

To **solve** (or as we frequently say, to **integrate**) a differential equation means:

a) to find its general solution or complete integral (if the initial conditions are not specified) or

b) to find a particular solution of the equation that will satisfy the given initial conditions (if such exist).

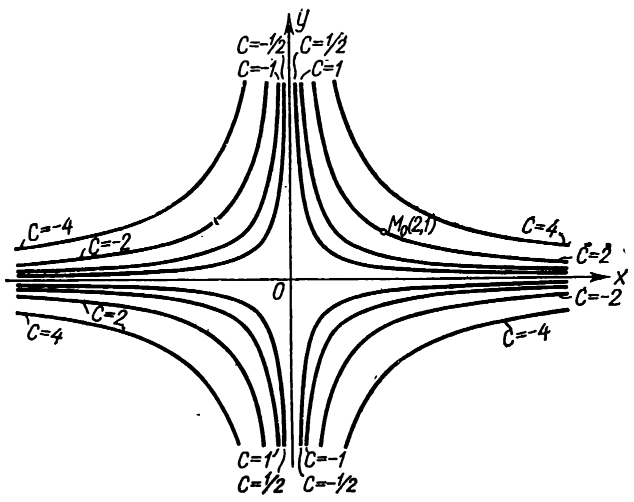


Fig. 245.

3. Let us now give a geometric interpretation of a first-order differential equation.

Let there be a differential equation solved for the derivative

$$\frac{dy}{dx}=f(x,\ y) \tag{1'}$$

and let $y=\varphi(x,\ C)$ be the general solution of this equation. This general solution determines the family of integral curves in the xy-plane.

For each point M with coordinates x and y, equation (1′) defines the value of the derivative $\frac{dy}{dx}$, or the slope of the tangent line to the integral curve passing through this point. Thus, the differential equation (1′) yields a collection of directions or, as we say, defines a *direction-field* in the xy-plane.

Consequently, from the geometric point of view, the problem of integrating a differential equation consists in finding the curves,

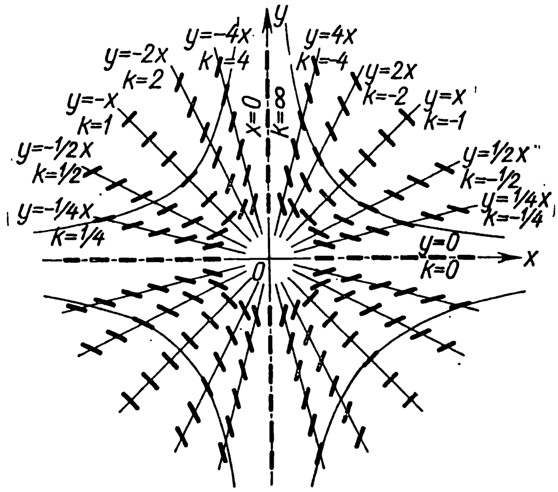


Fig. 246.

the direction of the tangents to which coincides with the direction-field at the corresponding points.

Fig. 246 shows a direction-field defined by the differential equation

$$\frac{dy}{dx} = -\frac{y}{x}.$$

4. Let us now consider the following problem.

Let there be given a family of functions that depends on a single parameter C:

$$y = \varphi(x, C), \tag{2}$$

and let only one curve of this family pass through each point of the plane (or some region in the plane).

For what differential equation is this family of functions a complete integral?

From relation (2), differentiating with respect to x, we find

$$\frac{dy}{dx} = \varphi'_x(x, C). \tag{3}$$

Since only one curve of the family passes through each point of the plane, for every number pair x and y, a unique value of

C is determined from equation (2). Putting this value of C into (3) we find $\frac{dy}{dx}$ as a function of x and y. This is what yields the differential equation that is satisfied by every function of the family (2).

Hence, to establish a relationship between x, y and $\frac{dy}{dx}$, that is, to write a differential equation whose general solution (complete integral) is given by formula (2), one has to eliminate C from relations (2) and (3).

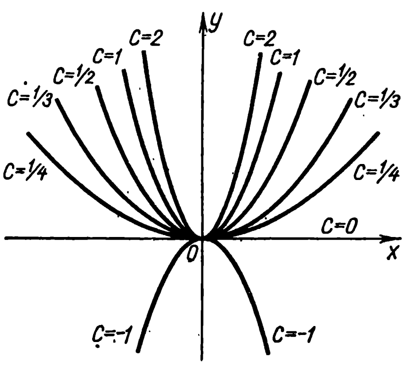


Fig. 247.

Example 2. Find the differential equation of the family of parabolas $y = Cx^2$ (Fig. 247).

Differentiating the equation of the family with respect to x, we get

$$\frac{dy}{dx} = 2Cx.$$

Putting the value $C = \frac{y}{x^2}$ into this equation from the equation of the family, we obtain a differentiable equation of the given family:

$$\frac{dy}{dx} = \frac{2y}{x}.$$

This differential equation is meaningful when $x \neq 0$; which is to say, in any region not containing points on the y-axis.

SEC. 4. EQUATIONS WITH SEPARATED AND SEPARABLE VARIABLES. THE PROBLEM OF THE DISINTEGRATION OF RADIUM

Let us consider a differential equation of the form

$$\frac{dy}{dx} = f_1(x) f_2(y), \tag{1}$$

where the right side is a product of a function dependent only on x by a function dependent only on y. We transform it in the following manner assuming that $f_2(y) \neq 0$:

$$\frac{1}{f_2(y)} dy = f_1(x)\, dx. \tag{1'}$$

Considering y a known function of x, equation (1′) may be regarded as the equality of two differentials, while the indefinite

integrals of them will differ by a constant term. Integrating the left side with respect to y and the right with respect to x, we obtain

$$\int \frac{1}{f_2(y)}\,dy = \int f_1(x)\,dx + C$$

which is a relationship connecting the solution of y, the independent variable x, and an arbitrary constant C; we have thus obtained a general solution (complete integral) of equation (1).

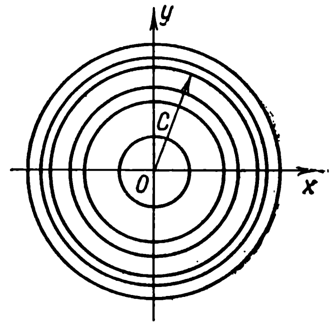


Fig. 248.

1. A type (1′) differential equation

$$M(x)\,dx + N(y)\,dy = 0 \qquad (2)$$

is called an equation with *separated variables*. From what has been proved, its complete integral is

$$\int M(x)\,dx + \int N(y)\,dy = C.$$

Example 1. Given an equation with separated variables:

$$x\,dx + y\,dy = 0.$$

Integrating we get the general solution:

$$\frac{x^2}{2} + \frac{y^2}{2} = C_1.$$

Since the left side of this equation is nonnegative, the right side is also nonnegative. Denoting $2C_1$ in terms of C^2, we will have

$$x^2 + y^2 = C^2.$$

This is the equation of a family of concentric circles (Fig. 248) with centre at the coordinate origin and radius C.

2. An equation of the form

$$M_1(x)\,N_1(y)\,dx + M_2(x)\,N_2(y)\,dy = 0 \qquad (3)$$

is called an equation with *variables separable*. It can be reduced *) to an equation with separated variables by dividing both sides by the expression $N_1(y)\,M_2(x)$:

$$\frac{M_1(x)\,N_1(y)}{N_1(y)\,M_2(x)}\,dx + \frac{M_2(x)\,N_2(y)}{N_1(y)\,M_2(x)}\,dy = 0$$

*) These transformations are permissible only in a region where neither $N_1(y)$ nor $M_2(x)$ vanish.

or

$$\frac{M_1(x)}{M_2(x)}dx+\frac{N_2(y)}{N_1(y)}dy=0,$$

that is, to an equation like (2).

Example 2. Given the equation

$$\frac{dy}{dx}=-\frac{y}{x}.$$

Separating variables, we have

$$\frac{dy}{y}=-\frac{dx}{x}.$$

Integrating we find

$$\int\frac{dy}{y}=-\int\frac{dx}{x}+C,$$

which is

$$\ln|y|=-\ln|x|+\ln|C|\,^{*)} \text{ or } \ln|y|=\ln\left|\frac{C}{x}\right|;$$

whence we get the general solution: $y=\frac{C}{x}$.

Example 3. Given the equation

$$(1+x)y\,dx+(1-y)x\,dy=0.$$

Separating variables we have

$$\frac{(1+x)}{x}dx+\frac{1-y}{y}dy=0; \quad \left(\frac{1}{x}+1\right)dx+\left(\frac{1}{y}-1\right)dy=0.$$

Integrating we obtain

$$\ln|x|+x+\ln|y|-y=C \text{ or } \ln|xy|+x-y=C.$$

This relation is the complete integral of the given equation.

Example 4. It is known that the decay rate of radium is directly proportional to its quantity at each given instant. Find the law of variation of a mass of radium as a function of the time if at $t=0$ the mass of radium was m_0.

The decay rate is determined as follows. Let there be mass m at time t, and mass $m+\Delta m$ at time $t+\Delta t$. During Δt mass Δm decays. The ratio $\frac{\Delta m}{\Delta t}$ is the mean rate of decay. The limit of this ratio as $\Delta t \to 0$

$$\lim_{\Delta t\to 0}\frac{\Delta m}{\Delta t}=\frac{dm}{dt}$$

is the *rate of decay* of radium at time t.

---

*) Having in view subsequent transformations, we denoted the arbitrary constant by $\ln|C|$, which is permissible since $\ln|C|$ (when $C\neq 0$) can take on any value from $-\infty$ to $+\infty$.

It is given that

$$\frac{dm}{dt} = -km, \tag{4}$$

where k is the constant of proportionality ($k>0$). We use the minus sign because the mass of radium diminishes with increasing time and therefore $\frac{dm}{dt}<0$.

Equation (4) is an equation with variables separable. Let us separate the variables:

$$\frac{dm}{m} = -k\,dt.$$

Solving the equation we obtain

$$\ln m = -kt - \ln C$$

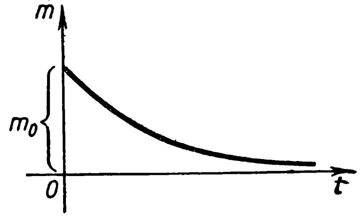


Fig. 249.

whence

$$\ln\frac{m}{C} = -kt,$$

$$m = Ce^{-kt}. \tag{5}$$

Since at $t=0$ the mass of radium was m_0, C must satisfy the relationship

$$m_0 = Ce^{-k_0} = C.$$

Putting the value of C into (5) we get the desired mass of radium as a function of time (Fig. 249):

$$m = m_0 e^{-kt}. \tag{6}$$

The constant k is determined from observations as follows. During time t_0 let $\alpha\%$ of the original mass of radium decay. Hence, the following relationship is fulfilled:

$$\left(1-\frac{\alpha}{100}\right)m_0 = m_0 e^{-kt_0}$$

whence

$$-kt_0 = \ln\left(1-\frac{\alpha}{100}\right)$$

or

$$k = -\frac{1}{t_0}\ln\left(1-\frac{\alpha}{100}\right).$$

Thus, it has been determined that for radium $k=0.00044$ (the unit of measure of time is one year).

Putting this value of k into (6) we obtain

$$m = m_0 e^{-0.00044t}.$$

Let us find the radium half-life, which is the interval of time during which half of the original mass of radium decays. Putting $\frac{m_0}{2}$ in place of m

in the latter formula, we get an equation for determining the half-life T:

$$\frac{m_0}{2}=m_0e^{-0.00044T}$$

whence

$$-0.00044T=-\ln 2$$

or

$$T=\frac{\ln 2}{0.00044}=1{,}590 \text{ years.}$$

Note. The simplest differential equation with separated variables is one of the form

$$\frac{dy}{dx}=f(x) \quad \text{or} \quad dy=f(x)\,dx.$$

Its complete integral is of the form

$$y=\int f(x)\,dx+C.$$

We dealt with the solution of equations of this kind in Ch. X.

SEC. 5. HOMOGENEOUS FIRST-ORDER EQUATIONS

Definition 1. The function $f(x, y)$ is called a *homogeneous function of degree n* in the variables x and y, if for any λ the following identity is true:

$$f(\lambda x, \lambda y)=\lambda^n f(x, y).$$

Example 1. The function $f(x, y)=\sqrt[3]{x^3+y^3}$ is a homogeneous function of degree one, since

$$f(\lambda x, \lambda y)=\sqrt[3]{(\lambda x)^3+(\lambda y)^3}=\lambda\sqrt[3]{x^3+y^3}=\lambda f(x, y).$$

Example 2. $f(x, y)=xy-y^2$ is a homogeneous function of degree two, since $(\lambda x)(\lambda y)-(\lambda y)^2=\lambda^2[xy-y^2]$.

Example 3. $f(x, y)=\frac{x^2-y^2}{xy}$ is a homogeneous function of zero degree, since $\frac{(\lambda x)^2-(\lambda y)^2}{(\lambda x)(\lambda y)}=\frac{x^2-y^2}{xy}$, that is, $f(\lambda x, \lambda y)=f(x, y)$ or $f(\lambda x, \lambda y)=\lambda^0 f(x, y)$.

Definition 2. An equation of the first order

$$\frac{dy}{dx}=f(x, y) \tag{1}$$

is called *homogeneous* in x and y if the function $f(x, y)$ is a homogeneous function of zero degree in x and y.

Solution of a homogeneous equation. It is given that $f(\lambda x, \lambda y) = f(x, y)$. Putting $\lambda = \frac{1}{x}$ in this identity, we have

$$f(x, y) = f\left(1, \frac{y}{x}\right).$$

Thus, a homogeneous function of zero degree is dependent only on the ratio of the arguments.

In this case, equation (1) takes the form

$$\frac{dy}{dx} = f\left(1, \frac{y}{x}\right). \tag{1'}$$

Making the substitution

$$u = \frac{y}{x}, \quad \text{or} \quad y = ux,$$

we get

$$\frac{dy}{dx} = u + \frac{du}{dx}x.$$

Putting this expression of the derivative into equation (1′), we obtain

$$u + x\frac{du}{dx} = f(1, u).$$

This is an equation with variables separable:

$$x\frac{du}{dx} = f(1, u) - u \quad \text{or} \quad \frac{du}{f(1, u) - u} = \frac{dx}{x}.$$

Integrating we find

$$\int \frac{du}{f(1, u) - u} = \int \frac{dx}{x} + C.$$

Putting the ratio $\frac{y}{x}$ in place of u after integration, we get the integral of equation (1′).

Example 4. Given the equation

$$\frac{dy}{dx} = \frac{xy}{x^2 - y^2}.$$

On the right is a zero-degree homogeneous function, which means that we have a homogeneous equation. Making the substitution $\frac{y}{x} = u$ we have

$$y = ux; \quad \frac{dy}{dx} = u + x\frac{du}{dx};$$

$$u + x\frac{du}{dx} = \frac{u}{1 - u^2}; \quad x\frac{du}{dx} = \frac{u^3}{1 - u^2}.$$

Separating variables we obtain

$$\frac{(1-u^2)\,du}{u^3}=\frac{dx}{x}; \quad \left(\frac{1}{u^3}-\frac{1}{u}\right)du=\frac{dx}{x};$$

Whence, integrating, we find

$$-\frac{1}{2u^2}-\ln|u|=\ln|x|+\ln|C| \quad \text{or} \quad -\frac{1}{2u^2}=\ln|uxC|.$$

Substituting $u=\frac{y}{x}$, we get the general solution of the original equation:

$$-\frac{x^2}{2y^2}=\ln|Cy|.$$

It is impossible here to get y as an explicit function of x in terms of elementary functions. Incidentally, it is very easy to express x in terms of y:

$$x=y\sqrt{-2C\ln|Cy|}.$$

Note. An equation of the type

$$M(x, y)\,dx+N(x, y)\,dy=0$$

will be homogeneous if, and only if, $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree. This follows from the fact that the ratio of two homogeneous functions of the same degree is a homogeneous function of degree zero.

Example 5. The equations

$$(2x+3y)\,dx+(x-2y)\,dy=0,$$
$$(x^2+y^2)\,dx-2xy\,dy=0$$

are homogeneous.

SEC. 6. EQUATIONS REDUCIBLE TO HOMOGENEOUS EQUATIONS

Equations of the following type are reducible to homogeneous equations:

$$\frac{dy}{dx}=\frac{ax+by+c}{a_1x+b_1y+c_1}. \tag{1}$$

If $c_1=c=0$, then equation (1) is obviously homogeneous. Now let c and c_1 (or one of them) be different from zero. Change the variables:

$$x=x_1+h, \quad y=y_1+k.$$

Then

$$\frac{dy}{dx}=\frac{dy_1}{dx_1}. \tag{2}$$

Putting into (2) the expressions x, y, and $\frac{dy}{dx}$, we obtain

$$\frac{dy_1}{dx_1}=\frac{ax_1+by_1+ah+bk+c}{a_1x_1+b_1y_1+a_1h+b_1k+c_1}. \tag{3}$$

Choose h and k so that the following equalities are fulfilled:

$$\left.\begin{aligned} ah+bk+c=0,\\ a_1h+b_1k+c_1=0.\end{aligned}\right\} \tag{4}$$

In other words, define h and k as solutions of a system of equations (4). Equation (3) then becomes homogeneous:

$$\frac{dy_1}{dx_1}=\frac{ax_1+by_1}{a_1x_1+b_1y_1}.$$

Solving this equation and passing once again to x and y by formulas (2), we obtain the solution of equation (1).
The system (4) has no solution if

$$\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}=0,$$

i. e., $ab_1=a_1b$. But if $\frac{a_1}{a}=\frac{b_1}{b}=\lambda$, that is, $a_1=\lambda a$, $b_1=\lambda b$, and, hence, equation (1) may be transformed to

$$\frac{dy}{dx}=\frac{(ax+by)+c}{\lambda(ax+by)+c_1}. \tag{5}$$

Then by substitution

$$z=ax+by \tag{6}$$

and the equation is reduced to one with variables separable.
Indeed,

$$\frac{dz}{dx}=a+b\frac{dy}{dx},$$

whence

$$\frac{dy}{dx}=\frac{1}{b}\frac{dz}{dx}-\frac{a}{b}. \tag{7}$$

Putting into (5) expressions (6) and (7), we get

$$\frac{1}{b}\frac{dz}{dx}-\frac{a}{b}=\frac{z+c}{\lambda z+c_1},$$

which is an equation with variables separable.

The device applied to integrating equation (1) is also applied to the integration of the equation

$$\frac{dy}{dx}=f\left(\frac{ax+by+c}{a_1x+b_1y+c_1}\right),$$

where f is an arbitrary continuous function.

Example 1. Given the equation

$$\frac{dy}{dx}=\frac{x+y-3}{x-y-1}.$$

To convert it into a homogeneous equation, make the substitution $x=x_1+h$; $y=y_1+k$. Then

$$\frac{dy_1}{dx_1}=\frac{x_1+y_1+h+k-3}{x_1-y_1+h-k-1}.$$

Solving the set of two equations

$$h+k-3=0; \quad h-k-1=0,$$

we find

$$h=2, \quad k=1.$$

As a result we get the homogeneous equation

$$\frac{dy_1}{dx_1}=\frac{x_1+y_1}{x_1-y_1},$$

which we solve by substitution:

$$\frac{y_1}{x_1}=u;$$

then

$$y_1=ux_1; \quad \frac{dy_1}{dx_1}=u+x_1\frac{du}{dx_1},$$

$$u+x_1\frac{du}{dx_1}=\frac{1+u}{1-u},$$

and we get an equation with variables separable:

$$x_1\frac{du}{dx_1}=\frac{1+u^2}{1-u}.$$

Separating the variables, we have

$$\frac{1-u}{1+u^2}du=\frac{dx_1}{x_1}.$$

Integrating we find

$$\arctan u-\frac{1}{2}\ln(1+u^2)=\ln x_1+\ln C,$$

$$\arctan u=\ln(\sqrt{1+u^2}x_1C)$$

or

$$Cx_1\sqrt{1+u^2}=e^{\arctan u}.$$

Putting $\frac{y_1}{x_1}$ in place of u, we obtain

$$C\sqrt{x_1^2+y_1^2}=e^{\arctan\frac{y_1}{x_1}}.$$

Passing to the variables x and y, we finally get

$$C\sqrt{(x-2)^2+(y-1)^2}=e^{\arctan\frac{y-1}{x-2}}.$$

Example 2. The equation

$$y'=\frac{2x+y-1}{4x+2y+5}$$

cannot be solved by the substitution $x=x_1+h$, $y=y_1+k$, since in this case the set of equations that serves to determine h and k is insolvable (here, the determinant $\begin{vmatrix}2 & 1\\ 4 & 2\end{vmatrix}$ of the coefficients of the variables is equal to zero).

This equation may be reduced to one with variables separable by the substitution

$$2x+y=z.$$

Then $y'=z'-2$ and the equation is reduced to the form

$$z'-2=\frac{z-1}{2z+5}$$

or

$$z'=\frac{5z+9}{2z+5}.$$

Solving it we find

$$\frac{2}{5}z+\frac{7}{25}\ln|5z+9|=x+C.$$

Since $z=2x+y$, we obtain the final solution of the initial equation in the form

$$\frac{2}{5}(2x+y)+\frac{7}{25}\ln|10x+5y+9|=x+C$$

or

$$10y-5x+7\ln|10x+5y+9|=C_1,$$

that is, as an implicit function y of x.

SEC. 7. FIRST-ORDER LINEAR EQUATIONS

Definition. *A first-order linear equation* is an equation that is linear in the unknown function and its derivative. It is of the form

$$\frac{dy}{dx}+P(x)y=Q(x), \tag{1}$$

where $P(x)$ and $Q(x)$ are given continuous functions of x (or are constants).

Solution of linear equation (1). Let us seek the solution of equation (1) in the form of a product of two functions of x:

$$y=u(x)v(x). \tag{2}$$

One of these functions may be arbitrary, while the other will be determined from equation (1).

Differentiating both sides of (2), we find

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}.$$

Putting the expression obtained of the derivative into (1), we have

$$u\frac{dv}{dx}+\frac{du}{dx}v+Puv=Q$$

or

$$u\left(\frac{dv}{dx}+Pv\right)+v\frac{du}{dx}=Q. \tag{3}$$

Let us choose the function v such that

$$\frac{dv}{dx}+Pv=0. \tag{4}$$

Separating the variables in this differential equation in the function v, we find

$$\frac{dv}{v}=-P\,dx.$$

Integrating we obtain

$$-\ln C_1+\ln v=-\int P\,dx$$

or

$$v=C_1e^{-\int P\,dx}.$$

Since for us it is sufficient to have some nonzero solution of equation (4), we take, as the function $v(x)$,

$$v(x)=e^{-\int P\,dx}, \tag{5}$$

where $\int P\,dx$ is some antiderivative. Obviously, $v(x)\neq 0$.

Putting the value of $v(x)$ which we have found into (3), we get (noting that $\frac{dv}{dx}+Pv=0$):

$$v(x)\frac{du}{dx}=Q(x),$$

or

$$\frac{du}{dx}=\frac{Q(x)}{v(x)},$$

whence

$$u=\int\frac{Q(x)}{v(x)}\,dx+C.$$

Substituting into formula (2), we finally get

$$y=v(x)\left[\int\frac{Q(x)}{v(x)}\,dx+C\right]$$

or

$$y = v(x)\int \frac{Q(x)}{v(x)}\,dx + Cv(x). \tag{6}$$

Note. It is obvious that expression (6) will not change if in place of the function $v(x)$ defined by (5) we take some function $v_1(x) = \bar{C}v(x)$. Indeed, putting $v_1(x)$ in (6) in place of $v(x)$, we get

$$y = \bar{C}v(x)\int \frac{Q(x)}{\bar{C}v(x)}\,dx = \overline{CC}v(x).$$

The $\bar{C}$'s in the first term cancel out; in the second term the product $\overline{CC}$ is an arbitrary constant, which we shall denote by C, and we again arrive at expression (6). If we denote $\int \frac{Q(x)}{v(x)}\,dx = \varphi(x)$, then expression (6) will take the form

$$y = v(x)\,\varphi(x) + Cv(x). \tag{6'}$$

It is obvious that this is a complete integral, since C may be chosen in such manner that the initial condition will be fulfilled:

$$\text{when } x = x_0, \quad y = y_0.$$

The value of C is determined from the equation

$$y_0 = v(x_0)\,\varphi(x_0) + Cv(x_0).$$

Example. Solve the equation

$$\frac{dy}{dx} - \frac{2}{x+1}y = (x+1)^3.$$

Solution. Putting

$$y = uv$$

we have

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v.$$

Putting the expression $\frac{dy}{dx}$ into the original equation, we obtain

$$u\frac{dv}{dx} + \frac{du}{dx}v - \frac{2}{x+1}uv = (x+1)^3,$$

$$u\left(\frac{dv}{dx} - \frac{2}{x+1}v\right) + v\frac{du}{dx} = (x+1)^3. \tag{7}$$

To determine v we get the equation

$$\frac{dv}{dx} - \frac{2}{x+1}v = 0,$$

that is,

$$\frac{dv}{v}=\frac{2dx}{x+1},$$

whence

$$\ln v=2\ln(x+1) \quad \text{or} \quad v=(x+1)^2.$$

Putting the expression of the function v into equation (7), we get the following equation for u:

$$(x+1)^2\frac{du}{dx}=(x+1)^3 \quad \text{or} \quad \frac{du}{dx}=(x+1),$$

whence

$$u=\frac{(x+1)^2}{2}+C.$$

Thus, the complete integral of the given equation will be of the form

$$y=\frac{(x+1)^4}{2}+C(x+1)^2.$$

The family obtained is the **general** solution. No matter what the initial condition (x_0, y_0), where $x_0\neq-1$, it is always possible to choose C so that the corresponding particular solution should satisfy the given initial condition. For example, the particular solution that satisfies the condition $y_0=3$ when $x_0=0$ is found as follows:

$$3=\frac{(0+1)^4}{2}+C(0+1)^2; \qquad C=\frac{5}{2}.$$

Consequently, the desired particular solution is

$$y=\frac{(x+1)^4}{2}+\frac{5}{2}(x+1)^2.$$

However, if the initial condition (x_0, y_0) is chosen so that $x_0=-1$, we will not find the particular solution that satisfies this condition. This is due to the fact that when $x_0=-1$ the function $P(x)=-\frac{2}{x+1}$ is discontinuous and, hence, the conditions of the theorem of the existence of a solution are not observed.

SEC. 8. BERNOULLI'S EQUATION

We consider an equation of the form*)

$$\frac{dy}{dx}+P(x)y=Q(x)y^n, \tag{1}$$

*) This equation results from the problem of the motion of a body provided the resistance of medium F depends on the velocity: $F=\lambda_1 v+\lambda_2 v^n$. The equation of motion will then assume the form $m\frac{dv}{dt}=-\lambda_1 v-\lambda_2 v^n$ or $\frac{dv}{dt}+\frac{\lambda_1}{m}v=-\frac{\lambda_2}{m}v^n$.

where $P(x)$ and $Q(x)$ are continuous functions of x (or constants), and $n \neq 0$ and $n \neq 1$ (otherwise we would have a linear equation).

This equation is called *Bernoulli's equation* and reduces to a linear equation by the following transformation.

Dividing all terms of the equation by y^n, we get

$$y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q. \tag{2}$$

Making the substitution

$$z=y^{-n+1},$$

we have

$$\frac{dz}{dx}=(-n+1)\,y^{-n}\frac{dy}{dx}.$$

Substituting into (2), we get

$$\frac{dz}{dx}+(-n+1)\,Pz=(-n+1)\,Q.$$

This is a linear equation.

Finding its complete integral and substituting the expression y^{-n+1} for z, we get the complete integral of the Bernoulli equation.

Example. Solve the equation

$$\frac{dy}{dx}+xy=x^3y^3. \tag{3}$$

Solution. Dividing all terms by y^3, we have

$$y^{-3}y'+xy^{-2}=x^3. \tag{4}$$

Introducing the new function

$$z=y^{-2},$$

we get

$$\frac{dz}{dx}=-2y^{-3}\frac{dy}{dx}.$$

Substituting into equation (4), we obtain

$$\frac{dz}{dx}-2xz=-2x^3. \tag{5}$$

This is a linear equation.

Let us find its complete integral:

$$z=uv;\ \frac{dz}{dx}=u\frac{dv}{dx}+\frac{du}{dx}v.$$

Put expressions z and $\frac{dz}{dx}$ into (5):

$$u\frac{dv}{dx}+\frac{du}{dx}v-2xuv=-2x^3$$

or

$$u\left(\frac{dv}{dx}-2xv\right)+v\frac{du}{dx}=-2x^3$$

Equate to zero the expression in the brackets:

$$\frac{dv}{dx}-2xv=0;\quad \frac{dv}{v}=2x\,dx;$$

$$\ln v=x^2;\quad v=e^{x^2}.$$

For u we get the equation

$$e^{x^2}\frac{du}{dx}=-2x^3.$$

Separating variables, we have

$$du=-2e^{-x^2}x^3\,dx,\quad u=-2\int e^{-x^2}x^3\,dx+C.$$

Integrating by parts, we find

$$u=x^2e^{-x^2}+e^{-x^2}+C;$$

$$z=uv=x^2+1+Ce^{-x^2}.$$

Consequently, the complete integral of the given equation is

$$y^{-2}=x^2+1+Ce^{-x^2},\quad \text{or}\quad y=\frac{1}{\sqrt{x^2+1+Ce^{-x^2}}}.$$

Note. Just as was done for linear equations, it may be shown that the solution of the Bernoulli equation may be sought in the form of a product of two functions:

$$y=u(x)\,v(x),$$

where $v(x)$ is some nonzero function that satisfies the equation $v'+Pv=0$.

SEC. 9. EXACT DIFFERENTIAL EQUATIONS

Definition. The equation

$$M(x,\ y)\,dx+N(x,\ y)\,dy=0 \tag{1}$$

is called *an exact differential equation* if $M(x, y)$ and $N(x, y)$ are continuous differentiable functions for which the following relationship is fulfilled

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}, \tag{2}$$

and $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous in some region.

Integrating exact differential equations. We shall prove that if the left side of equation (1) is an exact differential, then condi-

tion (2) is fulfilled, and, conversely, if condition (2) is fulfilled the left side of equation (1) is an exact differential of some function $u(x, y)$. That is, equation (1) is an equation of the form

$$du(x, y) = 0 \tag{3}$$

and, consequently, its complete integral is

$$u(x, y) = C.$$

Let us first assume that the left side of (1) is an exact differential of some function $u(x, y)$; that is,

$$M(x, y)\,dx + N(x, y)\,dy = du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy;$$

then

$$M = \frac{\partial u}{\partial x}; \quad N = \frac{\partial u}{\partial y}. \tag{4}$$

Differentiating the first relationship with respect to y, and the second with respect to x, we obtain

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x\,\partial y}; \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y\,\partial x}.$$

Assuming continuity of the second derivatives, we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

that is, (2) is a **necessary** condition for the left side of (1) to be an exact differential of some function $u(x, y)$. We shall show that this condition is also **sufficient:** if (2) is fulfilled then the left side of (1) is an exact differential of some function $u(x, y)$.

From the relation

$$\frac{\partial u}{\partial x} = M(x, y)$$

we find

$$u = \int_{x_0}^{x} M(x, y)\,dx + \varphi(y),$$

where x_0 is the abscissa of any point of the domain of existence of the solution.

When integrating with respect to x we consider y constant, and therefore the arbitrary constant of integration may be dependent on y. Let us choose a function $\varphi(y)$ so that the second of the

relations (4) is fulfilled. To do this, we differentiate *) both sides of the latter equation with respect to y and equate the result to $N(x, y)$:

$$\frac{\partial u}{\partial y}=\int_{x_0}^{x}\frac{\partial M}{\partial y}dx+\varphi'(y)=N(x, y);$$

but since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, we can write

$$\int_{x_0}^{x}\frac{\partial N}{\partial x}dx+\varphi'(y)=N;$$

that is, $N(x, y)\big|_{x_0}^{x}+\varphi'(y)=N(x, y)$

or

$$N(x, y)-N(x_0, y)+\varphi'(y)=N(x, y).$$

Hence,

$$\varphi'(y)=N(x_0, y)$$

or

$$\varphi(y)=\int_{y_0}^{y}N(x_0, y)\,dy+C_1.$$

Thus, the function $u(x, y)$ will have the form

$$u=\int_{x_0}^{x}M(x, y)\,dx+\int_{y_0}^{y}N(x_0, y)\,dy+C_1.$$

Here $P(x_0, y_0)$ is a point in the neighbourhood of which there is a solution of the differential equation (1).

Equating this expression to an arbitrary constant C, we get the complete integral of equation (1):

$$\int_{x_0}^{x}M(x, y)\,dx+\int_{y_0}^{y}N(x_0, y)\,dy=C. \tag{5}$$

---

*) The integral $\int_{x_0}^{x}M(x, y)\,dx$ is dependent on y. To find the derivative of this integral with respect to y, differentiate the integrand with respect to y: $\frac{\partial}{\partial y}\int_{x_0}^{x}M(x, y)\,dx=\int_{x_0}^{x}\frac{\partial M}{\partial y}dx$. This follows from Leibniz' theorem for differentiating a definite integral with respect to a parameter (see Sec. 10, Ch. XI).

Example. Given the equation

$$\frac{2x}{y^3}dx+\frac{y^2-3x^2}{y^4}dy=0.$$

Let us check to see whether this is an exact differential equation.

Denoting

$$M=\frac{2x}{y^3};\ N=\frac{y^2-3x^2}{y^4},$$

we have

$$\frac{\partial M}{\partial y}=-\frac{6x}{y^4};\ \frac{\partial N}{\partial x}=-\frac{6x}{y^4}.$$

For $y\neq 0$, condition (2) is fulfilled. Hence, the left side of this equation is an exact differential of some unknown function $u(x, y)$. Let us find this function.

Since $\frac{\partial u}{\partial x}=\frac{2x}{y^3}$, it follows that

$$u=\int\frac{2x}{y^3}dx+\varphi(y)=\frac{x^2}{y^3}+\varphi(y),$$

where $\varphi(y)$ is an as yet undefined function of y.

Differentiating this relation with respect to y and noting that

$$\frac{\partial u}{\partial y}=N=\frac{y^2-3x^2}{y^4},$$

we find

$$-\frac{3x^2}{y^4}+\varphi'(y)=\frac{y^2-3x^2}{y^4};$$

hence

$$\varphi'(y)=\frac{1}{y^2},\quad \varphi(y)=-\frac{1}{y}+C_1,$$

$$u(x, y)=\frac{x^2}{y^3}-\frac{1}{y}+C_1.$$

Thus the complete integral of the initial equation is

$$\frac{x^2}{y^3}-\frac{1}{y}=C.$$

SEC. 10. INTEGRATING FACTOR

Let the left side of the equation

$$M(x, y)\,dx+N(x, y)\,dy=0 \tag{1}$$

not be an exact differential. It is sometimes possible to choose a function $\mu(x, y)$ such that after multiplying all terms of the equation by it the left side of the equation is converted into an exact differential. The general solution of the equation thus obtained coincides with the general solution of the original equation; the function $\mu(x, y)$ is called the *integrating factor* of equation (1).

In order to find the integrating factor μ, do as follows. Multiply both sides of the given equation by the as yet unknown integrating factor μ:

$$\mu M\,dx+\mu N\,dy=0.$$

For this equation to be an exact differential equation, it is necessary and sufficient that the following relationship be fulfilled:

$$\frac{\partial(\mu M)}{\partial y}=\frac{\partial(\mu N)}{\partial x};$$

that is,

$$\mu\frac{\partial M}{\partial y}+M\frac{\partial\mu}{\partial y}=\mu\frac{\partial N}{\partial x}+N\frac{\partial\mu}{\partial x},$$

or

$$M\frac{\partial\mu}{\partial y}-N\frac{\partial\mu}{\partial x}=\mu\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right).$$

After dividing both sides of the latter equation by μ, we get

$$M\frac{\partial\ln\mu}{\partial y}-N\frac{\partial\ln\mu}{\partial x}=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}. \qquad (2)$$

It is obvious that any function $\mu(x, y)$ that satisfies this equation is the integrating factor of equation (1). Equation (2) is a partial differential equation in the unknown function μ dependent on the two variables x and y. It can be proved that under definite conditions it has an infinitude of solutions and that, consequently, equation (1) has an integrating factor. But in the general case, the problem of finding $\mu(x, y)$ from equation (2) is harder than the original problem of integrating equation (1). Only in certain particular cases does one manage to find the function $\mu(x, y)$.

For instance let equation (1) admit an integrating factor **dependent only on y.** Then

$$\frac{\partial\ln\mu}{\partial x}=0$$

and to find μ we obtain an **ordinary** differential equation

$$\frac{\partial\ln\mu}{\partial y}=\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$$

from which we determine (by a single quadrature) $\ln\mu$, and, hence, μ as well. It is clear that this may be done only if the expression $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ is not dependent on x.

Similarly, if the expression $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{N}$ is not dependent on y but only on x, then it is easy to find an integrating factor that **depends only on x.**

Example. Solve the equation

$$(y+xy^2)\,dx-x\,dy=0.$$

Solution. Here, $M=y+xy^2$; $N=-x$;

$$\frac{\partial M}{\partial y}=1+2xy;\quad \frac{\partial N}{\partial x}=-1;\quad \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}.$$

Thus, the left side of the equation is not an exact differential. Let us see whether this equation allows for an integrating factor dependent only on y or not. Noting that

$$\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}=\frac{-1-1-2xy}{y+xy^2}=-\frac{2}{y},$$

we conclude that the equation permits of an integrating factor dependent only on y. We find it:

$$\frac{\partial \ln \mu}{\partial y}=-\frac{2}{y};$$

whence

$$\ln \mu=-2\ln y,\ \text{i. e.,}\ \mu=\frac{1}{y^2}.$$

After multiplying through by the integrating factor μ, we obtain the equation

$$\left(\frac{1}{y}+x\right)dx-\frac{x}{y^2}\,dy=0$$

as an exact differential equation $\left(\dfrac{\partial M}{\partial y}=\dfrac{\partial M}{\partial x}=-\dfrac{1}{y^2}\right)$. Solving this equation, we find its complete integral:

$$\frac{x}{y}+\frac{x^2}{2}+C=0,$$

or

$$y=-\frac{2x}{x^2+2C}.$$

SEC. 11. THE ENVELOPE OF A FAMILY OF CURVES

Let there be an equation of the form

$$\Phi(x,\ y,\ C)=0, \tag{1}$$

where x and y are variable Cartesian coordinates and C is a parameter that can take on a variety of fixed values.

For each given value of the parameter C, equation (1) defines some **curve** in the xy-plane. Assigning to C all possible values, we obtain a **family of curves** dependent on a single parameter, or using the more common term, a **one-parameter family of curves.** Thus, equation (1) is the equation of a one-parameter family of curves (because it contains only one arbitrary constant).

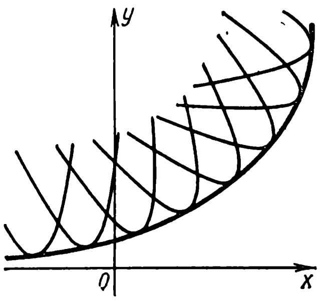


Fig. 250.

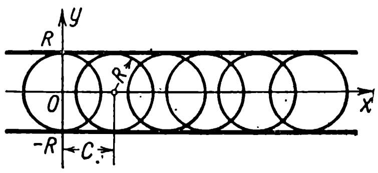


Fig. 251.

Definition. The line L is called the *envelope* of a one-parameter family of lines if at each point it touches some line of the family, and different lines of the given family touch the line L at different points (Fig. 250).

Example 1. Consider the family of lines

$$(x-C)^2+y^2=R^2,$$

where R is a constant and C is a parameter.

This is a family of circles of radius R with centres on the x-axis. This family will obviously have as envelopes the straight lines $y=R$ and $y=-R$ (Fig. 251).

Finding the equation of the envelope of a given family. Let there be given a family of curves,

$$\Phi(x, y, C)=0, \tag{1}$$

that depend on the parameter C.

Let us assume that this family has an envelope whose equation may be written in the form $y=\varphi(x)$, where $\varphi(x)$ is a continuous and differentiable function of x. We consider some point $M(x, y)$ lying on the envelope. This point also lies on some curve of the family (1). To this curve there corresponds a definite value of the parameter C, which value is determined from equation (1), for given (x, y): $C=C(x, y)$. Thus, for all points of the envelope the following equality is fulfilled:

$$\Phi(x, y, C(x, y))=0. \tag{2}$$

Suppose that $C(x, y)$ is a differentiable function that is not constant in any interval of the values of x and y under consideration.

From equation (2) of the envelope we find the slope of the tangent to the envelope at the point $M(x, y)$. Differentiate (2) with respect to x considering that y is a function of x:

$$\frac{\partial \Phi}{\partial x}+\frac{\partial \Phi}{\partial C}\frac{\partial C}{\partial x}+\left[\frac{\partial \Phi}{\partial y}+\frac{\partial \Phi}{\partial C}\frac{\partial C}{\partial y}\right]y'=0$$

or

$$\Phi'_x+\Phi'_y y'+\Phi'_C\left[\frac{\partial C}{\partial x}+\frac{\partial C}{\partial y}y'\right]=0. \tag{3}$$

The slope of the tangent to the curve of the family (1) at the point $M(x, y)$ is found from

$$\Phi'_x+\Phi'_y y'=0 \tag{4}$$

(on this curve, C is constant).

We assume that $\Phi'y \neq 0$, otherwise we would consider x as the function and y as the argument. Since the slope k of the envelope is equal to the slope k of the curve of the family, from (3) and (4) we obtain

$$\Phi'_C\left[\frac{\partial C}{\partial x}+\frac{\partial C}{\partial y}y'\right]=0.$$

But since on the envelope $C(x, y)\neq \text{const}$, it follows that

$$\frac{\partial C}{\partial x}+\frac{\partial C}{\partial y}y'\neq 0,$$

and so for its points the following equation holds:

$$\Phi'_C(x, y, C)=0. \tag{5}$$

Thus, the following two equations serve to determine the envelope:

$$\left.\begin{aligned}\Phi(x, y, C)=0,\\ \Phi'_C(x, y, C)=0.\end{aligned}\right\} \tag{6}$$

Conversely, if, by eliminating C from these equations, we get an equation $y=\varphi(x)$, where $\varphi(x)$ is a differentiable function, and $C\neq \text{const}$ on this curve, then $y=\varphi(x)$ is the equation of the envelope.

Note 1. If for the family (1) a certain function $y=\varphi(x)$ is the equation of the locus of singular points, that is, of points where $\Phi'_x=0$ and $\Phi'_y=0$, then the coordinates of these points also satisfy equations (6).

Indeed, the coordinates of singular points may be expressed in terms of the parameter C that enters into equation (1):

$$x=\lambda(C), \quad y=\mu(C). \tag{7}$$

If these expressions are substituted in equation (1), we get an identity in C:

$$\Phi[\lambda(C),\ \mu(C),\ C]=0.$$

Differentiating this identity with respect to C, we obtain

$$\Phi'_x\frac{d\lambda}{dc}+\Phi'_y\frac{d\mu}{dc}+\Phi'_C=0.$$

Since for any points the equalities $\Phi'_x=0$, $\Phi'_y=0$, are fulfilled, it follows that for them the equality $\Phi'_C=0$ is also fulfilled.

We have thus proved that the coordinate of singular points satisfy equations (6).

Summarising, equations (6) define either the envelope or the locus of singular points of the curves of the family (1), or a combination of both. Thus, after obtaining a curve that satisfies equations (6), one has further to find out whether it is an envelope or the locus of singular points.

Example 2. Find the envelope of the family of circles

$$(x-C)^2+y^2-R^2=0,$$

that are dependent on the single parameter C.

Solution. Differentiating the equation of the family with respect to C, we get

$$2(x-C)=0.$$

Eliminating C from these two equations, we obtain the equation

$$y^2-R^2=0 \quad \text{or} \quad y=\pm R.$$

It is clear, by geometric reasoning, that the pair of straight lines is the **envelope** (and not the locus of singular points, since the circles of a family do not have singular points).

Example 3. Find the envelope of the family of straight lines

$$x\cos\alpha+y\sin\alpha-p=0 \tag{a}$$

where α is a parameter.

Solution. Differentiating the given equation of the family with respect to α, we have

$$-x\sin\alpha+y\cos\alpha=0. \tag{b}$$

To eliminate the parameter α from equations (a) and (b), multiply the terms of the first by $\cos\alpha$, and of the second, by $\sin\alpha$, and then subtract the second from the first; we will then have

$$x=p\cos\alpha.$$

Putting this expression into (b), we find

$$y=p\sin\alpha.$$

Squaring the terms of the two latter equations and adding termwise, we get

$$x^2+y^2=p^2.$$

This is a circle. It is the **envelope** of the family (and not the locus of singular points, since straight lines do not have singular points) (Fig. 252).

Example 4. Find the envelope of the trajectories of shells fired from a gun with velocity v_0 at different angles of inclination of the barrel to the horizon. We shall consider that the gun is located at the coordinate origin and that the trajectories of the shells lie in the xy-plane (air resistance is disregarded).

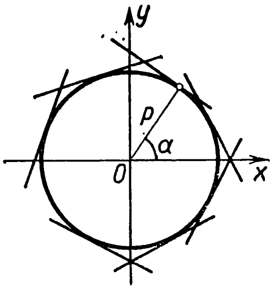


Fig. 252.

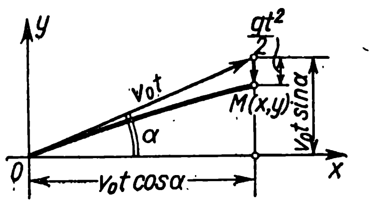


Fig. 253.

Solution. First find the equation of the trajectory of the shell for the case when the barrel makes an angle α with the positive x-axis. In flight, the shell participates simultaneously in two motions: a uniform motion with velocity v_0 in the direction of the barrel and a falling motion due to the force of gravity. Therefore, at each instant of time t the position of the shell M (Fig. 253) will be defined by the equations

$$x = v_0 t \cos\alpha,$$

$$y = v_0 t \sin\alpha - \frac{gt^2}{2}.$$

These are parametric equations of the trajectory (the parameter is the time t). Eliminating t, we get the equation of the trajectory in the form

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha};$$

Finally, introducing the notation $\tan\alpha = k$, $\frac{g}{2v^2} = a$, we get

$$y = kx - ax^2(1 + k^2). \tag{8}$$

This equation defines a parabola with vertical axis passing through the origin and with branches downwards. We obtain a variety of trajectories for the different values of k. Consequently, equation (8) is the equation of a one-parameter family of parabolas, which are the trajectories of a shell for different angles α and for a given initial velocity v_0 (Fig. 254).

Let us find the envelope of this family of parabolas. Differentiating with respect to k both sides of (8), we have

$$x - 2akx^2 = 0. \tag{9}$$

Eliminating k from equations (8) and (9), we get

$$y = \frac{1}{4a} - ax^2.$$

This is the equation of a parabola with vertex at the point $\left(0, \frac{1}{4a}\right)$, the axis of which coincides with the y-axis. It is not a locus of singular points [since parabolas (8) do not have singular points]. Thus, the parabola

$$y=\frac{1}{4a}-ax^2$$

is the envelope of the family of trajectories. It is called a **safety parabola** because no point outside it is in reach of a shell fired from a given gun with a given initial velocity v_0.

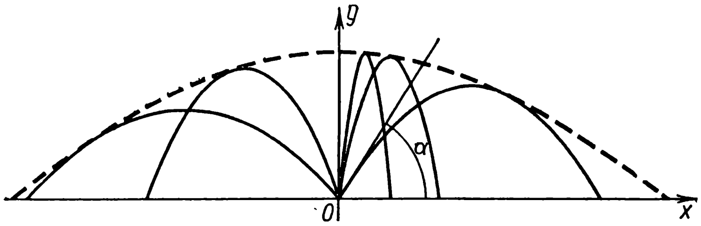


Fig. 254.

Example 5. Find the envelope of a family of semicubical parabolas

$$y^3-(x-C)^2=0.$$

Solution. Differentiate the given equation of the family with respect to the parameter C:

$$2(x-C)=0.$$

Eliminating the parameter C from the two equations, we get

$$y=0.$$

The x-axis is a locus of singular points—a cusp of the first kind (Fig. 255). Indeed, let us find the singular points of the curve

$$y^3-(x-C)^2=0$$

for a fixed value of C. Differentiating with respect to x and y, we find

$$F'_x=-2(x-C)=0;$$

$$F'_y=3y^2=0.$$

Solving the three foregoing equations simultaneously, we find the coordinates of the singular point: $x=C$, $y=0$; thus, each curve of the given family has a singular point on the x-axis.

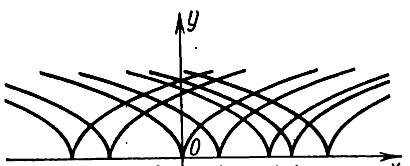


Fig. 255.

For continuous variation of the parameter C, the singular points will fill the entire x-axis.

Example 6. Find the envelope and locus of singular points of the family

$$(y-C)^2-\frac{2}{3}(x-C)^3=0. \qquad (10)$$

Solution. Differentiating both sides of (10) with respect to C, we find

$$-2(y-C)+\frac{2}{3}3(x-C)^2=0$$

or

$$y-C-(x-C)^2=0. \qquad (11)$$

Now eliminate the parameter C from (11) and from the equation (10) of the family:

$$y-C=(x-C)^2.$$

Putting the expression $y-C$ into the equation of the family, we get

$$(x-C)^4-\frac{2}{3}(x-C)^3=0$$

or

$$(x-C)^3\left[(x-C)-\frac{2}{3}\right]=0,$$

whence we obtain two possible values of C and two solutions of the problem corresponding to them.

First Solution:

$$C=x,$$

and so from (11) we find

$$y-x-(x-x)^2=0$$

or

$$y=x.$$

Second Solution:

$$C=x-\frac{2}{3}$$

and so from (11) we find

$$y-x+\frac{2}{3}-\left[x-x+\frac{2}{3}\right]^2=0$$

or

$$y=x-\frac{2}{9}.$$

We have obtained two straight lines: $y=x$ and $y=x-\frac{2}{9}$. The first is a locus of singular points, the second is an envelope (Fig. 256).

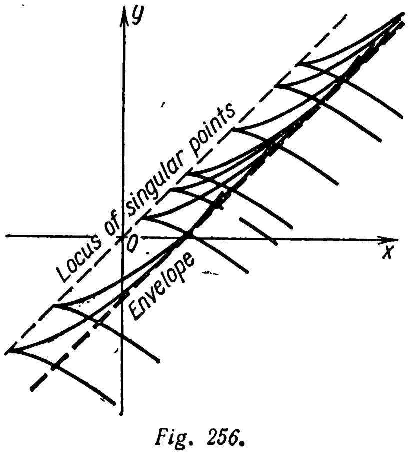


Fig. 256.

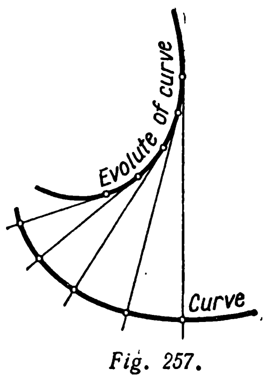


Fig. 257.

Note 2. In Sec. 7, Ch. VI, it was proved that the normal to a curve serves as a tangent to its evolute. Hence, the family of normals to a given curve is at the same time a family of tangents to its evolute. Thus, *the evolute of the curve is the envelope of the family of normals of this curve* (Fig. 257).

This remark enables us to point out another method for finding evolutes: to obtain the equation of an evolute, first find the family of all normals of the given curve and then find the envelope of this family.

SEC. 12. SINGULAR SOLUTIONS OF A FIRST-ORDER DIFFERENTIAL EQUATION

Let the differential equation

$$F\left(x,\ y,\ \frac{dy}{dx}\right)=0 \tag{1}$$

have a complete integral

$$\Phi(x,\ y,\ C)=0. \tag{2}$$

Let us assume that the family of integral curves that corresponds to equation (2) has an envelope. We shall prove that this envelope is also an integral curve of the differential equation (1).

Indeed, at each point the envelope touches some curve of the family; that is, it has a common tangent with it. Thus, at each common point the envelope and the curve of the family have the same values of x, y, y'.

But for a curve of the family, the numbers x, y, and y' satisfy equation (1). Consequently, the very same equation is satisfied by the abscissa, the ordinate and the slope of each point of the envelope. But this means that the envelope is an integral curve and its equation is a solution of the given differential equation.

Since, generally speaking, the envelope is not the curve of the family, its equation cannot be obtained from the complete integral (2) for any particular value of C. The solution of the differential equation which is not obtained from the complete integral for any value of C and which has as its graph the envelope of a family of integral curves entering into the general solution, is called a *singular solution* of the differential equation.

Let the complete integral be known:

$$\Phi(x,\ y,\ C)=0;$$

eliminating C from this equation and from the equation $\Phi'_C(x, y, C)=0$ we get $\psi(x, y)=0$. If this function satisfies the differential equation [and does not belong to the family (2)], then it is a singular integral.

It should be noted that at least two integral curves pass through each point of the curve that describes a singular solution; that is, *uniqueness of solution is violated at each point of a singular solution.*

Example. Find a singular solution of the equation

$$y^2(1+y'^2)=R^2.$$

Solution. Let us find its complete integral. We solve the equation for y':

$$\frac{dy}{dx}=\pm\frac{\sqrt{R^2-y^2}}{y}. \qquad (*)$$

Separating variables, we obtain

$$\frac{y\,dy}{\pm\sqrt{R^2-y^2}}=dx.$$

Whence, integrating, we find the complete integral:

$$(x-C)^2+y^2=R^2.$$

It is easy to see that the family of integral lines is a family of circles of radius R with centres on the x-axis. The pair of straight lines $y=\pm R$ will be the envelope of the family of curves.

The functions $y=\pm R$ satisfy the differential equation (1). This, consequently, is a singular integral.

SEC. 13. CLAIRAUT'S EQUATION

Let us consider the so-called *Clairaut equation:*

$$y=x\frac{dy}{dx}+\psi\left(\frac{dy}{dx}\right). \qquad (1)$$

It is integrated by introducing an auxiliary parameter. Put $\frac{dy}{dx}=p$; then equation (1) will take the form

$$y=xp+\psi(p). \qquad (1')$$

Differentiate, with respect to x, all the terms of this equation, bearing in mind that $p=\frac{dy}{dx}$ is a function of x:

$$p=x\frac{dp}{dx}+p+\psi'(p)\frac{dp}{dx}$$

or

$$[x+\psi'(p)]\frac{dp}{dx}=0.$$

Equating each factor to zero, we get

$$\frac{dp}{dx}=0 \qquad (2)$$

and

$$x+\psi'(p)=0. \tag{3}$$

1) Integrating (2) we obtain $p=C$ ($C=\text{const}$). Putting this value of p into (1′), we find its complete integral:

$$y=xC+\psi(C), \tag{4}$$

which, geometrically, is a **family of straight lines.**

2) If from (3) we find p as a function of x and put it into (1′), we obtain the function

$$y=xp(x)+\psi[p(x)], \tag{1''}$$

which may be readily shown to be the solution of equation (1).

Indeed, by virtue of (3) we have

$$\frac{dy}{dx}=p+[x+\psi'(p)]\frac{dp}{dx}=p.$$

And so, by substituting the function (1″) into equation (1) we get the identity

$$xp+\psi(p)=xp+\psi(p).$$

The solution of (1″) is not obtained from the complete integral (4) for any value of C. This is a **singular solution**; it is obtained by elimination of the parameter p from the equations

$$\left.\begin{array}{l} y=xp+\psi(p), \\ x+\psi'(p)=0, \end{array}\right\}$$

or, which is the same thing, by eliminating C from the equations

$$y=xC+\psi(C),$$

$$x+\psi'_C(C)=0.$$

Thus, *the singular solution of Clairaut's equation defines the envelope of a family of straight lines represented by the complete integral (4).*

Example. Find the general and singular solutions of the equation

$$y=x\frac{dy}{dx}+\frac{a\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}.$$

Solution. The general solution is obtained by substituting C for $\frac{dy}{dx}$:

$$y=xC+\frac{aC}{\sqrt{1+C^2}}.$$

To obtain the singular solution, differentiate the latter equation with respect to C:

$$x+\frac{a}{(1+C^2)^{\frac{3}{2}}}=0.$$

The singular solution (the equation of the envelope) is obtained in parametric form (where the parameter is C):

$$\begin{cases} x=-\dfrac{a}{(1+C^2)^{\frac{3}{2}}}, \\ y=\dfrac{aC^3}{(1+C^2)^{\frac{3}{2}}}. \end{cases}$$

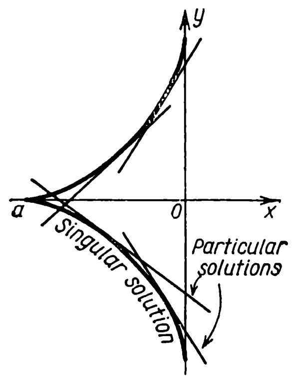


Fig. 258.

Eliminating C, we get a direct relationship between x and y. Raising both sides of each equation to the power $\frac{2}{3}$ and adding the resultant equations termwise, we get the singular solution in the following form:

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

This is an astroid. However, the envelope of the family (and, hence, the singular solution) is not the entire astroid, but only its left half (since it is evident from the parametric equations that $x\leqslant 0$) (Fig. 258).

SEC. 14. LAGRANGE'S EQUATION

The *Lagrange equation* is an equation of the form

$$y=x\varphi(y')+\psi(y') \tag{1}$$

where φ and ψ are known functions of $\frac{dy}{dx}$.

This equation is linear in y and x. Clairaut's equation, which was considered in the preceding section, is a particular case of the Lagrange equation when $\varphi(y')\equiv y'$. The Lagrange equation, like Clairaut's, is integrated by means of introducing an auxiliary parameter p. Put

$$y'=p;$$

then the initial equation is written in the form

$$y=x\varphi(p)+\psi(p). \tag{1'}$$

Differentiating with respect to x, we obtain

$$p=\varphi(p)+[x\varphi'(p)+\psi'(p)]\frac{dp}{dx}$$

or

$$p-\varphi(p)=[x\varphi'(p)+\psi'(p)]\frac{dp}{dx}. \qquad (1'')$$

From this equation we can straightway find certain solutions: namely, it becomes an identity for any constant value $p=p_0$ that satisfies the condition

$$p_0-\varphi(p_0)=0.$$

Indeed, for a constant value p the derivative $\frac{dp}{dx}\equiv 0$, and both sides of equation (1″) vanish.

The solution corresponding to each value $p=p_0$, that is, $\frac{dy}{dx}=p_0$ is a **linear** function of x $\left(\text{since the derivative } \frac{dy}{dx} \text{ is constant only in the case of linear functions}\right)$. To find this function it is sufficient to put into (1′) the value $p=p_0$:

$$y=x\varphi(p_0)+\psi(p_0).$$

If it turns out that this solution is not obtainable from the general solution for any value of the arbitrary constant, it will be a **singular solution.**

Let us now find the **general solution.** Write (1″) in the form

$$\frac{dx}{dp}-x\frac{\varphi'(p)}{p-\varphi(p)}=\frac{\psi'(p)}{p-\varphi(p)}$$

and regard x as a function of p. Then the equation obtained will be a linear differential equation in the function x of p.

Solving it, we find

$$x=\omega(p, C). \qquad (2)$$

Eliminating the parameter p from equations (1′) and (2), we get the **complete integral** (1) in the form $\Phi(x, y, C)=0$.

Example. Given the equation

$$y=xy'^2+y'^2. \qquad (\mathrm{I})$$

Putting $y'=p$ we have

$$y=xp^2+p^2. \qquad (\mathrm{I}')$$

Differentiating with respect to x, we get

$$p=p^2+[2xp+2p]\frac{dp}{dx}. \qquad (\mathrm{I}'')$$

Let us find the **singular solutions.** Since $p=p^2$ for $p_0=0$ and $p_1=1$, the solutions will be linear functions [see (I′)]:

$$y=x\cdot 0^2+0^2, \text{ that is, } y=0,$$

and

$$y=x+1.$$

When we find the complete integral, we will see whether these functions are particular or singular solutions. To find it, write equation (I″) in the form

$$\frac{dx}{dp}-x\frac{2p}{p-p^2}=\frac{2}{1-p}$$

and we shall regard x as a function of the independent variable p. Integrating this linear (in x) equation, we find

$$x=-1+\frac{C^2}{(p-1)^2}. \tag{II}$$

Eliminating p from equations (I′) and (II), we get the **complete integral**

$$y=(C+\sqrt{x+1})^2.$$

The **singular integral** of the initial equation is

$$y=0$$

since this solution is not obtainable from the general solution for any value of C.

However, the function $y=x+1$ is not a singular but a particular solution; it is obtained from the general solution when $C=0$.

SEC. 15. ORTHOGONAL AND ISOGONAL TRAJECTORIES

Suppose we have a one-parameter family of curves

$$\Phi(x, y, C)=0. \tag{1}$$

Lines intersecting all the curves of the given family (1) at a constant angle are called *isogonal trajectories*. If this angle is a right angle, they are *orthogonal trajectories*.

Orthogonal trajectories. Let us find the equation of orthogonal trajectories. Write the differential equation of the given family of curves, eliminating the parameter C from the equations

$$\Phi(x, y, C)=0$$

and

$$\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial y}\frac{dy}{dx}=0.$$

Let this differential equation be

$$F\left(x, y, \frac{dy}{dx}\right)=0. \tag{1′}$$

Here, $\frac{dy}{dx}$ is the slope of the tangent to some member of the family at the point $M(x, y)$. Since an orthogonal trajectory passing through the point $M(x, y)$ is perpendicular to the corresponding curve of the family, the slope of the tangent to it, $\frac{dy_T}{dx}$, is

connected with $\frac{dy}{dx}$ by the relationship (Fig. 259)

$$\frac{dy}{dx} = -\frac{1}{\frac{dy_T}{dx}}. \tag{2}$$

Putting this expression into equation (1′) and dropping the subscript T, we get a relationship between the coordinates of an arbitrary point (x, y) and the slope of the orthogonal trajectory at this point, that is, **a differential equation of orthogonal trajectories:**

$$F\left(x, y, -\frac{1}{\frac{dy}{dx}}\right) = 0. \tag{3}$$

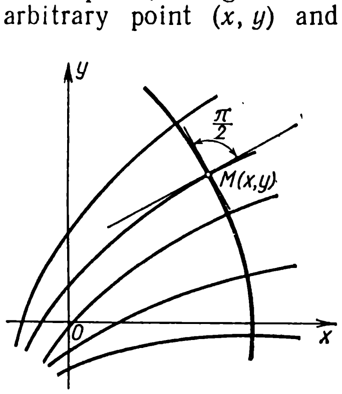


Fig. 259.

The complete integral of this equation

$$\Phi_1(x, y, C) = 0$$

yields a **family of orthogonal trajectories.**

A consideration of the plane flow of a fluid involves orthogonal trajectories.

Let us suppose that the fluid flow in a plane takes place in such manner that at each point of the xy-plane the velocity vector, $\boldsymbol{v}(x, y)$, of motion is defined. If this vector depends solely on the position of the point in the plane, but is independent of the time, the motion is called stationary or steady-state. We shall consider such motion. In addition, we shall assume that there exists a potential of velocities, that is, a function $u(x, y)$ such that the projections of the vector $\boldsymbol{v}(x, y)$ on the coordinate axis $v_x(x, y)$ and $v_y(x, y)$ are its partial derivatives with respect to x and y:

$$\frac{\partial u}{\partial x} = v_x \quad \frac{\partial u}{\partial y} = v_y. \tag{4}$$

The lines of the family

$$u(x, y) = C \tag{5}$$

are called *equipotential lines* (lines of equal potential).

The lines, the tangents to which at all points coincide with the vector $\boldsymbol{v}(x, y)$ in direction, are called *flow lines* and yield the trajectories of moving points.

We shall show that the flow lines are the orthogonal trajectories of a family of equipotential lines (Fig. 260).

Let φ be an angle formed by the velocity vector $\boldsymbol{v}$ with the x-axis. Then by relation (4)

$$\frac{\partial u(x, y)}{\partial x} = |\boldsymbol{v}| \cos\varphi; \quad \frac{\partial u(x, y)}{\partial y} = |\boldsymbol{v}| \sin\varphi,$$

whence we find the slope of the tangent to the flow line

$$\tan\varphi = \frac{\dfrac{\partial u(x, y)}{\partial y}}{\dfrac{\partial u(x, y)}{\partial x}}. \tag{6}$$

We obtain the slope of the tangent to the equipotential line by differentiating, with respect to x, relation (5):

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0,$$

whence

$$\frac{dy}{dx} = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}}. \tag{7}$$

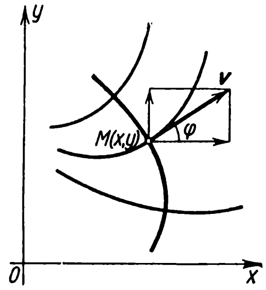


Fig. 260.

Thus, in magnitude and sign, the slope of the tangent to the equipotential line is the inverse of the slope of the tangent to the flow line. Whence it follows that equipotential lines and flow lines are mutually orthogonal.

In the case of an electric or magnetic field, the lines of force of the field serve as the orthogonal trajectories of the family of equipotential lines.

Example 1. Find the orthogonal trajectories of the family of parabolas

$$y = Cx^2.$$

Solution. Write the differential equation of the family

$$y' = 2Cx.$$

Eliminating C, we get

$$\frac{y'}{y} = \frac{2}{x}.$$

Substituting $-\frac{1}{y'}$ for y', we obtain a differential equation of the family of orthogonal trajectories

$$-\frac{1}{yy'} = \frac{2}{x}$$

or

$$y\,dy = -\frac{x\,dx}{2}.$$

Its complete integral is

$$\frac{x^2}{4} + \frac{y^2}{2} = C^2.$$

Hence, the orthogonal trajectories of the given family of parabolas will be represented by a certain family of ellipses with semi-axes $a = 2C$, $b = C\sqrt{2}$ (Fig. 261).

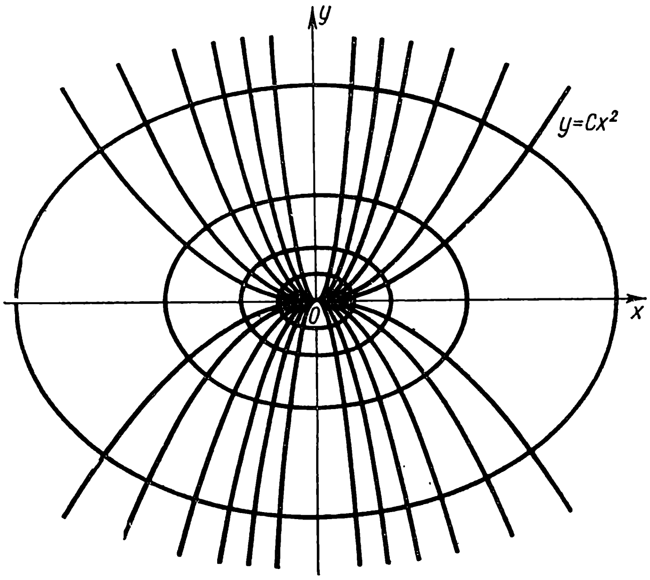


Fig. 261.

Isogonal trajectories. Let the trajectories cut the curves of a given family at an angle α, where $\tan\alpha = k$.

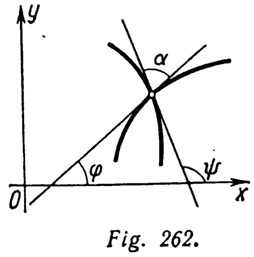


Fig. 262.

The slope $\frac{dy}{dx} = \tan\varphi$ (Fig. 262) of the tangent to a member of the family and the slope $\frac{dy_T}{dx} = \tan\psi$ to the isogonal trajectory are connected by the relationship

$$\tan\varphi = \tan(\psi - \alpha) = \frac{\tan\psi - \tan\alpha}{1 + \tan\alpha\tan\psi};$$

that is,

$$\frac{dy}{dx}=\frac{\frac{dy_T}{dx}-k}{k\frac{dy_T}{dx}+1}. \qquad (2')$$

Substituting this expression into equation (1′) and dropping the subscript T, we obtain the differential equation of isogonal trajectories.

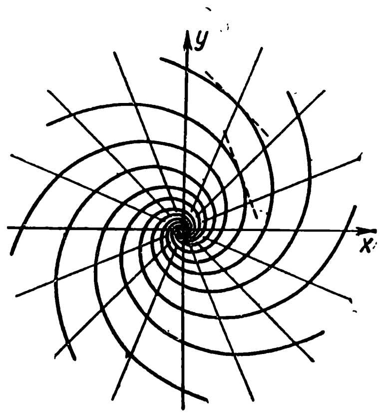


Fig. 263.

Example 2. Find the isogonal trajectories of a family of straight lines,

$$y=Cx, \qquad (8)$$

that cut the lines of the given family at an angle α, the tangent of which equals k: $\tan\alpha=k$.

Solution. Let us write the differential equation of the given family. Differentiating equation (8) with respect to x, we find

$$\frac{dy}{dx}=C.$$

On the other hand, from the same equation we have

$$C=\frac{y}{x}.$$

Consequently, the differential equation of the given family is of the form

$$\frac{dy}{dx}=\frac{y}{x}.$$

Utilising relationship (2′) we get the differential equation of isogonal trajectories

$$\frac{\frac{dy_T}{dx}-k}{k\frac{dy_T}{dx}+1}=\frac{y}{x}.$$

Whence, dropping the subscript T, we find

$$\frac{dy}{dx}=\frac{k+\frac{y}{x}}{1-k\frac{y}{x}}.$$

Integrating this homogeneous equation, we get the complete integral:

$$\ln\sqrt{x^2+y^2}=\frac{1}{k}\arctan\frac{y}{x}+\ln C, \qquad (9)$$

which defines the family of isogonal trajectories. To find out precisely which

curves enter into this family, let us change to polar coordinates:

$$\frac{y}{x}=\tan\varphi;\quad \sqrt{x^2+y^2}=\varrho.$$

Substituting these expressions into (9) we obtain

$$\ln\varrho=\frac{1}{k}\varphi+\ln C$$

or

$$\varrho=Ce^{\frac{\varphi}{k}}.$$

Consequently, the family of isogonal trajectories is a family of logarithmic spirals (Fig. 263).

SEC. 16. HIGHER-ORDER DIFFERENTIAL EQUATIONS (FUNDAMENTALS)

As has already been indicated above (see Sec. 2), a differential equation of the nth order may be written symbolically in the form

$$F(x, y, y', y'', \ldots, y^{(n)})=0 \tag{1}$$

or, if it can be solved for the nth derivative,

$$y^{(n)}=f(x, y, y', y'', \ldots, y^{(n-1)}). \tag{1'}$$

In this chapter we shall consider only such equations of higher order that may be solved for a higher derivative. For these equations we have a theorem on the existence and uniqueness of a solution, similar to the corresponding theorem on the solution of first-order equations.

Theorem. *If in the equation*

$$y^{(n)}=f(x, y, y', \ldots, y^{(n-1)})$$

the function $f(x, y, y', \ldots, y^{(n-1)})$ *and its partial derivatives with respect to the arguments* $y, y', \ldots, y^{(n-1)}$ *are continuous in some region containing the values* $x=x_0$, $y=y_0$, $y'=y'_0$, $\ldots$, $y^{(n-1)}=y_0^{(n-1)}$, *then there is one and only one solution,* $y=y(x)$, *of the equation that satisfies the conditions*

$$\left.\begin{array}{l} y_{x=x_0}=y_0, \\ y'_{x=x_0}=y'_0, \\ \cdots\cdots\cdots \\ y^{(n-1)}_{x=x_0}=y_0^{(n-1)}. \end{array}\right\} \tag{2}$$

These conditions are called *initial conditions.* The proof is beyond the scope of this book.

If we consider a second-order equation $y''=f(x, y, y')$, then the initial conditions for the solution, when $x=x_0$, will be

$$y=y_0, \quad y'=y'_0$$

where x_0, y_0, y'_0 are given numbers, which have the following geometric meaning: only one curve passes through a given point of the plane (x_0, y_0) with given tangent of the angle of inclination of the tangent line y'_0. From this it follows that if we want to assign different values of y'_0 for constant x_0 and y_0, we get an infinitude of integral curves with different angles of inclination passing through the given point.

We now introduce the concept of a general solution of an equation of the nth order.

Definition. The *general solution* of a differential equation of the nth order is the function

$$y=\varphi(x, C_1, C_2, \ldots, C_n),$$

which is dependent on n arbitrary constants C_1, C_2, ..., C_n and such that:

a) it satisfies the equation for any values of the constants C_1, C_2, ..., C_n;

b) for specified initial conditions

$$y_{x=x_0}=y_0,$$
$$y'_{x=x_0}=y'_0,$$
$$\cdots\cdots$$
$$y^{(n-1)}_{x=x_0}=y^{(n-1)}_0$$

the constants C_1, C_2, ..., C_n may be chosen so that the function $y=\varphi(x, C_1, C_2, \ldots, C_n)$ will satisfy these conditions (on the assumption that the initial values x_0, y_0, y'_0, ..., $y_0^{(n-1)}$ belong to the region where the conditions of the existence of a solution are fulfilled).

A relationship of the form $\Phi(x, y, C_1, C_2, \ldots, C_n)=0$, which implicitly defines the general solution, is called the *complete integral* of the differential equation.

Any function obtained from the general solution for specific values of the constants C_1, C_2, ..., C_n is called a *particular solution*. The graph of a particular solution is called an *integral curve* of the given differential equation.

To solve (integrate) a differential equation of the nth order means:

1) to find its general solution (if the initial conditions are not given) or

2) to find a particular solution of the equation that satisfies the given initial conditions (if there are such).

In the following sections we shall present methods of solving various equations of the nth order.

SEC. 17. AN EQUATION OF THE FORM $y^{(n)}=f(x)$

The simplest type of equation of the nth order is of the form

$$y^{(n)}=f(x). \tag{1}$$

Let us find the complete integral of this equation.

Integrating the left and right sides with respect to x, and taking into account that $y^{(n)}=(y^{(n-1)})'$, we obtain

$$y^{(n-1)}=\int_{x_0}^{x} f(x)\,dx+C_1,$$

where x_0 is any fixed value of x, and C_1 is the constant of integration.

Integrating once more we get

$$y^{(n-2)}=\int_{x_0}^{x}\left(\int_{x_0}^{x} f(x)\,dx\right)dx+C_1(x-x_0)+C_2.$$

Continuing, we finally get (after n integrations) the expression of the complete integral:

$$y=\int_{x_0}^{x}\ldots\int_{x_0}^{x} f(x)\,dx\ldots dx+\frac{C_1(x-x_0)^{n-1}}{(n-1)!}+C_2\frac{(x-x_0)^{n-2}}{(n-2)!}+\ldots+C_n.$$

In order to find a particular solution satisfying the initial conditions

$$y_{x=x_0}=y_0;\quad y'_{x=x_0}=y'_0;\quad \ldots;\quad y^{(n-1)}_{x=x_0}=y_0^{(n-1)},$$

it is sufficient to put

$$C_n=y_0,\quad C_{n-1}=y'_0,\quad \ldots,\quad C_1=y_0^{n-1}.$$

Example 1. Find the complete integral of the equation

$$y''=\sin(kx)$$

and a particular solution satisfying the initial conditions

$$y_{x=0}=0,\quad y'_{x=0}=1.$$

Solution.

$$y'=\int_0^x \sin kx\,dx+C_1=-\frac{\cos kx-1}{k}+C_1,$$

$$y=-\int_0^x\left(\frac{\cos kx-1}{k}\right)dx+\int_0^x C_1\,dx+C_2$$

or

$$y=-\frac{\sin kx}{k^2}+\frac{x}{k}+C_1x+C_2.$$

This is the complete integral. To find a particular solution satisfying the given initial conditions, it is sufficient to determine the corresponding values of C_1 and C_2.

From the condition $y_{x=0}=0$, we find $C_2=0$.

From the condition $y'_{x=0}=1$, we find $C_1=0$.

Thus, the desired particular solution is of the form

$$y=-\frac{\sin kx}{k^2}+x\left(\frac{1}{k}+1\right).$$

Differential equations of this kind are encountered in the theory of the bending of girders.

Example 2. Let us consider an elastic prismatic girder bending under the action of external forces both continuously distributed (weight, load), and concentrated. Let the x-axis be horizontal along the axis of the girder in its underformed state and let the y-axis be directed vertically downwards (Fig. 264).

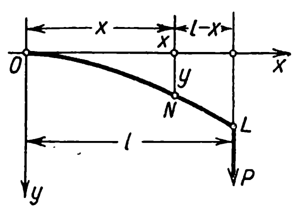


Fig. 264.

Each force acting on the girder (the load of the girder, and the reaction of the supports, for instance) has a moment, relative to some cross section of the girder, equal to the product of the force by the distance of the point of application of the force from the given cross section. The sum, $M(x)$, of the moments of all the forces applied to that part of the girder situated to one side of the given cross section with abscissa x is called the bending moment of the girder relative to the given cross section. In courses of strength of materials, it is proved that the bending moment of the girder is

$$\frac{EJ}{R},$$

where E is the so-called modulus of elasticity which depends on the material of the girder, J is the moment of inertia of the cross-sectional area of the girder relative to the horizontal line passing through the centre of gravity of the cross-sectional area, and R is the radius of curvature of the axis of the bent girder, which radius is expressed by the formula (Sec. 6, Ch. VI)

$$R=\frac{(1+y'^2)^{3/2}}{y''}.$$

Thus, the differential equation of the bent axis of a girder has the form

$$\frac{y''}{(1+y'^2)^{3/2}}=\frac{M(x)}{EJ}. \tag{2}$$

If we consider that the deformations are small and that the tangents to the axis of the girder, when bent, form a small angle with the x-axis, we can disregard the square of the small quantity y'^2 and consider

$$R=\frac{1}{y''}.$$

Then the differential equation of the bent girder will have the form

$$y''=\frac{M(x)}{EJ}, \tag{2'}$$

but this equation is of the form of (1).

Example 3. A girder is fixed in place at the extremity O and is subjected to the action of a concentrated vertical force P applied to the end of the girder L at a distance l from O (Fig. 264). The weight of the girder is ignored.

We consider a cross section at the point $N(x)$. The bending moment relative to section N is, in the given case, equal to

$$M(x)=(l-x)P.$$

The differential equation (2′) has the form

$$y''=\frac{P}{EJ}(l-x).$$

The initial conditions are: for $x=0$ the deflection y is equal to zero and the tangent to the bent axis of the girder coincides with the x-axis; that is,

$$y_{x=0}=0, \quad y'_{x=0}=0.$$

Integrating the equation, we find

$$y'=\frac{P}{EJ}\int_0^x (l-x)\,dx=\frac{P}{EJ}\left(lx-\frac{x^2}{2}\right);$$

$$y=\frac{P}{2EJ}\left(lx^2-\frac{x^3}{3}\right). \tag{3}$$

In particular, from formula (3) we determine the deflection h at the extremity of the girder L:

$$h=y_{x=l}=\frac{Pl^3}{3EJ}.$$

SEC. 18. SOME TYPES OF SECOND-ORDER DIFFERENTIAL EQUATIONS REDUCIBLE TO FIRST-ORDER EQUATIONS

I. *An equation of the type*

$$\frac{d^2y}{dx^2}=f\left(x, \frac{dy}{dx}\right) \tag{1}$$

does not explicitly contain the unknown function y.

Solution. Let us denote the derivative $\frac{dy}{dx}$ in terms of p, that is, we set $\frac{dy}{dx}=p$. Then $\frac{d^2y}{dx^2}=\frac{dp}{dx}$.

Putting these expressions of the derivatives into equation (1), we get a first-order equation,

$$\frac{dp}{dx}=f(x,\ p),$$

in the unknown function p of x. Integrating this equation, we find its general solution:

$$p=p(x,\ C_1),$$

and then from the relation $\frac{dy}{dx}=p$ we get the complete integral of equation (1):

$$y=\int p(x,\ C_1)\,dx+C_2.$$

Example 1. Let us consider the differential equation of a catenary (see Sec. 1):

$$\frac{d^2y}{dx^2}=\frac{1}{a}\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Set

$$\frac{dy}{dx}=p;$$

then

$$\frac{d^2y}{dx^2}=\frac{dp}{dx},$$

and we get a first-order differential equation in the auxiliary function p of x:

$$\frac{dp}{dx}=\frac{1}{a}\sqrt{1+p^2}.$$

Separating variables, we have

$$\frac{dp}{\sqrt{1+p^2}}=\frac{dx}{a},$$

whence

$$\ln(p+\sqrt{1+p^2})=\frac{x}{a}+C_1,$$

$$p=\frac{1}{2}\left(e^{\frac{x}{a}+C_1}-e^{-\left(\frac{x}{a}-C_1\right)}\right).$$

But since $p=\frac{dy}{dx}$, the latter relation is a differential equation in the sought-for function y. Integrating it, we obtain the equation of a catenary (see

Sec. 1):

$$y=\frac{a}{2}\left(e^{\frac{x}{a}+C_1}+e^{-\left(\frac{x}{a}+C_1\right)}\right)+C_2.$$

Let us find the particular solution that satisfies the following initial conditions:

$$y_{x=0}=a,$$
$$y'_{x=0}=0.$$

The first condition yields $C_2=0$, the second, $C_1=0$.
We finally obtain

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right).$$

Note. We can similarly integrate the equation

$$y^{(n)}=f(x,\ y^{(n-1)}).$$

Setting $y^{(n-1)}=p$, we get for a determination of p the first-order equation

$$\frac{dp}{dx}=f(x,\ p).$$

From here we get p as a function of x, and from the relation $y^{(n-1)}=p$ we find y (see Sec. 17).

II. *An equation of the type.*

$$\frac{d^2y}{dx^2}=f\left(y,\ \frac{dy}{dx}\right) \tag{2}$$

does not contain the independent variable x explicitly. To solve it, we again set

$$\frac{dy}{dx}=p, \tag{3}$$

but now *we shall consider p as a function of y* (and not of x, as before). Then

$$\frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\frac{dy}{dx}=\frac{dp}{dy}p.$$

Putting into (2) the expressions $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, we get a first-order equation in the auxiliary function p:

$$p\frac{dp}{dy}=f(y,\ p). \tag{4}$$

Integrating it, we find p as a function of y and the arbitrary constant C_1:

$$p=p(y,\ C_1).$$

Substituting this value in (3), we get a first-order differential equation for the function y of x:

$$\frac{dy}{dx}=p(y,\ C_1).$$

Separating variables, we have

$$\frac{dy}{p(y,\ C_1)}=dx.$$

Integrating this equation, we get the complete integral of the initial equation:

$$\Phi(x,\ y,\ C_1,\ C_2)=0.$$

Example 2. Find the complete integral of the equation

$$3y''=y^{-\frac{5}{3}}.$$

Solution. Put $p=\frac{dy}{dx}$ and consider p as a function of y. Then $y''=p\frac{dp}{dy}$ and we get a first-order equation for the auxiliary function p:

$$3p\frac{dp}{dy}=y^{-\frac{5}{3}}.$$

Integrating this equation, we find

$$p^2=C_1-y^{-\frac{2}{3}} \quad \text{or} \quad p=\pm\sqrt{C_1-y^{-2/3}}.$$

But $p=\frac{dy}{dx}$; consequently, for a determination of y we get the equation

$$\pm\frac{dy}{\sqrt{C_1-y^{-2/3}}}=dx, \quad \text{or} \quad \frac{y^{1/3}dy}{\pm\sqrt{C_1y^{2/3}-1}}=dx,$$

whence

$$x+C_2=\pm\int\frac{y^{1/3}dy}{\sqrt{C_1y^{2/3}-1}}.$$

To compute the latter integral we make the substitution

$$C_1y^{2/3}-1=t^2.$$

Then

$$y^{1/3}=(t^2+1)^{\frac{1}{2}}\frac{1}{C_1^{1/2}};$$

$$dy=3t\,(t^2+1)^{1/2}\frac{1}{C_1^{3/2}}\,dt.$$

Consequently,

$$\int \frac{y^{1/3}\,dy}{\sqrt{C_1 y^{2/3}-1}} = \frac{1}{C_1^2}\int \frac{3t\,(t^2+1)}{t}\,dt = \frac{3}{C_1^2}\left(\frac{t^3}{3}+t\right) =$$

$$= \frac{1}{C_1^2}\sqrt{C_1 y^{2/3}-1}\,(C_1 y^{2/3}+2).$$

Finally we get

$$x + C_2 = \pm \frac{1}{C_1^2}\sqrt{C_1 y^{2/3}-1}\,(C_1 y^{2/3}+2).$$

Example 3. Let a point move along the x-axis under the action of a force that depends solely on the position of the point. The differential equation of motion will be

$$m\frac{d^2x}{dt^2} = F(x).$$

At $t=0$ let $x=x_0$, $\frac{dx}{dt}=v_0$.

Multiplying both sides of the equation by $\frac{dx}{dt}dt$ and integrating from 0 to t, we have

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - \frac{1}{2}mv_0^2 = \int_{x_0}^{x} F(x)\,dx$$

or

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \left[-\int_{x_0}^{x} F(x)\,dx\right] = \frac{1}{2}mv_0^2 = \text{const}.$$

The first term of this equation is the kinetic energy, the second term, the potential energy of the moving point. From this equation if follows that the sum of the kinetic and potential energy remains constant throughout the time of motion.

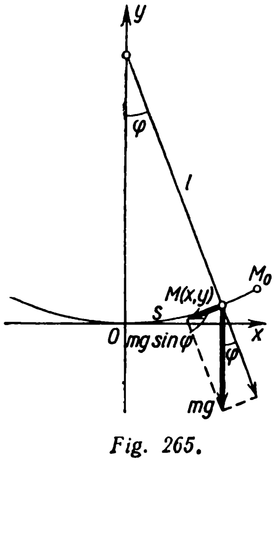


Fig. 265.

The problem of a simple pendulum. Let there be a material point of mass m, which is in motion (by the force of gravity) along the circle L lying in the vertical plane. Let us find the equation of motion of the point neglecting resistance forces (friction, air resistance, etc.).

Putting the origin at the lowest point of the circle, we put the x-axis along the tangent to the circle (Fig. 265).

Denote by l the radius of the circle, by s the arc length from the origin O to the variable point M where the mass m is located; this length is taken with the appropriate sign ($s>0$, if the point M is on the right of O; $s<0$ if M is on the left of O).

Our problem consists in establishing s as a function of the time t.

Let us decompose the force of gravity mg into tangential and normal components. The former,

equal to $-mg\sin\varphi$, produces motion, the latter is cancelled by the reaction of the curve along which the mass m is moving.

Thus, the equation of motion is of the form

$$m\frac{d^2s}{dt^2}=-mg\sin\varphi.$$

Since the angle $\varphi=\frac{s}{l}$ for a circle, we get the equation

$$\frac{d^2s}{dt^2}=-g\sin\frac{s}{l}.$$

This is a Type II differential equation (since it does not contain the independent variable t explicitly).

Let us integrate it in the appropriate fashion:

$$\frac{ds}{dt}=p,\quad \frac{d^2s}{dt^2}=\frac{dp}{ds}p.$$

Hence,

$$p\frac{dp}{ds}=-g\sin\frac{s}{l}$$

or

$$p\,dp=-g\sin\frac{s}{l}\,ds,$$

whence

$$p^2=2gl\cos\frac{s}{l}+C_1.$$

Let us denote by s_0 the greatest arc length to which the point M swings. For $s=s_0$ the velocity of the point is zero:

$$\left.\frac{ds}{dt}\right|_{s=s_0}=\left.p\right|_{s=s_0}=0.$$

This enables us to determine C_1:

$$0=2gl\cos\frac{s_0}{l}+C_1,$$

whence

$$C_1=-2gl\cos\frac{s_0}{l}.$$

Therefore,

$$p^2=\left(\frac{ds}{dt}\right)^2=2gl\left(\cos\frac{s}{l}-\cos\frac{s_0}{l}\right)$$

or, applying to the latter expression the formula for the difference of cosines,

$$\left(\frac{ds}{dt}\right)^2=4gl\sin\frac{s+s_0}{2l}\sin\frac{s_0-s}{2l}, \tag{5}$$

or *)

$$\frac{ds}{dt}=2\sqrt{gl}\sqrt{\sin\frac{s+s_0}{2l}\sin\frac{s_0-s}{2l}}. \tag{6}$$

This is an equation with variables separable. Separating the variables, we get

$$\frac{ds}{\sqrt{\sin\frac{s+s_0}{2l}\sin\frac{s_0-s}{2l}}}=2\sqrt{gl}\,dt. \tag{7}$$

We shall assume, for the time being, that $s\neq s_0$, then the denominator of the fraction is different from zero. If we consider that $s=0$ for $t=0$, then from (7) we get

$$\int_0^s\frac{ds}{\sqrt{\sin\frac{s+s_0}{2l}\sin\frac{s_0-s}{2l}}}=2\sqrt{gl}\cdot t. \tag{8}$$

This is the equation that yields s as a function of t. The integral on the left cannot be expressed in terms of elementary functions; neither can the function s of t. Let us consider this problem approximately. We shall assume that the angles $\frac{s_0}{l}$ and $\frac{s}{l}$ are small. The angles $\frac{s+s_0}{2l}$ and $\frac{s_0-s}{2l}$ will not exceed $\frac{s_0}{l}$. In (6) let us replace, approximately, the sines of the angles by the angles

$$\frac{ds}{dt}=2\sqrt{gl}\sqrt{\frac{s+s_0}{2l}\,\frac{s_0-s}{2l}}$$

or

$$\frac{ds}{dt}=\sqrt{\frac{g}{l}}\sqrt{(s_0^2-s^2)}. \tag{6'}$$

Separating variables, we get (assuming, for the time being, that $s\neq s_0$)

$$\frac{ds}{\sqrt{s_0^2-s^2}}=\sqrt{\frac{g}{l}}\,dt. \tag{7'}$$

Again we consider that $s=0$ when $t=0$. Integrating the latter equation, we get

$$\int_0^s\frac{ds}{\sqrt{s_0^2-s^2}}=\sqrt{\frac{g}{l}}\,t \tag{8'}$$

or

$$\arcsin\frac{s}{s_0}=\sqrt{\frac{g}{l}}\,t,$$

*) We put the plus sign in front of the root. From the note at the end of the solution it follows that there is no need to consider the case with the minus sign.

whence

$$s = s_0 \sin \sqrt{\frac{g}{l}}\, t. \tag{9}$$

Note. When solving, we assumed that $s \neq s_0$. But it is clear, by direct substitution, that the function (9) is the solution of equation (6') for any value of t.

Let it be recalled that the solution (9) is an approximate solution of equation (5), since equation (6) was replaced by the approximate equation (6').

Equation (9) shows that the point M (which may be regarded as the extremity of the pendulum) performs harmonic oscillations with a period $T = 2\pi \sqrt{\frac{l}{g}}$. This period is independent of the amplitude s_0.

Example 4. Escape-velocity problem.

Determine the smallest velocity with which a body must be thrown vertically upwards so that it will not return to the earth. Air resistance is neglected.

Solution. Denote the mass of the earth and the mass of the body by M and m respectively. By Newton's law of gravitation, the force of attraction f acting on the body m is

$$f = k\frac{M \cdot m}{r^2},$$

where r is the distance between the centre of the earth and the centre of gravity of the body, and k is the gravitational constant.

The differential equation of motion of this body with mass m will be

$$m\frac{d^2r}{dt^2} = -k\frac{M \cdot m}{r^2}$$

or

$$\frac{d^2r}{dt^2} = -k\frac{M}{r^2} \tag{10}$$

The minus sign indicates that the acceleration is negative. The differential equation (10) is an equation of type (2). We shall solve it for the following initial conditions:

$$\text{for } t = 0 \quad r = R, \quad \frac{dr}{dt} = v_0.$$

Here, R is the radius of the earth and v_0 is the launching velocity. We denote

$$\frac{dr}{dt} = v, \quad \frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr}\cdot\frac{dr}{dt} = v\frac{dv}{dr},$$

where v is the velocity of motion. Putting this into (10), we get

$$v\frac{dv}{dr} = -k\frac{M}{r^2}.$$

Separating variables, we obtain

$$v\,dv = -kM\frac{dr}{r^2}.$$

Integrating this equation, we find

$$\frac{v^2}{2} = +kM\frac{1}{r} + C_1. \tag{11}$$

From the condition that $v=v_0$ at the earth's surface (for $r=R$), we determine C_1:

$$\frac{v_0^2}{2}=+kM\frac{1}{R}+C_1$$

or

$$C_1=-\frac{kM}{R}+\frac{v_0^2}{2}.$$

We put the value of C_1 into (11):

$$\frac{v^2}{2}=+kM\frac{1}{r}-\frac{kM}{R}+\frac{v_0^2}{2}$$

or

$$\frac{v^2}{2}=kM\frac{1}{r}+\left(\frac{v_0^2}{2}-\frac{kM}{R}\right). \tag{12}$$

It is given that the body should move so that the velocity is always positive; hence, $\frac{v^2}{2}>0$. Since for a boundless increase of r the quantity $\frac{kM}{r}$ becomes arbitrarily small, the condition $\frac{v^2}{2}>0$ will be fulfilled for any r only for the case

$$\frac{v_0^2}{2}-\frac{kM}{R}\geqslant 0 \tag{13}$$

or

$$v_0\geqslant\sqrt{\frac{2kM}{R}}.$$

Hence, the lowest velocity will be determined by the equation

$$v_0=\sqrt{\frac{2kM}{R}}, \tag{14}$$

where

$$k=6.66\cdot 10^{-8}\ \text{cm}^3/\text{gm}\cdot\text{sec}^2,$$
$$R=63\cdot 10^7\ \text{cm}.$$

At the earth's surface, for $r=R$, the acceleration of gravity is g ($g=981$ cm/sec^2). For this reason, from (10) we obtain

$$g=k\frac{M}{R^2}$$

or

$$M=\frac{gR^2}{k}.$$

Putting this value of M into (14) we obtain

$$v_0=\sqrt{2gR}=\sqrt{2\cdot 981\cdot 63\cdot 10^7}\approx 11.2\cdot 10^5\,\frac{\text{cm}}{\text{sec}}=11.2\,\frac{\text{km}}{\text{sec}}.$$

SEC. 19. GRAPHICAL METHOD OF INTEGRATING SECOND-ORDER DIFFERENTIAL EQUATIONS

Let us find out the geometric meaning of a second-order differential equation. Suppose we have an equation

$$y''=f(x,y,y'). \tag{1}$$

Denote by φ the angle formed by the positive x-axis and the tangent to a curve; then

$$\frac{dy}{dx}=\tan\varphi. \tag{2}$$

To find the geometric significance of the second derivative, recall the formula that determines the radius of curvature of a curve at a given point*):

$$R=\frac{(1+y'^2)^{3/2}}{y''}.$$

Whence

$$y''=\frac{(1+y'^2)^{3/2}}{R}.$$

But

$$y'=\tan\varphi;\quad 1+y'^2=1+\tan^2\varphi=\sec^2\varphi;\quad (1+y'^2)^{3/2}= \\ =|\sec^3\varphi|=\frac{1}{|\cos^3\varphi|},$$

therefore

$$y''=\frac{1}{R\,|\cos^3\varphi|}. \tag{3}$$

Now putting into (1) the expressions obtained for y and y'', we have

$$\frac{1}{R\,|\cos^3\varphi|}=f(x,y,\tan\varphi)$$

or

$$R=\frac{1}{|\cos^3\varphi|\cdot f(x,y,\tan\varphi)}. \tag{4}$$

It is thus evident that a second-order differential equation determines the magnitude of the radius of curvature of an integral curve if the coordinates of the point and the direction of the tangent to this point are specified.

*) Up till now we have always considered the radius of curvature *positive*; in this section we shall consider it a number that can take on both positive and negative values: if the curve is convex ($y''<0$), we consider the radius of curvature negative ($R<0$); if the curve is concave ($y''>0$), it is positive ($R>0$).

From the foregoing there follows a method of approximate construction of an integral curve by means of a smooth curve composed of arcs of circles. *)

To illustrate, let it be required to find the solution of equation (1) that satisfies the following initial conditions:

$$y_{x=x_0}=y_0;\quad y'_{x=x_0}=y'_0.$$

Through the point $M_0(x_0, y_0)$ draw a ray M_0T_0 with slope $y'=$ $=\tan\varphi_0=y'_0$ (Fig. 266). From equation (4) we find the magnitude of $R=R_0$. Lay off a segment M_0C_0, equal to R_0, perpendicular to M_0T_0, and from the point C_0 (as centre) strike an arc $\widehat{M_0M_1}$ with radius R_0. It should be noted that if $R_0<0$, then the segment M_0C_0 must be drawn in that direction so that the arc of the circle is convex upwards, and for $R_0>0$, convex down (see footnote on page 527).

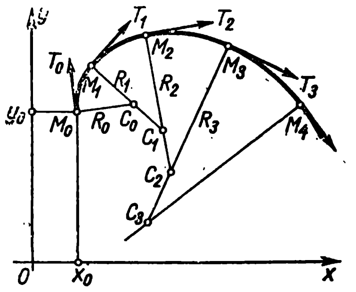


Fig. 266.

Then let x_1, y_1 be the coordinates of the point M_1 which lies on the constructed arc and is sufficiently close to the point M_0 while $\tan\varphi_1$ is the slope of the tangent M_1T_1 to the circle drawn at M_1. From equation (4) we find the value of $R=R_1$ that corresponds to M_1. Draw the segment M_1C_1, perpendicular to M_1T_1, equal to R_1, and from C_1 (as centre) strike an arc $\widehat{M_1M_2}$ with radius R_1. Then on this arc take a point $M_2(x_2, y_2)$ close to M_1 and continue construction as before until we get a sufficiently large piece of the curve consisting of the arcs of circles. From the foregoing it is clear that this curve is approximately an integral curve that passes throught the point M_0.

Obviously, the smaller the arcs $\widehat{M_0M_1}$, $\widehat{M_1M_2}$, $\ldots$, the closer the constructed curve will be to the integral curve.

SEC. 20. HOMOGENEOUS LINEAR EQUATIONS. DEFINITIONS AND GENERAL PROPERTIES

Definition 1. An nth-order differential equation is called *linear* if it is of the first degree in the unknown function y and its

*) A curve is called *smooth* if it has tangents at all points and the angle of inclination of the tangent is a continuous function of the arc length s.

derivatives $y', \ldots, y^{(n-1)}, y^{(n)}$; that is, if it is of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = f(x), \tag{1}$$

where $a_0, a_1, a_2, \ldots, a_n$ and $f(x)$ are given functions of x or constants, and $a_0 \neq 0$ for all values of x from the domain in which we consider equation (1). From now on we shall presume that the functions $a_0, a_1, \ldots, a_n$ and $f(x)$ are continuous for all values of x and that the coefficient $a_0 = 1$ (if it is not equal to 1 we can divide all terms of the equation by it). The function $f(x)$ on the right side of the equation is called the *right-hand member of the equation*.

If $f(x) \not\equiv 0$, then the equation is called *nonhomogeneous* linear or an equation with a *right-hand member*. But if $f(x) \equiv 0$ then the equation has the form

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0 \tag{2}$$

and is called *homogeneous* linear or an equation *without a right-hand member* (the left side of this equation is a homogeneous function of the first degree in $y, y', y'', \ldots, y^{(n)}$).

Let us determine some of the basic properties of homogeneous linear equations, confining our proof to second-order equations.

Theorem 1. *If y_1 and y_2 are two particular solutions of a homogeneous linear equation of the second order*

$$y'' + a_1 y' + a_2 y = 0, \tag{3}$$

then $y_1 + y_2$ is also a solution of this equation.

Proof. Since y_1 and y_2 are solutions of the equation, we have

and
$$\left.\begin{aligned} y_1'' + a_1 y_1' + a_2 y_1 = 0 \\ y_2'' + a_1 y_2' + a_2 y_2 = 0. \end{aligned}\right\} \tag{4}$$

Putting into equation (3) the sum $y_1 + y_2$ and taking into account the identities (4), we will have

$$(y_1 + y_2)'' + a_1 (y_1 + y_2)' + a_2 (y_1 + y_2) =$$
$$= (y_1'' + a_1 y_1' + a_2 y_1) + (y_2'' + a_1 y_2' + a_2 y_2) = 0 + 0 = 0.$$

Thus, $y_1 + y_2$ is a solution of the equation.

Theorem 2. *If y_1 is a solution of equation (3) and C is a constant, then Cy_1 is also a solution of (3).*

Proof. Substituting into (3) the expression Cy_1, we get

$$(Cy_1)'' + a_1 (Cy_1)' + a_2 (Cy_1) = C[y_1'' + a_1 y_1' + a_2 y_1] = C \cdot 0 = 0;$$

and the theorem is thus proved.

Definition 2. The two solutions of equation (3), y_1 and y_2, are called *linearly independent on an interval* $[a, b]$ if their ratio on this interval is not a constant; that is, if

$$\frac{y_1}{y_2} \neq \text{const.}$$

Otherwise the solutions are called *linearly dependent*. In other words, two solutions, y_1 and y_2, are called *linearly dependent* on an interval $[a, b]$ if there exists a **constant** number λ such that $\frac{y_1}{y_2}=\lambda$ when $a \leqslant x \leqslant b$. In this case, $y_1=\lambda y_2$.

Example 1. Let there be an equation $y''-y=0$. It is easy to verify that the functions e^x, e^{-x}, $3e^x$, $5e^{-x}$ are solutions of this equation. Here, the functions e^x and e^{-x} are linearly independent on any interval because the ratio $\frac{e^x}{e^{-x}}=e^{2x}$ does not remain constant as x varies. But the functions e^x and $3e^x$ are linearly dependent, since $\frac{3e^x}{e^x}=3=\text{const.}$

Definition 3. If y_1 and y_2 are functions of x, the determinant

$$W(y_1, y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_1'y_2$$

is called the *Wronskian* of the given functions.

Theorem 3. *If the functions y_1 and y_2 are linearly dependent on an interval $[a, b]$, then the Wronskian on this interval is identically zero.*

Indeed, if $y_2=\lambda y_1$ where $\lambda=\text{const}$, then $y_2'=\lambda y_1'$ and

$$W(y_1, y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} y_1 & \lambda y_1 \\ y_1' & \lambda y_1' \end{vmatrix}=\lambda\begin{vmatrix} y_1 & y_1 \\ y_1' & y_1' \end{vmatrix}=0.$$

Theorem 4. *If the Wronskian $W(y_1, y_2)$, formed for the solutions y_1 and y_2 of the homogeneous linear equation (3), is not zero for some value $x=x_0$ on an interval $[a, b]$ where the coefficients of the equation are continuous, then it does not vanish for any value of x whatsoever on this interval.*

Proof. Since y_1 and y_2 are two solutions of equation (3), we have

$$y_1''+a_1y_1'+a_2y_1=0, \quad y_2''+a_1y_2'+a_2y_2=0.$$

Multiplying the terms of the first equation by y_2, the terms of the second equation by $-y_1$, and adding, we get

$$(y_1''y_2-y_1y_2'')+a_1(y_1'y_2-y_1y_2')=0. \tag{5}$$

The difference in the second brackets is the Wronskian $W(y_1, y_2)$. The expression in the first brackets is a derivative of the Wronskian $W'(y_1, y_2)$:

$$W'(y_1y_2) = (y_1'y_2 - y_1y_2')' = y_1''y_2 + y_1'y_2' - y_1'y_2' - y_1y_2'' = y_1''y_2 - y_1y_2''.$$

Thus, equation (5) assumes the form

$$W' = -a_1W. \qquad (6)$$

Separating variables (for $W \neq 0$), we obtain

$$\frac{W'}{W} = -a_1.$$

Integrating, we find

$$\ln W = -\int_{x_0}^{x} a_1\, dx + \ln C$$

or

$$\ln\frac{W}{C} = -\int_{x_0}^{x} a_1\, dx,$$

whence

$$W = Ce^{-\int_{x_0}^{x} a_1 dx}, \qquad (7)$$

it is given that

$$W_{x=x_0} = Ce^0 = C \neq 0.$$

But then from (7) it follows that $W \neq 0$ for any values of x, because the exponential function does not vanish for any finite value of the argument.

Note 1. If the Wronskian is zero for some value $x = x_0$, then it is also zero for any value x in the interval under consideration. This follows directly from (7): if $W = 0$ when $x = x_0$, then

$$(W)_{x=x_0} = C = 0;$$

consequently, $W \equiv 0$, no matter what the value of the upper limit of x in formula (7).

Theorem 5. *If the solutions y_1 and y_2 of equation (3) are linearly independent on an interval $[a, b]$, then the Wronskian W, formed for these solutions, does not vanish at any point of the given interval.*

We shall hint at the proof of this theorem without giving it completely.

Suppose that $W=0$ at some point of the interval; then, by Theorem 3, the Wronskian will be zero at all points of $[a, b]$:

$$W=0$$

or

$$y_1y_2'-y_1'y_2=0.$$

Let us first consider those subintervals in $[a, b]$ where $y_1 \neq 0$. Then

$$\frac{y_1y_2'-y_1'y_2}{y_1^2}=0$$

or

$$\left(\frac{y_2}{y_1}\right)'=0.$$

Consequently, on each of these subintervals $\frac{y_2}{y_1}$ is a constant

$$\frac{y_2}{y_1}=\lambda=\text{const.}$$

Taking advantage of the existence and uniqueness theorem, it may be shown that $y_2=\lambda y_1$ for all points of the interval $[a, b]$ including those where $y_1=0$; but this is impossible since it is given that y_2 and y_1 are linearly independent. Thus, the Wronskian does not vanish for any single point of $[a, b]$.

Theorem 6. *If y_1 and y_2 are two linearly independent solutions of equation (3), then*

$$y=C_1y_1+C_2y_2, \tag{8}$$

where C_1 and C_2 are arbitrary constants, is its general solution.

Proof. From Theorems 1 and 2 it follows that the function

$$C_1y_1+C_2y_2$$

is a solution of equation (3) for any values of C_1 and C_2.

We shall now prove that no matter what the initial conditions $y_{x=x_0}=y_0$, $y'_{x=x_0}=y'_0$, it is possible to choose the values of the arbitrary constants C_1 and C_2 so that the corresponding particular solution $C_1y_1+C_2y_2$ should satisfy the given initial conditions.

Substituting the initial conditions into (8), we have

$$\left.\begin{aligned} y_0&=C_1y_{10}+C_2y_{20},\\ y'_0&=C_1y'_{10}+C_2y'_{20}, \end{aligned}\right\} \tag{9}$$

where we put

$$(y_1)_{x=x_0}=y_{10}; \quad (y_2)_{x=x_0}=y_{20}; \quad (y'_1)_{x=x_0}=y'_{10}; \quad (y'_2)_{x=x_0}=y'_{20}.$$

From the system (9) we can determine C_1 and C_2, since the determinant of this system

$$\begin{vmatrix} y_{10} & y_{20} \\ y'_{10} & y'_{20} \end{vmatrix} = y_{10}y'_{20} - y'_{10}y_{20}$$

is the Wronskian for $x = x_0$ and, hence, is not equal to 0 (by virtue of the linear independence of the solutions y_1 and y_2). The particular solution obtained from the family (8) for the found values of C_1 and C_2 satisfies the given initial conditions. Thus, the theorem is proved.

Example 2. The equation

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0,$$

whose coefficients $a_1 = \frac{1}{x}$ and $a_2 = \frac{1}{x^2}$ are continuous on any interval that does not contain the point $x = 0$, permits of the particular solutions

$$y_1 = x, \quad y_2 = \frac{1}{x}$$

(this is readily verified by substitution). Hence, its general solution is of the form

$$y = C_1 x + C_2 \frac{1}{x}.$$

Note 2. There are no general methods for finding (in finite form) the general solution of a linear equation with variable coefficients. However, such a method exists for an equation with constant coefficients. It will be given in the next section. For the case of equations with variable coefficients, certain devices will be given in Chapter XVI (Series) that will enable us to find approximate solutions satisfying definite initial conditions.

Here we shall prove a theorem that will enable us to find the general solution of a second-order differential equation with variable coefficients if one of its particular solutions is known. Since it is sometimes possible to find or guess one particular solution directly, this theorem will prove useful in many cases.

Theorem 7. *If we know one particular solution of a second-order homogeneous linear equation, the finding of the general solution reduces to integrating the functions.*

Proof. Let y_1 be some known particular solution of the equation

$$y'' + a_1 y' + a_2 y = 0.$$

We find another particular solution of the given equation so that y_1 and y_2 are linearly independent. Then the general solution will be expressed by the formula $y = C_1 y_1 + C_2 y_2$, where C_1 and C_2 are arbitrary constants. By virtue of formula (7) (see proof of

Theorem 4), we can write

$$y_2'y_1-y_2y_1'=Ce^{-\int a_1dx}.$$

Thus, for a determination of y_2 we obtain a first-order linear equation. Integrate it as follows. Divide all terms by y_1^2:

$$\frac{y_2'y_1-y_2y_1'}{y_1^2}=\frac{1}{y_1^2}Ce^{-\int a_1dx}$$

or

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right)=\frac{1}{y_1^2}Ce^{-\int a_1dx};$$

whence

$$\frac{y_2}{y_1}=\int\frac{Ce^{-\int a_1dx}}{y_1^2}\,dx+C_2.$$

Since we are seeking a particular solution, we get (by putting $C_2=0$ and $C=1$)

$$y_2=y_1\int\frac{e^{-\int a_1dx}}{y_1^2}\,dx. \qquad (10)$$

It is obvious that y_1 and y_2 are linearly independent solutions since $\frac{y_2}{y_1}\neq\text{const}$.

Thus, the general solution of the initial equation is of the form

$$y=C_1y_1+C_2y_1\int\frac{e^{-\int a_1dx}}{y_1^2}\,dx. \qquad (11)$$

Example 3. Find the general solution of the equation

$$(1-x^2)y''-2xy'+2y=0.$$

Solution. It is evident, by direct verification, that this equation has a particular solution $y_1=x$. Let us find the second particular solution y_2, so that y_1 and y_2 should be linearly independent.

Noting that in our case $a_1=\frac{-2x}{1-x^2}$, we have, by (10),

$$y=x\int\frac{e^{\int\frac{2x\,dx}{1-x^2}}}{x^2}\,dx=x\int\frac{e^{-\ln(1-x^2)}}{x^2}\,dx=x\int\frac{dx}{x^2(1-x^2)}=$$
$$=x\int\left(\frac{1}{x^2}+\frac{1}{2(1-x)}+\frac{1}{2(1+x)}\right)dx=x\left[-\frac{1}{x}+\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right].$$

Consequently, the general solution is of the form

$$y=C_1x+C_2\left(\frac{1}{2}x\ln\left|\frac{1+x}{1-x}\right|-1\right).$$

SEC. 21. SECOND-ORDER HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

We have a second-order homogeneous linear equation

$$y'' + py' + qy = 0, \tag{1}$$

where p and q are real constants. To find the complete integral of this equation, it is sufficient (as has already been proved) to find two linearly independent particular solutions.

Let us look for the particular solutions in the form

$$y = e^{kx}, \text{ where } k = \text{const}; \tag{2}$$

then

$$y' = ke^{kx}; \quad y'' = k^2 e^{kx}.$$

Substituting the expressions of the derivatives into equation (1), we find

$$e^{kx}(k^2 + pk + q) = 0.$$

Since $e^{kx} \neq 0$, it means that

$$k^2 + pk + q = 0. \tag{3}$$

Thus, if k satisfies equation (3), then e^{kx} will be a solution of (1). Equation (3) is called an *auxiliary equation* with respect to equation (1).

The auxiliary equation is a quadratic equation with two roots; let us denote them by k_1 and k_2. Then

$$k_1 = -\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}; \quad k_2 = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}.$$

The following cases are possible:

I. k_1 and k_2 are real numbers and not equal ($k_1 \neq k_2$);

II. k_1 and k_2 are complex numbers;

III. k_1 and k_2 are real and equal numbers ($k_1 = k_2$).

Let us consider each case separately.

I. The roots of the auxiliary equation are real and distinct, $k_1 \neq k_2$. Here, the particular solutions are the functions

$$y_1 = e^{k_1 x}; \quad y_2 = e^{k_2 x}.$$

These solutions are linearly independent because

$$\frac{y_2}{y_1} = \frac{e^{k_2 x}}{e^{k_1 x}} = e^{(k_2 - k_1)x} \neq \text{const}.$$

Hence, the complete integral has the form

$$y = C_1 e^{k_1 x} + C_2 e^{k_2 x}.$$

Example 1. Given the equation

$$y''+y'-2y=0.$$

The auxiliary equation is of the form

$$k^2+k-2=0.$$

We find the roots of the auxiliary equation:

$$k_{1,2}=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+2};$$

$$k_1=1,\quad k_2=-2.$$

The complete integral is

$$y=C_1e^x+C_2e^{-2x}.$$

II. The roots of the auxiliary equation are complex. Since complex roots are conjugate in pairs, we write

$$k_1=\alpha+i\beta;\quad k_2=\alpha-i\beta,$$

where

$$\alpha=-\frac{p}{2};\quad \beta=\sqrt{q-\frac{p^2}{4}}.$$

The particular solutions may be written in the form

$$y_1=e^{(\alpha+i\beta)x};\quad y_2=e^{(\alpha-i\beta)x}. \tag{4}$$

These are complex functions of a real argument that satisfy the differential equation (1) (see Sec. 4, Ch. VII).

It is obvious that if some complex function of a real argument

$$y=u(x)+iv(x) \tag{5}$$

satisfies (1), then this equation is satisfied by the functions $u(x)$ and $v(x)$.

Indeed, putting expression (5) into (1), we have

$$[u(x)+iv(x)]''+p[u(x)+iv(x)]'+q[u(x)+iv(x)]\equiv 0$$

or

$$(u''+pu'+qu)+i(v''+pv'+qv)\equiv 0.$$

But a complex function is equal to zero if, and only if, the real part and the imaginary part are equal to zero; that is,

$$u''+pu'+qu=0,$$
$$v''+pv'+qv=0.$$

Thus we have proved that $u(x)$ and $v(x)$ are solutions of the equation.

Let us rewrite the complex solutions (4) in the form of a sum of the real part and the imaginary part:

$$y_1=e^{\alpha x}\cos\beta x+ie^{\alpha x}\sin\beta x,$$
$$y_2=e^{\alpha x}\cos\beta x-ie^{\alpha x}\sin\beta x.$$

From what has been proved, the particular solutions of (1) are the real functions

$$\tilde{y}_1 = e^{\alpha x} \cos \beta x, \qquad (6')$$

$$\tilde{y}_2 = e^{\alpha x} \sin \beta x. \qquad (6'')$$

The functions $\tilde{y}_1$ and $\tilde{y}_2$ are linearly independent, since

$$\frac{\tilde{y}_1}{\tilde{y}_2} = \frac{e^{\alpha x} \cos \beta x}{e^{\alpha x} \sin \beta x} = \cot \beta x \neq \text{const}.$$

Consequently, the general solution of equation (1) in the case of complex roots of the auxiliary equation is of the form

$$y = A\tilde{y}_1 + B\tilde{y}_2 = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x$$

or

$$y = e^{\alpha x} (A \cos \beta x + B \sin \beta x), \qquad (7)$$

where A and B are arbitrary constants.

Example 2. Given the equation

$$y'' + 2y' + 5y = 0.$$

Find the complete integral and a particular solution that satisfies the initial conditions $y_{x=0} = 0$, $y'_{x=0} = 1$. Construct the graph.

Solution. 1) We write the auxiliary equation

$$k^2 + 2k + 5 = 0$$

and find its roots:

$$k_1 = -1 + 2i, \quad k_2 = -1 - 2i.$$

Thus, the complete integral is

$$y = e^{-x} (A \cos 2x + B \sin 2x).$$

2) We find a particular solution that satisfies the given initial conditions and determine the corresponding values of A and B.

From the first condition we find

$$0 = e^{-0} (A \cos 2 \cdot 0 + B \sin 2 \cdot 0), \text{ whence } A = 0.$$

Noting that

$$y' = e^{-x} 2B \cos 2x - e^{-x} B \sin 2x$$

we obtain from the second condition

$$1 = 2B, \quad \text{so} \quad B = \frac{1}{2}.$$

Thus, the desired particular solution is

$$y = \frac{1}{2} e^{-x} \sin 2x.$$

Its graph is shown in Fig. 267.

III. The roots of the auxiliary equation are real and equal. Here, $k_1 = k_2$.

One particular solution, $y_1 = e^{k_1 x}$, is obtained from earlier reasoning. We must find the second particular solution, which is

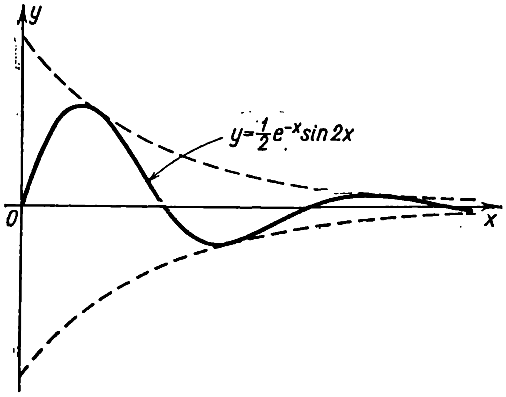


Fig. 267.

linearly independent of the first (the function $e^{k_2 x}$ is identically equal to $e^{k_1 x}$ and therefore cannot be regarded as the second particular solution).

We shall seek the second particular solution in the form

$$y_2 = u(x)\,e^{k_1 x}$$

where $u(x)$ is the unknown function to be determined.

Differentiating, we find

$$y_2' = u'e^{k_1 x} + k_1 u e^{k_1 x} = e^{k_1 x}(u' + k_1 u),$$

$$y_2'' = u''e^{k_1 x} + 2k_1 u' e^{k_1 x} + k_1^2 u e^{k_1 x} = e^{k_1 x}(u'' + 2k_1 u' + k_1^2).$$

Putting the expressions of the derivatives into (1), we obtain

$$e^{k_1 x}\left[u'' + (2k_1 + p)\,u' + (k_1^2 + pk_1 + q)\,u\right] = 0.$$

Since k_1 is a multiple root of the auxiliary equation, we have

$$k_1^2 + pk_1 + q = 0.$$

In addition, $k_1 = k_2 = -\frac{p}{2}$ or $2k_1 = -p$, $2k_1 + p = 0$.

Hence, in order to find $u(x)$ we must solve the equation $e^{k_1 x}u'' = 0$ or $u'' = 0$. Integrating, we get $u = Ax + B$. In particular, we can set $A = 1$ and $B = 0$; then

$$u = x.$$

Thus, for the second particular solution we can take

$$y_2 = xe^{k_1 x}.$$

This solution is linearly independent of the first, since $\frac{y_2}{y_1} = x \neq \text{const}$. Therefore, the following function is the complete integral:

$$y = C_1 e^{k_1 x} + C_2 x e^{k_1 x} = e^{k_1 x}(C_1 + C_2 x).$$

Example 3. Given the equation

$$y'' - 4y' + 4y = 0.$$

Write the auxiliary equation $k^2 - 4k + 4 = 0$. Find its roots: $k_1 = k_2 = 2$. The complete integral is then

$$y = C_1 e^{2x} + C_2 x e^{2x}.$$

SEC. 22. HOMOGENEOUS LINEAR EQUATIONS OF THE *N*TH ORDER WITH CONSTANT COEFFICIENTS

Let us consider a homogeneous linear equation of the *n*th order:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0. \quad (1)$$

We shall assume that $a_1, a_2, \ldots, a_n$ are constants. Before giving a method for solving equation (1), we introduce a definition that will be needed later on.

Definition 1. If for all x of the interval $[a, b]$ we have the equality

$$\varphi_n(x) = A_1 \varphi_1(x) + A_2 \varphi_2(x) + \ldots + A_{n-1}\varphi_{n-1}(x),$$

where $A_1, A_2, \ldots, A_n$ are constants, not all equal to zero, then we say that $\varphi_n(x)$ *is expressed linearly in terms of the functions* $\varphi_1(x), \varphi_2(x), \ldots, \varphi_{n-1}(x)$.

Definition 2. n functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_{n-1}(x), \varphi_n(x)$ *are called linearly independent* if not one of the functions is expressed linearly in terms of the rest.

Note 1. From the definitions it follows that if the functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ are linearly dependent, there will be found constants $C_1, C_2, \ldots, C_n$, not all equal to zero, such that for all x of the interval $[a, b]$ the following identity will be fulfilled:

$$C_1 \varphi_1(x) + C_2 \varphi_2(x) + \ldots + C_n \varphi_n(x) \equiv 0.$$

Examples:

1. The functions $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = 3e^x$ are linearly dependent, since for $C_1 = 1$, $C_2 = 0$, $C_3 = -\frac{1}{3}$ we have the identity

$$C_1 e^x + C_2 e^{2x} + C_3 3e^x \equiv 0.$$

2. The functions $y_1=1$, $y_2=x$, $y_3=x^2$ are linearly independent, since the expression

$$C_1 1+C_2x+C_3x^2$$

will not be identically zero for any C_1, C_2, C_3 that are not simultaneously equal to zero.

3. The functions $y_1=e^{k_1x}$, $y_2=e^{k_2x}$, ..., $y_n=e^{k_nx}$ ..., where $k_1, k_2, \ldots, k_n, \ldots$ are different numbers which are linearly independent. (This assertion is given without proof.)

Let us now solve equation (1). For this equation, the following theorem holds.

Theorem. *If the functions $y_1, y_2, \ldots, y_n$ are linearly independent solutions of equation (1), then its general solution is*

$$y=C_1y_1+C_2y_2+\ldots+C_ny_n, \tag{2}$$

where $C_1, \ldots, C_n$ are arbitrary constants.

If the coefficients of equation (1) are constant, the general solution is found in the same way as in the case of second-order equations.

1) We form the auxiliary equation

$$k^n+a_1k^{n-1}+a_2k^{n-2}+\ldots+a_n=0.$$

2) We find the roots of the auxiliary equation

$$k_1, k_2, \ldots, k_n.$$

3) From the character of the roots we write out the particular linearly independent solutions, taking note of the fact that:

a) to every real root k of order one there corresponds a particular solution e^{kx};

b) to every pair of complex conjugate roots $k^{(1)}=\alpha+i\beta$ and $k^{(2)}=\alpha-i\beta$ there correspond two particular solutions $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$;

c) to every real root k of multiplicity r there correspond r linearly independent particular solutions

$$e^{kx},\ xe^{kx},\ \ldots,\ x^{r-1}e^{kx};$$

d) to each pair of complex conjugate roots $k^{(1)}=\alpha+i\beta$, $k^{(2)}=\alpha-i\beta$ of multiplicity μ there correspond 2μ particular solutions:

$$e^{\alpha x}\cos\beta x,\quad xe^{\alpha x}\cos\beta x,\ \ldots,\ x^{\mu-1}e^{\alpha x}\cos\beta x,$$
$$e^{\alpha x}\sin\beta x,\quad xe^{\alpha x}\sin\beta x,\ \ldots,\ x^{\mu-1}e^{\alpha x}\sin\beta x.$$

The number of these particular solutions is exactly equal to the degree of the auxiliary equation (that is, to the order of the given linear differential equation). It may be proved that these solutions are linearly independent.

4) After finding n linearly independent particular solutions $y_1, y_2, \ldots, y_n$ we construct the general solution of the given linear equation:

$$y = C_1y_1 + C_2y_2 + \ldots + C_ny_n,$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Example 4. Find the general solution of the equation

$$y^{IV} - y = 0.$$

Solution. Form the auxiliary equation

$$k^4 - 1 = 0.$$

Find the roots of the auxiliary equation:

$$k_1 = 1, \quad k_2 = -1, \quad k_3 = i, \quad k_4 = -i.$$

Write the complete integral

$$y = C_1e^x + C_2e^{-x} + A\cos x + B\sin x,$$

where C_1, C_2, A, B are arbitrary constants.

Note 2. From the foregoing it follows that the whole difficulty in solving homogeneous linear differential equations with constant coefficients lies in the solution of the auxiliary equation.

SEC. 23. NONHOMOGENEOUS SECOND-ORDER LINEAR EQUATIONS

Let there be a nonhomogeneous second-order linear equation

$$y'' + a_1y' + a_2y = f(x). \tag{1}$$

The structure of the general solution of such an equation is determined by the following theorem.

Theorem 1. *The general solution of the nonhomogeneous equation (1) is represented as the sum of some particular solution of the equation y^* and the general solution $\bar{y}$ of the corresponding homogeneous equation*

$$\bar{y}'' + a_1\bar{y}' + a_2\bar{y} = 0. \tag{2}$$

Proof. We need to prove that the sum

$$y = \bar{y} + y^* \tag{3}$$

is **the general solution** of equation (1). Let us first prove that the function (3) is a **solution** of (1).

Substituting the sum $\bar{y} + y^*$ into (1) in place of y, we get

$$(\bar{y} + y^*)'' + a_1(\bar{y} + y^*)' + a_2(\bar{y} + y^*) = f(x)$$

or

$$(\bar{y}'' + a_1\bar{y}' + a_2\bar{y}) + (y^{*''} + a_1 y^{*'} + a_2 y^*) = f(x). \tag{4}$$

Since $\bar{y}$ is a solution of (2), the expression in the first brackets is identically zero. Since y^* is a solution of (1), the expression in the second brackets is equal to $f(x)$. Consequently, (4) is an identity. Thus, the first part of the theorem is proved.

We shall now prove that expression (3) is the **general** solution of equation (1); in other words, we shall prove that the arbitrary constants that enter into the expression may be chosen so that the following initial conditions are satisfied:

$$\left.\begin{aligned} y_{x=x_0} &= y_0, \\ y'_{x=x_0} &= y'_0, \end{aligned}\right\} \tag{5}$$

no matter what the numbers x_0, y_0 and y'_0 [provided that x_0 is taken from the region where the functions a_1, a_2 and $f(x)$ are continuous].

Noting that $\bar{y}$ may be given in the form

$$\bar{y} = C_1 y_1 + C_2 y_2,$$

where y_1 and y_2 are linearly independent solutions of equation (2), and C_1 and C_2 are arbitrary constants, we can rewrite (3) in the form

$$y = C_1 y_1 + C_2 y_2 + y^*. \tag{3'}$$

Then, by the conditions (5), we will have *)

$$C_1 y_{10} + C_2 y_{20} + y_0^* = y_0,$$
$$C_1 y'_{10} + C_2 y'_{20} + y_0^{*'} = y'_0.$$

From this system of equations we have to determine C_1 and C_2. Rewriting the system in the form

$$\left.\begin{aligned} C_1 y_{10} + C_2 y_{20} &= y_0 - y_0^*, \\ C_1 y'_{10} + C_2 y'_{20} &= y'_0 - y_0^{*'}, \end{aligned}\right\} \tag{6}$$

we note that the determinant of this system is the Wronskian for the functions y_1 and y_2 at the point $x = x_0$. Since it is given that these functions are linearly independent, the Wronskian is not zero; consequently, system (6) has a definite solution, C_1

*) Here, y_{10}, y_{20}, y_0^*, y'_{10}, y'_{20}, $y_0^{*'}$ denote the numerical values of the functions y_1, y_2, y^*, y'_1, y'_2, $y^{*'}$ when $x = x_0$.

and C_2; in other words, there exist values C_1 and C_2 such that formula (3) defines the solution of equation (1) which satisfies the given initial conditions. The theorem is completely proved.

Thus, if we know the general solution $\bar{y}$ of the homogeneous equation (2), the basic difficulty, when integrating the nonhomogeneous equation (1), lies in finding some particular solution y^*.

We shall give a general method for finding the particular solutions of a nonhomogeneous equation.

The method of variation of arbitrary constants (parameters). We write the general solution of the homogeneous equation (2):

$$y = C_1y_1 + C_2y_2. \tag{7}$$

We shall seek a particular solution of the nonhomogeneous equation (1) in the form (7), considering C_1 and C_2 as some (as yet) undetermined **functions** of x.

Differentiate (7):

$$y' = C_1y_1' + C_2y_2' + C_1'y_1 + C_2'y_2.$$

Now choose the needed functions C_1 and C_2 so that the following equation is fulfilled:

$$C_1'y_1 + C_2'y_2 = 0. \tag{8}$$

If we take note of this additional condition, the first derivative y' will take the form

$$y' = C_1y_1' + C_2y_2'.$$

Differentiating this expression, we find y'':

$$y'' = C_1y_1'' + C_2y_2'' + C_1'y_1' + C_2'y_2'.$$

Putting y, y' and y'' into (1), we get

$$C_1y_1'' + C_2y_2'' + C_1'y_1' + C_2'y_2' + a_1(C_1y_1' + C_2y_2') + \\ + a_2(C_1y_1 + C_2y_2) = f(x)$$

or

$$C_1(y_1'' + a_1y_1' + a_2y_1) + C_2(y_2'' + a_1y_2' + a_2y_2) + C_1'y_1' + C_2'y_2' = f(x).$$

The expressions in the first two brackets vanish, since y_1 and y_2 are solutions of the homogeneous equation. Hence, the latter equation takes on the form

$$C_1'y_1' + C_2'y_2' = f(x). \tag{9}$$

Thus, the function (7) will be a solution of the nonhomogeneous equation (1) provided the functions C_1 and C_2 satisfy the system of equations (8) and (9); that is, if

$$C_1'y_1 + C_2'y_2 = 0, \quad C_1'y_1' + C_2'y_2' = f(x).$$

Since the determinant of this system is the Wronskian for the linearly independent functions y_1 and y_2, it is not equal to zero. Hence, in solving the system we will find C_1' and C_2' as definite functions of x:

$$C_1' = \varphi_1(x), \quad C_2' = \varphi_2(x).$$

Integrating, we obtain

$$C_1 = \int \varphi_1(x)\,dx + \overline{C}_1; \quad C_2 = \int \varphi_2(x)\,dx + \overline{C}_2,$$

where $\overline{C}_1$ and $\overline{C}_2$ are constants of integration.

Substituting the expressions obtained of C_1 and C_2 into (7), we find an integral that is dependent on the two arbitrary constants $\overline{C}_1$ and $\overline{C}_2$; that is, we find the general solution of the nonhomogeneous equation *).

Example. Find the general solution of the equation

$$y'' - \frac{y'}{x} = x.$$

Solution. Let us find the general solution of the homogeneous equation

$$y'' - \frac{y'}{x} = 0.$$

Since

$$\frac{y''}{y'} = \frac{1}{x} \quad \text{we have} \quad \ln y' = \ln x + \ln C; \quad y' = Cx;$$

and so

$$y = C_1x^2 + C_2.$$

For the latter expression to be a solution of the given equation, we have to define C_1 and C_2 as functions of x from the system

$$C_1'x^2 + C_2'\cdot 1 = 0, \quad 2C_1'x + C_2'\cdot 0 = x.$$

Solving this system, we find

$$C_1' = \frac{1}{2}, \quad C_2' = -\frac{1}{2}x^2,$$

whence, after integration, we get

$$C_1 = \frac{x}{2} + \overline{C}_1, \quad C_2 = -\frac{x^3}{6} + \overline{C}_2.$$

Putting the functions obtained into the formula $y = C_1x^2 + C_2$, we get the general solution of the nonhomogeneous equation

$$y = \overline{C}_1x^2 + \overline{C}_2 + \frac{x^3}{2} - \frac{x^3}{6}$$

or $y = \overline{C}_1x^2 + \overline{C}_2 + \frac{x^3}{3}$, where $\overline{C}_1$ and $\overline{C}_2$ are arbitrary constants.

---

*) If we put $\overline{C}_1 = \overline{C}_2 = 0$, we get a particular solution of equation (1).

When seeking particular solutions, it is useful the take advantage of the results of the following theorem.

Theorem 2. *Let the nonhomogeneous equation*

$$y''+a_1y'+a_2y=f_1(x)+f_2(x) \tag{10}$$

be such that the right side is a sum of two functions, $f_1(x)$ and $f_2(x)$. *If* y_1 *is a particular solution of the equation*

$$y''+a_1y'+a_2y=f_1(x), \tag{11}$$

and y_2 *is a particular solution of the equation*

$$y''+a_1y'+a_2y=f_2(x), \tag{12}$$

then y_1+y_2 *is a particular solution* *) *of equation* (10).

Proof. Substituting the expression y_1+y_2 into (10), we get

$$(y_1+y_2)''+a_1(y_1+y_2)'+a_2(y_1+y_2)=f_1(x)+f_2(x)$$

or

$$(y_1''+a_1y_1'+a_2y_1)+(y_2''+a_1y_2'+a_2y_2)=f_1(x)+f_2(x). \tag{13}$$

From equations (11) and (12) it follows that equality (13) is an identity. And the theorem is proved.

SEC. 24. NONHOMOGENEOUS SECOND-ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Suppose we have the equation

$$y''+py'+qy=f(x) \tag{1}$$

where p and q are real numbers.

A general method for finding the solution of a nonhomogeneous equation was given in the preceding section. In the case of an equation with constant coefficients, it is sometimes easier to find a particular solution without resorting to integration. Let us consider several such possibilities for equation (1).

I. Let the right side of (1) be the product of an exponential function by a polynomial; that is, of the form

$$f(x)=P_n(x)e^{\alpha x}, \tag{2}$$

where $P_n(x)$ is a polynomial of degree n. Then the following particular cases are possible:

*) Obviously, the appropriate theorem remains true for any number of terms on the right side.

a) The number α *is not a root* of the auxiliary equation

$$k^2+pk+q=0.$$

In this case, the particular solution must be sought for in the form

$$y^*=(A_0x^n+A_1x^{n-1}+\ldots+A_n)e^{\alpha x}=Q_n(x)e^{\alpha x}. \qquad (3)$$

Indeed, substituting y^* into equation (1) and cancelling $e^{\alpha x}$ out of all terms, we will have

$$Q_n''(x)+(2\alpha+p)Q_n'(x)+(\alpha^2+p\alpha+q)Q_n(x)=P_n(x). \qquad (4)$$

$Q_n(x)$ is a polynomial of degree n, $Q_n'(x)$ is a polynomial of degree $n-1$, and $Q_n''(x)$ is a polynomial of degree $n-2$. Thus, n-degree polynomials are found on the left and right of the equality sign. Equating the coefficients of the same degrees of x (the number of unknown coefficients is $n+1$), we get a system of $n+1$ equations for determining the unknown coefficients A_0, A_1, $A_2, \ldots, A_n$.

b) The number α is a **simple (single) root** of the auxiliary equation.

If in this case we should seek the particular solution in the form (3), then on the left side of (4) we would have a polynomial of degree $n-1$, since the coefficient of $Q_n(x)$, that is, $\alpha^2+p\alpha+q$ is equal to zero, and the polynomials $Q_n'(x)$ and $Q_n''(x)$ have degrees less than n. Hence, (4) would not be an identity, no matter what the $A_0, A_1, \ldots, A_n$. For this reason, the particular solution in this case has to be taken in the form of a polynomial of degree $n+1$, but without the absolute term (since the absolute term of this polynomial vanishes upon differentiation)*):

$$y^*=xQ_n(x)e^{\alpha x}.$$

c) The number α is a **double root** of the auxiliary equation. Then, as a result of the substitution of the function $Q_n(x)e^{\alpha x}$ into the differential equation, the degree of the polynomial is diminished by two units. Indeed, if α is the root of the auxiliary equation, then $\alpha^2+p\alpha+q=0$; moreover, since α is a double root, it follows that $2\alpha=-p$ (since by a familiar theorem of elementary algebra, the sum of the roots of a reduced quadratic equation is equal to the coefficient of the unknown in the first degree with sign reversed). And so $2\alpha+p=0$.

*) We remark that all the results given above also hold for the case when α is a complex number (this follows from the rules of differentiation of the function e^{mx}, where m is any complex number; see Sec. 4, Ch. VII).

Consequently, on the left side of (4) there remains $Q_n''(x)$, that is, a polynomial of degree $n-2$. To obtain a polynomial of degree n as a result of substitution, one should seek the particular solution in the form of a product of $e^{\alpha x}$ by the $(n+2)nd$ degree polynomial. Then the absolute term of this polynomial and the first-degree term will vanish upon differentiation; for this reason, they need not be included in the particular solution.

Thus, when α is a double root of the auxiliary equation, the particular solution may be taken in the form

$$y^* = x^2 Q_n(x) e^{\alpha x}.$$

Example 1. Find the general solution of the equation

$$y'' + 4y' + 3y = x.$$

Solution. The general solution of the corresponding homogeneous equation is

$$\bar{y} = C_1 e^{-x} + C_2 e^{-3x}.$$

Since the right-hand side of the given nonhomogeneous equation is of the form xe^{0x} [that is, of the form $P_1(x)e^{0x}$], and 0 is not a root of the auxiliary equation $k^2+4k+3=0$, it follows that we should seek the particular solution in the form $y^* = Q_1(x)e^{0x}$; in other words, we put

$$y^* = A_0 x + A_1.$$

Substituting this expression into the given equation, we will have

$$4A_0 + 3(A_0 x + A_1) = x.$$

Equating the coefficients of identical degrees of x, we get

$$3A_0 = 1, \quad 4A_0 + 3A_1 = 0,$$

whence

$$A_0 = \frac{1}{3}; \quad A_1 = -\frac{4}{9}.$$

Consequently,

$$y^* = \frac{1}{3}x - \frac{4}{9}.$$

The general solution of $y = \bar{y} + y^*$ will be

$$y = C_1 e^{-x} + C_2 e^{-3x} + \frac{1}{3}x - \frac{4}{9}.$$

Example 2. Find the general solution of the equation

$$y'' + 9y = (x^2+1)e^{3x}.$$

Solution. The general solution of the homogeneous equation is readily found:

$$\bar{y} = C_1 \cos 3x + C_2 \sin 3x.$$

The right side of the given equation $(x^2+1)e^{3x}$ has the form

$$P_2(x)e^{3x},$$

Since the coefficient 3 in the exponent is not a root of the auxiliary equation, we seek the particular solution in the form

$$y^* = Q_2(x)e^{3x} \quad \text{or} \quad y^* = (Ax^2 + Bx = C)e^{3x}.$$

Substituting this expression in the differential equation, we will have

$$[9(Ax^2 + Bx + C) + 6(2Ax + B) + 2A + 9(Ax^2 + Bx + C)]e^{3x} = (x^2 + 1)e^{3x}.$$

Cancelling out e^{3x} and equating the coefficients of identical degrees of x, we obtain

$$18A = 1, \quad 12A + 18B = 0, \quad 2A + 6B + 18C = 1,$$

whence $A = \frac{1}{18}$; $B = -\frac{1}{27}$; $C = \frac{5}{81}$. Consequently, the particular solution is

$$y^* = \left(\frac{1}{18}x^2 - \frac{1}{27}x + \frac{5}{81}\right)e^{3x}$$

and the general solution is

$$y = C_1 \cos 3x + C_2 \sin 3x + \left(\frac{1}{18}x^2 - \frac{1}{27}x + \frac{5}{81}\right)e^{3x}.$$

Example 3. To solve the equation

$$y'' - 7y' + 6y = (x - 2)e^x.$$

Solution. Here, the right side is of the form $P_1(x)e^{1x}$ and the coefficient 1 in the exponent is a simple root of the auxiliary polynomial. Hence, we seek the particular solution in the form $y^* = xQ_1(x)e^x$ or

$$y^* = x(Ax + B)e^x;$$

putting this expression in the equation, we get

$$[(Ax^2 + Bx) + (4Ax + 2B) + 2A - 7(Ax^2 + Bx) - 7(2Ax + B) + \\ + 6(Ax^2 + Bx)]e^x = (x - 2)e^x$$

or

$$(-10Ax - 5B + 2A)e^x = (x - 2)e^x.$$

Equating the coefficients of identical degrees of x, we get

$$-10A = 1, \quad -5B + 2A = -2,$$

whence $A = -\frac{1}{10}$, $B = \frac{9}{25}$. Consequently, the particular solution is

$$y^* = x\left(-\frac{1}{10}x + \frac{9}{25}\right)e^x$$

and the general solution is

$$y = C_1e^{6x} + C_2e^x + x\left(-\frac{1}{10}x + \frac{9}{25}\right)e^x.$$

II. Let the right side have the form

$$f(x) = P(x)e^{\alpha x}\cos\beta x + Q(x)e^{\alpha x}\sin\beta x, \tag{5}$$

where $P(x)$ and $Q(x)$ are polynomials.

This case may be considered by the technique used in the preceding case, if we pass from trigonometric functions to exponential

functions. Replacing $\cos\beta x$ and $\sin\beta x$ by exponential functions using Euler's formulas (see Sec. 5, Ch. VII), we obtain

$$f(x) = P(x)e^{\alpha x}\frac{e^{i\beta x}+e^{-i\beta x}}{2} + Q(x)e^{\alpha x}\frac{e^{i\beta x}-e^{-i\beta x}}{2i}$$

or

$$f(x) = \left[\frac{1}{2}P(x) + \frac{1}{2i}Q(x)\right]e^{(\alpha+i\beta)x} + \left[\frac{1}{2}P(x) - \frac{1}{2i}Q(x)\right]e^{(\alpha-i\beta)x}. \quad (6)$$

Here, the square brackets contain polynomials whose degrees are equal to the highest degree of the polynomials $P(x)$ and $Q(x)$. We have thus obtained the right side of the form considered in Case I.

It is proved (we omit the proof) that it is possible to find particular solutions which do not contain complex numbers.

Thus, if the right side of equation (1) is of the form

$$f(x) = P(x)e^{\alpha x}\cos\beta x + Q(x)e^{\alpha x}\sin\beta x, \quad (7)$$

where $P(x)$ and $Q(x)$ are polynomials in x, then the form of the particular solution is determined as follows:

a) if the number $\alpha+i\beta$ is not a root of the auxiliary equation, then the particular solution of equation (1) should be sought in the form

$$y^* = U(x)e^{\alpha x}\cos\beta x + V(x)e^{\alpha x}\sin\beta x, \quad (8)$$

where $U(x)$ and $V(x)$ are polynomials of degree equal to the highest degree of the polynomials $P(x)$ and $Q(x)$;

b) if the number $\alpha+i\beta$ is a root of the auxiliary equation, we then write the particular solution in the form

$$y^* = x[U(x)e^{\alpha x}\cos\beta x + V(x)e^{\alpha x}\sin\beta x]. \quad (9)$$

Here, in order to avoid mistakes we must note that these forms of particular solutions, (8) and (9), are obviously retained when one of the polynomials $P(x)$ and $Q(x)$ on the right side of equation (1) is identically zero; that is, when the right side is of the form

$$P(x)e^{\alpha x}\cos\beta x \quad \text{or} \quad Q(x)e^{\alpha x}\sin\beta x.$$

Let us further consider an important special case. Let the right side of a second-order linear equation have the form

$$f(x) = M\cos\beta x + N\sin\beta x, \quad (7')$$

where M and N are constants.

a) If βi is not a root of the auxiliary equation, the particular solution should be sought in the form

$$y^* = A\cos\beta x + B\sin\beta x. \quad (8')$$

b) If βi is a root of the auxiliary equation, then the particular solution should be sought in the form

$$y^* = x(A\cos\beta x + B\sin\beta x). \tag{9'}$$

We remark that the function (7') is a special case of the function (7) $[P(x)=M,\ (Q)x=N,\ \alpha=0]$; the functions (8') and (9') are special cases of the functions (8) and (9).

Example 4. Find the complete integral of the nonhomogeneous linear equation

$$y'' + 2y' + 5y = 2\cos x.$$

Solution. The auxiliary equation $k^2+2k+5=0$ has roots $k_1=-1+2i$; $k_2=-1-2i$. Therefore, the complete integral of the corresponding homogeneous equation is

$$\bar{y} = e^{-x}(C_1\cos 2x + C_2\sin 2x).$$

We seek the particular solution of the nonhomogeneous equation in the form

$$y^* = A\cos x + B\sin x,$$

where A and B are constant coefficients to be determined.

Putting y^* into the given equation, we will have

$$-A\cos x - B\sin x + 2(-A\sin x + B\cos x) + 5(A\cos x + B\sin x) = 2\cos x.$$

Equating the coefficients of $\cos x$ and $\sin x$, we get two equations for determining A and B:

$$-A+2B+5A=2;\quad -B-2A+5B=0,$$

whence $A=\frac{1}{5}$; $B=\frac{1}{10}$.

The general solution of the given equation is $y=\bar{y}+y^*$, that is,

$$y = e^{-x}(C_1\cos 2x + C_2\sin 2x) + \frac{1}{5}\cos x + \frac{1}{10}\sin x.$$

Example 5. To solve the equation

$$y'' + 4y = \cos 2x.$$

Solution. The auxiliary equation has roots $k_1=2i$, $k_2=-2i$; therefore, the general solution of the homogeneous equation is of the form

$$\bar{y} = C_1\cos 2x + C_2\sin 2x.$$

We seek the particular solution of the nonhomogeneous equation in the form

$$y^* = x(A\cos 2x + B\sin 2x).$$

Then

$$y^{*\prime} = 2x(-A\sin 2x + B\cos 2x) + (A\cos 2x + B\sin 2x),$$
$$y^{*\prime\prime} = -4x(-A\cos 2x - B\sin 2x) + 4(-A\sin 2x + B\cos 2x).$$

Putting these expressions of the derivatives into the given equation and equating the coefficients of $\cos 2x$ and $\sin 2x$, we get a system of equations for determining A and B:

$$4B=1;\quad -4A=0,$$

whence $A=0$ and $B=\frac{1}{4}$. Thus, the complete integral of the given equation is

$$y=C_1\cos 2x+C_2\sin 2x+\frac{1}{4}x\sin 2x.$$

Example 6. To solve the equation

$$y''-y=3e^{2x}\cos x.$$

Solution. The right side of the equation has the form

$$f(x)=e^{2x}(M\cos x+N\sin x),$$

and $M=3$, $N=0$. The auxiliary equation $k^2-1=0$ has roots $k_1=1$, $k_2=-1$. The general solution of the homogeneous equation is

$$\bar{y}=C_1e^x+C_2e^{-x}.$$

Since the number $\alpha+i\beta=2+i\cdot 1$ is not a root of the auxiliary equation, we seek the particular solution in the form

$$y^*=e^{2x}(A\cos x+B\sin x).$$

Putting this expression into the equation, we get (after collecting like terms)

$$(2A+4B)e^{2x}\cos x+(-4A+2B)e^{2x}\sin x=3e^{2x}\cos x.$$

Equating the coefficients of $\cos x$ and $\sin x$, we obtain

$$2A+4B=3,\quad -4A+2B=0.$$

Whence $A=\frac{3}{10}$ and $B=\frac{3}{5}$. Consequently, the particular solution is

$$y^*=e^{2x}\left(\frac{3}{10}\cos x+\frac{3}{5}\sin x\right),$$

and the general solution is

$$y=C_1e^x+C_2e^{-x}+e^{2x}\left(\frac{3}{10}\cos x+\frac{3}{5}\sin x\right).$$

SEC. 25. HIGHER-ORDER NONHOMOGENEOUS LINEAR EQUATIONS

Let us consider the equation

$$y^{(n)}+a_1y^{(n-1)}+\ldots+a_ny=f(x), \tag{1}$$

where a_1, a_2, ..., a_n, $f(x)$ are continuous functions of x (or constants).

Suppose we know the general solution

$$\bar{y}=C_1y_1+C_2y_2+\ldots+C_ny_n \tag{2}$$

of the corresponding homogeneous equation

$$\bar{y}^{(n)}+a_1y^{(n-1)}+a_2y^{(n-2)}+\ldots+a_ny=0. \tag{3}$$

As in the case of a second-order equation, the following assertion holds for equation (1).

Theorem. *If $\bar{y}$ is the general solution of the homogeneous equation (3) and y^* is a particular solution of the nonhomogeneous equation (1), then*

$$Y = \bar{y} + y^*$$

is the general solution of the nonhomogeneous equation.

Thus, the problem of integrating equation (1), as in the case of a second-order equation, reduces to finding a particular solution of the nonhomogeneous equation.

As in the case of a second-order equation, the particular solution of equation (1) may be found by the method of variation of parameters, considering $C_1, C_2, \ldots, C_n$ in expression (2) as functions of x.

We form the system of equations (cf. Sec. 23):

$$\left.\begin{array}{l} C_1'y_1 + C_2'y_2 + \ldots + C_n'y_n = 0, \\ C_1'y_1' + C_2'y_2' + \ldots + C_n'y_n' = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ C_1'y_1^{(n-2)} + C_2'y_2^{(n-2)} + \ldots + C_n'y_n^{(n-2)} = 0, \\ C_1'y_1^{(n-1)} + C_2'y_2^{(n-1)} + \ldots + C_n'y_n^{(n-1)} = f(x). \end{array}\right\} \quad (4)$$

This system of equations with the unknown functions $C_1', C_2', \ldots, C_n'$ has very definite solutions. (The determinant of the coefficients of $C_1', C_2', \ldots, C_n'$ is the Wronskian formed for the particular solutions $y_1, y_2, \ldots, y_n$ of a homogeneous equation, and since these particular solutions are, by definition, linearly independent, the Wronskian is not zero.)

Thus, the system (4) may be solved for the functions $C_1', C_2', \ldots, C_n'$. Finding them and integrating, we obtain

$$C_1 = \int C_1'\,dx + \bar{C}_1; \quad C_2 = \int C_2'\,dx + \bar{C}_2; \quad \ldots; \quad C_n = \int C_n'\,dx + \bar{C}_n,$$

where $\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_n$ are the constants of integration.

We shall prove that in such a case the expression

$$y^* = C_1y_1 + C_2y_2 + \ldots + C_ny_n \quad (5)$$

is the general solution of the nonhomogeneous equation (1).

Differentiate expression (5) n times, each time taking into account equations (4); this yields

$$\begin{array}{l} y^* = C_1y_1 + C_2y_2 + C_3y_3 + \ldots + C_ny_n, \\ y^{*\prime} = C_1y_1' + C_2y_2' + C_3y_3' + \ldots + C_ny_n', \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y^{*(n-1)} = C_1y_1^{(n-1)} + C_2y_2^{(n-1)} + \ldots + C_ny_n^{(n-1)}, \\ y^{*(n)} = C_1y_1^{(n)} + C_2y_2^{(n)} + \ldots + C_ny_n^{(n)} + f(x). \end{array}$$

Multiplying the terms of the first, second, ... and, finally, second to the last equation by a_n, a_{n-1}, ..., a_1, respectively, and adding, we get

$$y^{*(n)}+a_1y^{*(n-1)}+\ldots+a_ny^*=f(x),$$

since y_1, y_2, ..., y_n are particular solutions of the homogeneous equation; for this reason, the sums of the terms obtained in adding vertical columns are equal to zero.

Hence, the function $y^*=C_1y_1+\ldots+C_ny_n$ [where C_1, ..., C_n are functions of x determined from equations (4)] is a solution of the nonhomogeneous equation (1), and since this solution depends on the n arbitrary constants $\bar{C}_1$, $\bar{C}_2$, ..., $\bar{C}_n$, it is the general solution.

The proposition is thus proved.

For the case of a higher-order nonhomogeneous equation with constant coefficients (cf. Sec. 24), the particular solutions are found more easily, namely:

I. Let there be a function on the right side of the differential equation: $f(x)=P(x)e^{\alpha x}$, where $P(x)$ is a polynomial in x; then we have to distinguish two cases:

a) if α is not a root of the auxiliary equation, then the particular solution may be sought in the form

$$y^*=Q(x)e^{\alpha x},$$

where $Q(x)$ is a polynomial of the same degree as $P(x)$, but with undetermined coefficients;

b) if α is a root of multiplicity μ of the auxiliary equation, then the particular solution of the nonhomogeneous equation may be sought in the form

$$y^*=x^{\mu}Q(x)e^{\alpha x},$$

where $Q(x)$ is a polynomial of the same degree as $P(x)$.

II. Let the right side of the equation have the form

$$f(x)=M\cos\beta x+N\sin\beta x,$$

where M and N are constants. Then the form of the particular solution will be determined as follows:

a) if the number βi is not a root of the auxiliary equation, then the particular solution has the form

$$y^*=A\cos\beta x+B\sin\beta x,$$

where A and B are constant undetermined coefficients;

b) if the number βi is a root of the auxiliary equation of multiplicity μ, then

$$y^*=x^{\mu}(A\cos\beta x+B\sin\beta x).$$

III. Let

$$f(x)=P(x)e^{\alpha x}\cos\beta x+Q(x)e^{\alpha x}\sin\beta x,$$

where $P(x)$ and $Q(x)$ are polynomials in x. Then:

a) if the number $\alpha+\beta i$ is not a root of the auxiliary polynomial, then we seek the particular solution in the form

$$y^*=U(x)e^{\alpha x}\cos\beta x+V(x)e^{\alpha x}\sin\beta x,$$

where $U(x)$ and $V(x)$ are polynomials of degree equal to the highest degree of the polynomials $P(x)$ and $Q(x)$;

b) if the number $\alpha+\beta i$ is a root of multiplicity μ of the auxiliary polynomial, then we seek the particular solution in the form

$$y^*=x^{\mu}[U(x)e^{\alpha x}\cos\beta x+V(x)e^{\alpha x}\sin\beta x],$$

where $U(x)$ and $V(x)$ have the same meaning as in Case *a*.

General remarks on Cases II and III. Even when the right side of the equation contains an expression with only $\cos\beta x$ or only $\sin\beta x$, we must seek the solution in the form indicated, that is, with sine and cosine. In other words, from the fact that the right side does not contain $\cos\beta x$ or $\sin\beta x$, it does not in the least follow that the particular solution of the equation does not contain these functions. This was evident when we considered Examples 4, 5, 6 of the preceding section, and also Example 2 of the present section.

Example 1. Find the general solution of the equation

$$y^{IV}-y=x^3+1.$$

Solution. The auxiliary equation $k^4-1=0$ has the roots

$$k_1=1,\quad k_2=-1,\quad k_3=i,\quad k_4=-i.$$

We find the general solution of the homogeneous equation (see Example 4, Sec. 22):

$$\bar{y}=C_1e^x+C_2e^{-x}+C_3\cos x+C_4\sin x.$$

We seek the particular solution of the nonhomogeneous equation in the form

$$y^*=A_0x^3+A_1x^2+A_2x+A_3.$$

Differentiating y^* four times and substituting the expressions obtained into the given equation, we get

$$-A_0x^3-A_1x^2-A_2x-A_3=x^3+1.$$

Equating the coefficients of identical degrees of x, we have

$$-A_0=1;\quad -A_1=0;\quad -A_2=0;\quad -A_3=1.$$

Hence

$$y^*=-x^3-1.$$

The complete integral of the nonhomogeneous equation is found from the formula $y=\bar{y}+y^*$:

$$y=C_1e^x+C_2e^{-x}+C_3\cos x+C_4\sin x-x^3-1.$$

Example 2. To solve the equation

$$y^{IV}-y=5\cos x.$$

Solution. The auxiliary equation $k^4-1=0$ has the roots $k_1=1$, $k_2=-1$, $k_3=i$, $k_4=-i$. Hence, the general solution of the corresponding homogeneous equation is

$$\bar{y}=C_1e^x+C_2e^{-x}+C_3\cos x+C_4\sin x.$$

Further, the right side of the given nonhomogeneous equation has the form

$$f(x)=M\cos x+N\sin x,$$

where $M=5$ and $N=0$.

Since i is a simple root of the auxiliary equation, we seek the particular solution in the form

$$y^*=x(A\cos x+B\sin x).$$

Putting this expression into the equation, we find

$$4A\sin x-4B\cos x=5\cos x,$$

whence

$$4A=0, \quad -4B=5$$

or $A=0$, $B=-\frac{5}{4}$. Consequently, the particular solution of the differential equation is

$$y^*=-\frac{5}{4}x\sin x$$

and the general solution is

$$y=C_1e^x+C_2e^{-x}+C_3\cos x+C_4\sin x-\frac{5}{4}x\sin x.$$

SEC. 26. THE DIFFERENTIAL EQUATION OF MECHANICAL VIBRATIONS

In this and the following sections we shall consider a problem in applied mechanics, and investigate and solve it by means of linear differential equations.

Let a load of mass Q be at rest on an elastic spring (Fig. 268). We denote by y the deviation of the load from the equilibrium position. We shall consider deviation downwards as positive, upwards as negative. In the equilibrium position, the force of the weight is balanced by the elasticity of the spring. Let us suppose that the force that tends to return the load to equilibrium (the so-called restoring force) is proportional to the deflection, that is, equal to ky, where k is some constant for the given spring (the so-called "spring rigidity")*).

*) Springs whose restoring force is proportional to the deflection are called springs with a "linear characteristic".

Let us suppose that the motion of the load Q is restricted by a resistance force operating in a direction opposite to that of motion and proportional to the velocity of motion of the load relative to the lower point of the spring; that is, a force $-\lambda v = -\lambda \frac{dy}{dt}$, where $\lambda = \text{const} > 0$ (shock absorber). Write the differential equation of the motion of the load on the spring. By Newton's second law we have

$$Q\frac{d^2y}{dt^2} = -ky - \lambda\frac{dy}{dt} \tag{1}$$

(here, k and λ are positive numbers). We thus have a homogeneous linear differential equation of the second order with constant coefficients.

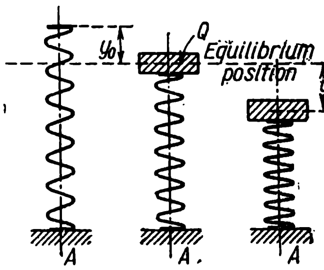


Fig. 268.

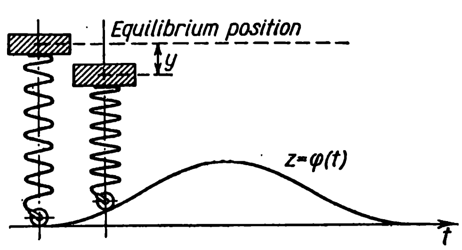


Fig. 269.

This equation may be rewritten as follows:

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0, \tag{1'}$$

where

$$p = \frac{\lambda}{Q}; \quad q = \frac{k}{Q}.$$

Let it further be assumed that the lower point of the spring A executes vertical motions under the law $z = \varphi(t)$. This will occur, for instance, if the lower end of the spring is attached to a roller, which moves over an uneven spot together with the spring and the load (Fig. 269).

In this case the restoring force will be equal not to $-ky$, but to $-k[y + \varphi(t)]$, the force of resistance will be $-\lambda[y' + \varphi'(t)]$, and in place of equation (1) we will have the equation

$$Q\frac{d^2y}{dt^2} + \lambda\frac{dy}{dt} + ky = -k\varphi(t) - \lambda\varphi'(t) \tag{2}$$

or

$$\frac{d^2y}{dt^2}+p\frac{dy}{dt}+qy=f(t), \tag{2'}$$

where

$$f(t)=-\frac{k\varphi(t)+\lambda\varphi'(t)}{Q}.$$

We thus have a nonhomogeneous second-order differential equation.

Equation (1′) is called an equation of *free oscillations*, equation (2′) is an equation of *forced oscillations*.

SEC. 27. FREE OSCILLATIONS

Let us first consider the equation of free oscillations

$$y''+py'+qy=0.$$

We write the corresponding auxiliary equation

$$k^2+pk+q=0$$

and find its roots:

$$k_1=-\frac{p}{2}+\sqrt{\frac{p^2}{4}-q}; \quad k_2=-\frac{p}{2}-\sqrt{\frac{p^2}{4}-q}.$$

1) Let $\frac{p^2}{4}>q$. Then the roots k_1 and k_2 are real negative numbers. The general solution is expressed in terms of exponential functions:

$$y=C_1e^{k_1t}+C_2e^{k_2t} \quad (k_1<0,\ k_2<0). \tag{1}$$

From this formula it follows that the deviation of y for any initial conditions approaches zero asymptotically if $t\to\infty$. In the given case, there will be no oscillations, since the forces of resistance are great compared to the coefficient of rigidity of the spring k.

2) Let $\frac{p^2}{4}=q$; then the roots k_1 and k_2 are equal $\left(\text{and are also equal to the negative number } -\frac{p}{2}\right)$. Therefore, the general solution will be

$$y=C_1e^{-\frac{p}{2}t}+C_2te^{-\frac{p}{2}t}=(C_1+C_2t)e^{-\frac{pt}{2}}. \tag{2}$$

Here the deviation also approaches zero as $t\to\infty$, but not so rapidly as in the preceding case (due to the factor C_1+C_2t).

3) Let $p=0$ (no resistance). The auxiliary equation is of the form

$$k^2+q=0,$$

and its roots are $k_1=\beta i$; $k_2=-\beta i$, where $\beta=\sqrt{q}$. The general solution is

$$y=C_1\cos\beta t+C_2\sin\beta t. \tag{3}$$

In the latter formula, we replace the arbitrary constants C_1 and C_2 with others. We introduce the constants A and φ_0, which are connected with C_1 and C_2 by the relations

$$C_1=A\sin\varphi_0,\quad C_2=A\cos\varphi_0.$$

A and φ_0 are defined as follows in terms of C_1 and C_2:

$$A=\sqrt{C_1^2+C_2^2},\quad \varphi_0=\operatorname{arc\,tan}\frac{C_1}{C_2}.$$

Substituting the values of C_1 and C_2 into formula (3), we get

$$y=A\sin\varphi_0\cos\beta t+A\cos\varphi_0\sin\beta t$$

or

$$y=A\sin(\beta t+\varphi_0). \tag{3'}$$

These oscillations are called *harmonic*. The integral curves are sine curves. The time interval T, during which the argument of the sine varies by 2π, is called the *period* of oscillation; here, $T=\frac{2\pi}{\beta}$. The *frequency* is the number of oscillations during time 2π; here, the frequency is β; A is the greatest deviation from equilibrium and is called the *amplitude*; φ_0 is the *initial phase*. The graph of the function (3′) is shown in Fig. 270.

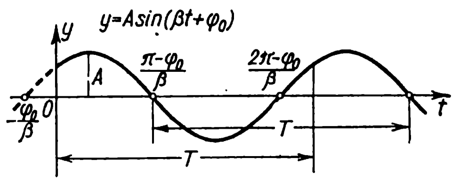


Fig. 270.

4) Let $p\neq 0$ and $\frac{p^2}{4}<q$.

In this case, the roots of the auxiliary equation are complex numbers:

$$k_1=\alpha+i\beta,\quad k_2=\alpha-i\beta,$$

where

$$\alpha=-\frac{p}{2}<0,\quad \beta=\sqrt{q-\frac{p^2}{4}}.$$

The complete integral has the form

$$y=e^{\alpha t}(C_1\cos\beta t+C_2\sin\beta t) \tag{4}$$

or

$$y = Ae^{\alpha t} \sin(\beta t + \varphi_0). \qquad (4')$$

Here, for the amplitude we have to consider the quantity $Ae^{\alpha t}$ which depends on the time. Since $\alpha < 0$, it approaches zero as

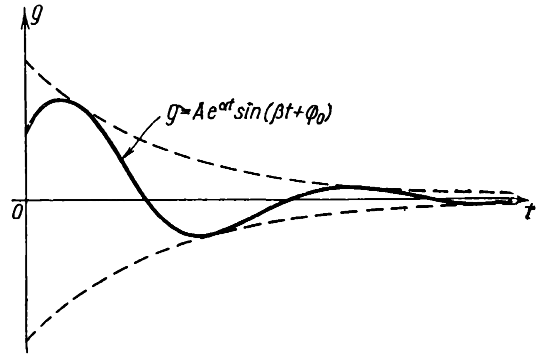


Fig. 271.

$t \to \infty$, which means that here we are dealing with *damped oscillations.* The graph of damped oscillations is shown in Fig. 271.

SEC. 28. FORCED OSCILLATIONS

The equation of forced oscillations has the form

$$y'' + py' + qy = f(t).$$

Let us consider an important practical case when the disturbing external force is periodic and varies under the law

$$f(t) = a \sin \omega t;$$

then the equation will have the form

$$y'' + py' + qy = a \sin \omega t. \qquad (1)$$

1) Let us first presume that $p \neq 0$ and $\frac{p^2}{4} < q$, that is, the roots of the auxiliary equation are the complex numbers $\alpha \pm i\beta$. In this case [see formulas (4) and (4'), Sec. 27], the general solution of the homogeneous equation has the form

$$\bar{y} = Ae^{\alpha t} \sin(\beta t + \varphi_0). \qquad (2)$$

We seek a particular solution of the nonhomogeneous equation in the form

$$y^* = M \cos \omega t + N \sin \omega t. \qquad (3)$$

Putting this expression of y^* into the original differential equation, we find the values of M and N:

$$M=\frac{-p\omega a}{(q-\omega^2)^2+p^2\omega^2}; \quad N=\frac{(q-\omega^2)\,a}{(q-\omega^2)^2+p^2\omega}.$$

Before putting these values of M and N into (3), let us introduce the new constants A^* and φ^*, setting

$$M=A^*\sin\varphi^*, \quad N=A^*\cos\varphi^*,$$

that is

$$A^*=\sqrt{M^2+N^2}=\frac{a}{\sqrt{(q-\omega^2)^2+p^2\omega^2}}, \quad \tan\varphi^*=\frac{M}{N}.$$

Then the particular solution of the nonhomogeneous equation may be written in the form

$$y^*=A^*\sin\varphi^*\cos\omega t+A^*\cos\varphi^*\sin\omega t=A^*\sin(\omega t+\varphi^*),$$

or, finally,

$$y^*=\frac{a}{\sqrt{(q-\omega^2)^2+p^2\omega^2}}\sin(\omega t+\varphi^*).$$

The complete integral of equation (1) is $y=\bar{y}+y^*$ or

$$y=Ae^{-\alpha t}\sin(\beta t+\varphi_0)+\frac{a}{\sqrt{(q-\omega^2)^2+p^2\omega^2}}\sin(\omega t+\varphi^*).$$

The first term of the sum on the right side (the solution of the homogeneous equation) represents damped oscillations; it diminishes with increasing t and, consequently, after some interval of time the second term (which determines the forced oscillations) will acquire prime importance. The frequency ω of these oscillations is equal to the frequency of the external force $f(t)$; the amplitude of the forced vibrations is the greater, the less p and the closer ω^2 is to q.

Let us investigate more closely the dependence of the amplitude of forced vibrations on the frequency ω for various values of p. For this, we denote the amplitude of forced vibrations by $D(\omega)$:

$$D(\omega)=\frac{a}{\sqrt{(q-\omega^2)^2+p^2\omega^2}}.$$

Putting $q=\beta_1^2$ (for $p=0$, β_1 would be equal to its natural frequency), we have

$$D(\omega)=\frac{a}{\sqrt{(\beta_1^2-\omega^2)^2+p^2\omega^2}}=\frac{a}{\beta_1^2\sqrt{\left(1-\frac{\omega^2}{\beta_1^2}\right)^2+\frac{p^2}{\beta_1^2}\frac{\omega^2}{\beta_1^2}}},$$

Introducing the notation

$$\frac{\omega}{\beta_1}=\lambda; \quad \frac{p}{\beta_1}=\gamma,$$

where λ is the ratio of the frequency of the disturbing force to the frequency of free oscillations of the system, and the constant γ is independent of the disturbing force, we find that the magnitude of the amplitude will be expressed by the formula

$$\overline{D}(\lambda)=\frac{a}{\beta_1^2\sqrt{(1-\lambda^2)^2+\gamma^2\lambda^2}}. \tag{4}$$

Let us find the maximum of this function. It will obviously be for that value of λ for which the square of the denominator has a minimum. But the minimum of the function

$$\sqrt{(1-\lambda^2)^2+\gamma^2\lambda^2} \tag{5}$$

is reached when

$$\lambda=\sqrt{1-\frac{\gamma^2}{2}}$$

and is equal to

$$\gamma=\sqrt{1-\frac{\gamma^2}{4}}.$$

Hence, the maximum amplitude is equal to

$$\overline{D}_{\max}=\frac{a}{\beta_1^2\gamma\sqrt{1-\frac{\gamma^2}{4}}}.$$

The graphs of the function $\overline{D}(\lambda)$ for various values of γ are shown in Fig. 272 (in constructing the graphs we put $a=1$, $\beta_1=1$ for the sake of definiteness). These curves are called resonance curves.

From formula (5) it follows that for small γ the maximum value of amplitude is attained for values of λ close to unity, that is, when the frequency of the external force is close to the frequency of free oscillations. If $\gamma=0$ (thus, $p=0$), that is, if there is no resistance to motion, the amplitude of forced vibrations increases without bound as $\lambda \to 1$ or as $\omega \to \beta_1=\sqrt{q}$:

$$\lim_{\substack{\lambda\to 1\\(\gamma=0)}} \overline{D}(\lambda)=\infty.$$

At $\omega^2=q$ we have resonance.

2) Now let us suppose that $p=0$; that is, we consider the equation of elastic oscillations without resistance but with a periodic external force:

$$y'' + qy = a \sin \omega t.$$

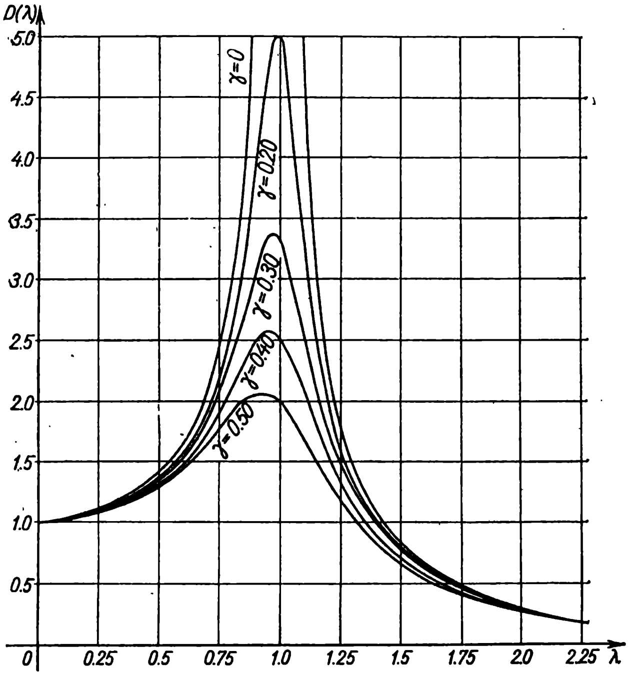


Fig. 272.

The general solution of the homogeneous equation is

$$\bar{y} = C_1 \cos \beta t + C_2 \sin \beta t \quad (\beta^2 = q).$$

If $\beta \neq \omega$, that is, if the frequency of the external force is not equal to the natural frequency, then the particular solution of the nonhomogeneous equation will have the form

$$y^* = M \cos \omega t + N \sin \omega t. \tag{6}$$

Putting this expression into the original equation, we find

$$M = 0, \quad N = \frac{a}{q - \omega^2}.$$

The general solution is

$$y = A \sin(\beta t + \varphi_0) + \frac{a}{q - \omega^2} \sin \omega t.$$

Thus, motion results from the superposition of a natural oscillation with frequency β and a forced vibration with frequency ω.

If $\beta=\omega$, that is, the natural frequency coincides with the frequency of the external force, then function (3) is not a solution of equation (6). In this case, in accord with the results of Sec. 24, we have to seek the particular solution in the form

$$y^* = t(M\cos\omega t + N\sin\omega t). \quad (7)$$

Substituting this expression into the equation, we find M and N:

$$M=-\frac{a}{2\omega};\; N=0.$$

Consequently,

$$y^* = -\frac{a}{2\beta}t\cos\omega t.$$

The general solution will have the form

$$y = A\sin(\beta t+\varphi_0) - \frac{a}{2\omega}t\cos\beta t.$$

The second term on the right side shows that in this case the amplitude increases without bound with the time t. This phenomenon, which occurs when the natural frequency of the system coincides with the frequency of the external force, is called *resonance*.

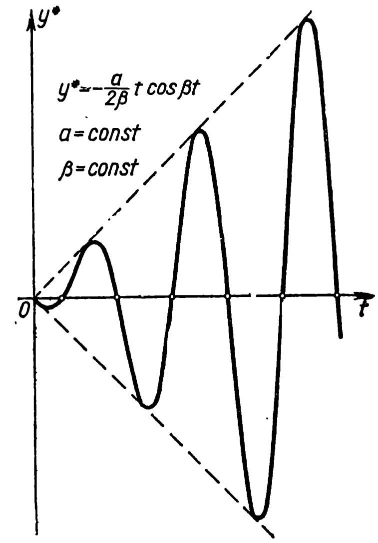


Fig. 273.

The graph of the function y^* is shown in Fig. 273.

SEC. 29. SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

In the solution of many problems it is required to find the functions $y_1=y_1(x)$, $y_2=y_2(x)$, ..., $y_n=y_n(x)$, which satisfy a system of differential equations containing the argument x, the unknown functions $y_1, y_2, \ldots, y_n$ and their derivatives.

Consider the following system of first-order equations:

$$\left.\begin{aligned} \frac{dy_1}{dx}&=f_1(x, y_1, y_2, \ldots, y_n), \\ \frac{dy_2}{dx}&=f_2(x, y_1, y_2, \ldots, y_n), \\ &\ldots\ldots\ldots\ldots \\ \frac{dy_n}{dx}&=f_n(x, y_1, y_2, \ldots, y_n), \end{aligned}\right\} \quad (1)$$

where $y_1, y_2, \ldots, y_n$ are unknown functions and x is the argument.

A system of this kind, where the left sides of the equations contain first-order derivatives, while the right sides do not contain derivatives, is called *normal*.

To integrate the system means to determine the functions $y_1, y_2, \ldots, y_n$, which satisfy the system of equations (1) and the given initial conditions:

$$(y_1)_{x=x_0}=y_{10},\ (y_2)_{x=x_0}=y_{20},\ \ldots,\ (y_n)_{x=x_0}=y_{n0}. \qquad (2)$$

Integration of a system like (1) is performed as follows. Differentiate the first equation of (1) with respect to x:

$$\frac{d^2y_1}{dx^2}=\frac{\partial f_1}{\partial x}+\frac{\partial f_1}{\partial y_1}\frac{dy_1}{dx}+\ldots+\frac{\partial f_1}{\partial y_n}\frac{dy_n}{dx}.$$

Replacing the derivatives $\frac{dy_1}{dx}, \frac{dy_2}{dx}, \ldots, \frac{dy_n}{dx}$ with their expressions $f_1, f_2, \ldots, f_n$ from equations (1), we get the equation

$$\frac{d^2y_1}{dx^2}=F_2(x, y_1, \ldots, y_n).$$

Differentiating this equation and then doing as before, we obtain

$$\frac{d^3y_1}{dx^3}=F_3(x, y_1, y_2, \ldots, y_n).$$

Continuing in the same fashion, we finally get the equation

$$\frac{d^ny_1}{dx^n}=F_n(x, y_1, \ldots, y_n).$$

We thus get the following system:

$$\left.\begin{aligned}\frac{dy_1}{dx}&=f_1(x, y_1, \ldots, y_n),\\ \frac{d^2y_1}{dx^2}&=F_2(x, y_1, \ldots, y_n),\\ &\ldots\ldots\ldots\ldots\\ \frac{d^ny_1}{dx^n}&=F_n(x, y_1, \ldots, y_n).\end{aligned}\right\} \qquad (3)$$

From the first $n-1$ equations we determine (if this is possible) $y_2, y_3, \ldots, y_n$ and express them in terms of x, y_1 and the derivatives $\frac{dy_1}{dx}, \frac{d^2y_1}{dx^2}, \ldots, \frac{d^{n-1}y_1}{dx^{n-1}}$:

$$\left.\begin{aligned}y_2&=\varphi_2(x, y_1, y_1', \ldots, y_1^{(n-1)}),\\ y_3&=\varphi_3(x, y_1, y_1', \ldots, y_1^{(n-1)}),\\ &\ldots\ldots\ldots\ldots\\ y_n&=\varphi_n(x, y_1, y_1', \ldots, y_1^{(n-1)}).\end{aligned}\right\} \qquad (4)$$

Putting these expressions into the last of the equations (3), we get an nth-order equation for determining y_1:

$$\frac{d^n y_1}{dx^n} = \Phi(x, y_1, y_1', \ldots, y_1^{(n-1)}). \tag{5}$$

Solving this equation, we find y_1:

$$y_1 = \psi_1(x, C_1, C_2, \ldots, C_n). \tag{6}$$

Differentiating the latter expression $n-1$ times, we find the derivatives $\frac{dy_1}{dx}, \frac{d^2y_1}{dx^2}, \ldots, \frac{d^{n-1}y_1}{dx^{n-1}}$ as functions of $x, C_1, C_2, \ldots, C_n$.

Substituting these functions into equations (4), we determine $y_2, y_3, \ldots, y_n$:

$$\left.\begin{array}{l} y_2 = \psi_2(x, C_1, C_2, \ldots, C_n), \\ \ldots\ldots\ldots\ldots\ldots \\ y_n = \psi_n(x, C_1, C_2, \ldots, C_n). \end{array}\right\} \tag{7}$$

For this solution to satisfy the given initial conditions (2), it remains for us to find [from equations (6) and (7)] the appropriate values of the constants $C_1, C_2, \ldots, C_n$ (like we did in the case of a **single** differential equation).

Note 1. If the system (1) is linear in the unknown functions, then equation (5) is also linear.

Example 1. Integrate the system

$$\frac{dy}{dx} = y + z + x, \quad \frac{dz}{dx} = -4y - 3z + 2x \tag{a}$$

with the initial conditions

$$(y)_{x=0} = 1, \quad (z)_{x=0} = 0. \tag{b}$$

Solution. 1) Differentiating the first equation with respect to x, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{dz}{dx} + 1.$$

Putting the expressions $\frac{dy}{dx}$ and $\frac{dz}{dx}$ from equations (a) into this equation, we get

$$\frac{d^2y}{dx^2} = (y + z + x) + (-4y - 3z + 2x) + 1$$

or

$$\frac{d^2y}{dx^2} = -3y - 2z + 3x + 1. \tag{c}$$

2) From the first equation of system (a) we find

$$z = \frac{dy}{dx} - y - x \tag{d}$$

and put it into the equation just obtained; we get

$$\frac{d^2y}{dx^2}=-3y-2\left(\frac{dy}{dx}-y-x\right)+3x+1$$

or

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=5x+1. \qquad \text{(e)}$$

The general solution of this equation is

$$y=(C_1+C_2x)e^{-x}+5x-9 \qquad \text{(f)}$$

and from (d) we have

$$z=(C_2-2C_1-2C_2x)e^{-x}-6x+14. \qquad \text{(g)}$$

Choosing the constants C_1 and C_2 so that the initial conditions (b) are satisfied,

$$(y)_{x=0}=1, \quad (z)_{x=0}=0,$$

we get, from equations (f) and (g),

$$1=C_1-9, \quad 0=C_2-2C_1+14,$$

whence $C_1=10$ and $C_2=6$.

Thus, the solution that satisfies the given initial conditions (b) has the form

$$y=(10+6x)e^{-x}+5x-9, \quad z=(-14-12x)e^{-x}-6x+14.$$

Note 2. In the foregoing we assumed that from the first $n-1$ equations of the system (3) it is possible to determine the functions $y_2, y_3, \ldots, y_n$. It may happen that the variables $y_2, \ldots, y_n$ are eliminated not from n, but from a smaller number of equations. Then to determine y_1 we will have an equation of order less than n.

Example 2. Integrate the system

$$\frac{dx}{dt}=y+z; \quad \frac{dy}{dt}=x+z; \quad \frac{dz}{dt}=x+y.$$

Solution. Differentiating the first equation with respect to t, we find

$$\frac{d^2x}{dt^2}=\frac{dy}{dt}+\frac{dz}{dt}=(x+z)+(x+y),$$

$$\frac{d^2x}{dt^2}=2x+y+z.$$

Eliminating the variables y and z from the equations

$$\frac{dx}{dt}=y+z; \quad \frac{d^2x}{dt^2}=2x+y+z,$$

we get a second-order equation in x:

$$\frac{d^2x}{dt^2}-\frac{dx}{dt}-2x=0.$$

Integrating this equation, we obtain its general solution:

$$x=C_1e^{-t}+C_2e^{2t}. \qquad (\alpha)$$

Whence we find

$$\frac{dx}{dt}=-C_1e^{-t}+2C_2e^{2t} \text{ and } y=\frac{dx}{dt}-z=-C_1e^{-t}+2C_2e^{2t}-z. \qquad (\beta)$$

Putting into the third of the given equations the expressions that have been found for x and y, we get an equation for determining z:

$$\frac{dz}{dt}+z=3C_2e^{2t}.$$

Integrating this equation, we find

$$z=C_3e^{-t}+C_2e^{2t}. \qquad (\gamma)$$

But then, from equation (β), we get

$$y=-(C_1+C_3)e^{-t}+C_2e^{2t}.$$

Equations (α), (β), and (γ) give the general solution of the given system.

The differential equations of the system can contain higher-order derivatives. This then yields a system of differential equations of higher order.

For instance, the problem of the motion of a material point under the action of a force F reduces to a system of three second-order differential equations. Let F_x, F_y, F_z be the projections of the force F on the coordinate axes. The position of the point at any instant of time t is determined by its coordinates x, y, and z. Hence, x, y, z are functions of t. The projections of the velocity vector of the point on the axes will be $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$.

Suppose that the force F and, hence, its projections F_x, F_y, F_z depend on the time t, the position x, y, z of the point, and on the velocity of motion of the point, that is, on $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$.

In this problem the following three functions are the sought-for functions;

$$x=x(t), \quad y=y(t), \quad z=z(t).$$

These functions are determined from equations of dynamics (Newton's law):

$$\left.\begin{aligned} m\frac{d^2x}{dt^2}&=F_x\left(t,\ x,\ y,\ z,\ \frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt}\right),\\ m\frac{d^2y}{dt^2}&=F_y\left(t,\ x,\ y,\ z,\ \frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt}\right),\\ m\frac{d^2z}{dt^2}&=F_z\left(t,\ x,\ y,\ z,\ \frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt}\right).\end{aligned}\right\} \qquad (8)$$

We thus have a system of three second-order differential equations. In the case of plane motion, that is, motion in which the trajectory is a plane

curve (lying, for example, in the xy-plane), we get a system of two equations for determining the functions $x(t)$ and $y(t)$:

$$m\frac{d^2x}{dt^2}=F_x\left(t,\ x,\ y,\ \frac{dx}{dt},\ \frac{dy}{dt}\right), \tag{9}$$

$$m\frac{d^2y}{dt^2}=F_y\left(t,\ x,\ y,\ \frac{dx}{dt},\ \frac{dy}{dt}\right). \tag{10}$$

It is possible to solve a system of differential equations of higher order by reducing it to a system of first-order equations. Using equations (9) and (10) as examples, we shall show how this is done. We introduce the notation

$$\frac{dx}{dt}=u,\quad \frac{dy}{dt}=v.$$

Then

$$\frac{d^2x}{dt^2}=\frac{du}{dt},\qquad \frac{d^2y}{dt^2}=\frac{dv}{dt}.$$

The system of two second-order equations (9) and (10) with two unknown functions $x(t)$ and $y(t)$ is replaced by a system of four first-order equations with four unknown functions x, y, u, v:

$$\frac{dx}{dt}=u,$$

$$\frac{dy}{dt}=v,$$

$$m\frac{du}{dt}=F_x(t,\ x,\ y,\ u,\ v),$$

$$m\frac{dv}{dt}=F_y(t,\ x,\ y,\ u,\ v).$$

We remark in conclusion that the general method that we have considered of solving the system may, in certain specific cases, be replaced by some artificial technique that gets the result faster.

Example 3. To find the general solution of the following system of differential equations:

$$\frac{d^2y}{dx^2}=z,$$

$$\frac{d^2z}{dx^2}=y.$$

Solution. Differentiate, with respect to x, both sides of the first equation twice:

$$\frac{d^4y}{dx^4}=\frac{d^2z}{dx^2}.$$

But $\frac{d^2z}{dx^2}=y$, and so we get a fourth-order equation:

$$\frac{d^4y}{dx^4}=y.$$

Integrating this equation, we obtain its general solution (see Sec. 22, Example 4):

$$y=C_1e^x+C_2e^{-x}+C_3\cos x+C_4\sin x.$$

Finding $\frac{d^2y}{dx^2}$ from this equation and putting it into the first equation, we find z:

$$z = C_1e^x + C_2e^{-x} - C_3\cos x - C_4\sin x.$$

SEC. 30. SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Suppose we have the following system of differential equations:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \\ &\ldots\ldots\ldots\ldots\ldots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n, \end{aligned}\right\} \qquad (1)$$

where the coefficients a_{ij} are constants. Here, t is the argument, and $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ are the unknown functions. The system (1) is a *system of homogeneous linear differential equations with constant coefficients.*

As was pointed out in the preceding section, this system may be solved by reducing it to a single equation of order n, which in the given instance will be linear (this was indicated in Note 1 of the preceding section). But system (1) may be solved in another way, without reducing it to an equation of the nth order. This method makes it possible to analyse the character of the solutions more clearly.

We seek a particular solution of the system in the following form:

$$x_1 = \alpha_1 e^{kt}, \quad x_2 = \alpha_2 e^{kt}, \ \ldots, \ x_n = \alpha_n e^{kt}. \qquad (2)$$

It is required to determine the constants α_1, α_2, ..., α_n and k in such a way that the functions $\alpha_1 e^{kt}$, $\alpha_2 e^{kt}$, ..., $\alpha_n e^{kt}$ should satisfy the system of equations (1). Putting them into system (1), we get:

$$\left.\begin{aligned} k\alpha_1 e^{kt} &= (a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots + a_{1n}\alpha_n)\,e^{kt}, \\ k\alpha_2 e^{kt} &= (a_{21}\alpha_1 + a_{22}\alpha_2 + \ldots + a_{2n}\alpha_n)\,e^{kt}, \\ &\ldots\ldots\ldots\ldots\ldots \\ k\alpha_n e^{kt} &= (a_{n1}\alpha_1 + a_{n2}\alpha_2 + \ldots + a_{nn}\alpha_n)\,e^{kt}. \end{aligned}\right\} \qquad (2)$$

Cancel out e^{kt}. Transposing all terms to one side and collecting coefficients of $\alpha_1, \alpha_2, \ldots, \alpha_n$, we get a system of equations:

$$\left.\begin{aligned} &(a_{11}-k)\alpha_1+a_{12}\alpha_2+\ldots+a_{1n}\alpha_n=0,\\ &a_{21}\alpha_1+(a_{22}-k)\alpha_2+\ldots+a_{2n}\alpha_n=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &a_{n1}\alpha_1+a_{n2}\alpha_2+\ldots+(a_{nn}-k)\alpha_n=0. \end{aligned}\right\} \qquad (3)$$

Choose $\alpha_1, \alpha_2, \ldots, \alpha_n$ and k such that will satisfy the system (3). This is a system of linear algebraic equations in $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let us form the determinant of the system (3):

$$\Delta(k)=\begin{vmatrix} a_{11}-k & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22}-k & \ldots & a_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ a_{n1} & a_{n2} & \ldots & (a_{nn}-k) \end{vmatrix} \qquad (4)$$

If k is such that the determinant Δ is different from zero, then the system (3) has only trivial solutions $\alpha_1=\alpha_2=\ldots=\alpha_n=0$ and, hence, formulas (2) yield only trivial solutions:

$$x_1(t)=x_2(t)=\ldots=x_n(t)\equiv 0.$$

Thus, we obtain nontrivial solutions (2) only for k such that the determinant (4) vanishes. We arrive at an equation of order n for determining k:

$$\begin{vmatrix} a_{11}-k & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22}-k & \ldots & a_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ a_{n1} & a_{n2} & \ldots & a_{nn}-k \end{vmatrix}=0. \qquad (5)$$

This equation is called the **characteristic equation** of the system (1), and its roots are the **roots of the characteristic equation.**

Let us consider a few cases.

I. The roots of the characteristic equation are real and distinct. Denote by $k_1, k_2, \ldots, k_n$ the roots of the characteristic equation. For each root k_i write the system (3) and determine the coefficients

$$\alpha_1^{(i)},\ \alpha_2^{(i)},\ \ldots,\ \alpha_n^{(i)}.$$

It may be shown that one of them is arbitrary; it may be considered equal to unity. Thus we obtain:

for the root k_1 the following solution of the system (1)

$$x_1^{(1)}=\alpha_1^{(1)}e^{k_1t},\quad x_2^{(1)}=\alpha_2^{(1)}e^{k_1t},\ \ldots,\ x_n^{(1)}=\alpha_n^{(1)}e^{k_1t};$$

for the root k_2 the solution of the system (1)

$$x_1^{(2)}=\alpha_1^{(2)}e^{k_2t},\quad x_2^{(2)}=\alpha_2^{(2)}e^{k_2t},\ \dots,\ x_n^{(2)}=\alpha_n^{(2)}e^{k_2t};$$

. . . .

for the root k_n the solution of the system (1)

$$x_1^{(n)}=\alpha_1^{(n)}e^{k_nt},\ x_2^{(n)}=\alpha_2^{(n)}e^{k_nt},\ \dots,\ x_n^{(n)}=\alpha_n^{(n)}e^{k_nt}.$$

By direct substitution into the equations we see that the system of functions

$$\left.\begin{aligned} x_1&=C_1\alpha_1^{(1)}e^{k_1t}+C_2\alpha_1^{(2)}e^{k_2t}+\dots+C_n\alpha_1^{(n)}e^{k_nt},\\ x_2&=C_1\alpha_2^{(1)}e^{k_1t}+C_2\alpha_2^{(2)}e^{k_2t}+\dots+C_n\alpha_2^{(n)}e^{k_nt},\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ x_n&=C_1\alpha_n^{(1)}e^{k_1t}+C_2\alpha_n^{(2)}e^{k_2t}+\dots+C_n\alpha_n^{(n)}e^{k_nt}, \end{aligned}\right\} \tag{6}$$

where $C_1, C_2, \dots, C_n$ are arbitrary constants, is likewise a solution of the system of differential equations (1). This is the **general solution of system** (1). It may readily be shown that one can find values of the constants such that the solution will satisfy the given initial conditions.

Example 1. Find the general solution of the system of equations

$$\frac{dx_1}{dt}=2x_1+2x_2,\ \frac{dx_2}{dt}=x_1+3x_2.$$

Solution. Form the characteristic equation

$$\begin{vmatrix} 2-k & 2\\ 1 & 3-k\end{vmatrix}=0$$

or $k^2-5k+4=0$. Find its roots:

$$k_1=1,\quad k_2=4.$$

Seek the solution of the system in the form

$$x_1^{(1)}=\alpha_1^{(1)}e^t,\quad x_2^{(1)}=\alpha_2^{(1)}e^t$$

and

$$x_1^{(2)}=\alpha_1^{(2)}e^{4t},\ x_2^{(2)}=\alpha_2^{(2)}e^{4t}.$$

Form the system (3) for the root $k_1=1$ and determine $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$:

$$\left.\begin{aligned}(2-1)\,\alpha_1^{(1)}+2\alpha_2^{(1)}&=0,\\ 1\alpha_1^{(1)}+(3-1)\,\alpha_2^{(1)}&=0\end{aligned}\right\}$$

or

$$\alpha_1^{(1)}+2\alpha_2^{(1)}=0,$$
$$\alpha_1^{(1)}+2\alpha_2^{(1)}=0,$$

whence $\alpha_2^{(1)}=-\frac{1}{2}\alpha_1^{(1)}$. Putting $\alpha_1^{(1)}=1$, we get $\alpha_2^{(1)}=-\frac{1}{2}$. Thus, we obtain the solution of the system:

$$x_1^{(1)}=e^t,\quad x_2^{(1)}=-\frac{1}{2}e^t.$$

Now form the system (3) for the root $k_2=4$ and determine $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$:

$$-2\alpha_1^{(2)}+2\alpha_2^{(2)}=0,$$
$$\alpha_1^{(2)}-2\alpha_2^{(2)}=0,$$

whence $\alpha_1^{(2)}=\alpha_2^{(2)}$ and $\alpha_1^{(2)}=1$, $\alpha_2^{(2)}=1$. We obtain the second solution of the system:

$$x_1^{(2)}=e^{4t},\quad x_2^{(2)}=e^{4t}.$$

The general solution of the system will be [see (6)]

$$x_1=C_1e^t+C_2e^{4t},$$
$$x_2=-\frac{1}{2}C_1e^t+C_2e^{4t}.$$

II. The roots of the characteristic equation are distinct, but include complex roots. Among the roots of the characteristic equation let there be two complex conjugate roots:

$$k_1=\alpha+i\beta,\quad k_2=\alpha-i\beta.$$

To these roots will correspond the solutions

$$x_j^{(1)}=\alpha_j^{(1)}e^{(\alpha+i\beta)t}\quad (j=1,\ 2,\ \ldots,n), \tag{7}$$
$$x_j^{(2)}=\alpha_j^{(2)}e^{(\alpha-i\beta)t}\quad (j=1,\ 2,\ \ldots,n). \tag{8}$$

The coefficients $\alpha_j^{(1)}$ and $\alpha_j^{(2)}$ are determined from the system of equations (3).

Just as in Sec. 21, it may be shown that the real and imaginary parts of the complex solution are also solutions. We thus obtain two particular solutions:

$$\left.\begin{aligned}\bar{x}_j^{(1)}&=e^{\alpha t}(\lambda_j^{(1)}\cos\beta x+\lambda_j^{(2)}\sin\beta x),\\ \bar{x}_j^{(2)}&=e^{\alpha t}(\bar{\lambda}_j^{(1)}\sin\beta x+\bar{\lambda}_j^{(2)}\cos\beta x),\end{aligned}\right\} \tag{9}$$

where $\lambda_j^{(1)}$, $\lambda_j^{(2)}$, $\bar{\lambda}_j^{(1)}$, $\bar{\lambda}_j^{(2)}$ are real numbers determined in terms of $\alpha_j^{(1)}$ and $\alpha_j^{(2)}$.

Appropriate combinations of functions (9) will enter into the general solution of the system.

Example 2. Find the general solution of the system

$$\frac{dx_1}{dt} = -7x_1 + x_2,$$
$$\frac{dx_2}{dt} = -2x_1 - 5x_2.$$

Solution. Form the characteristic equation

$$\begin{vmatrix} -7-k & 1 \\ -2 & -5-k \end{vmatrix} = 0$$

or $k^2 + 12k + 37 = 0$ and find its roots:

$$k_1 = -6 + i, \quad k_2 = -6 - i.$$

Substituting $k_1 = -6 + i$ into the system (3), we find

$$\alpha_1^{(1)} = 1, \quad \alpha_2^{(1)} = 1 + i.$$

We write the solution (7):

$$x_1^{(1)} = 1e^{(-6+i)t}, \quad x_2^{(1)} = (1+i)\,e^{(-6+i)t}. \qquad (7')$$

Putting $k_2 = -6 - i$ into system (3), we find

$$\alpha_1^{(2)} = 1, \quad \alpha_2^{(2)} = 1 - i.$$

We get a second system of solutions (8):

$$x_1^{(2)} = e^{(-6-i)t}, \quad x_2^{(2)} = (1-i)\,e^{(-6-i)t}. \qquad (8')$$

Rewrite the solution (7'):

$$x_1^{(1)} = e^{-6t}(\cos t + i \sin t),$$
$$x_2^{(1)} = (1+i)\,e^{-6t}(\cos t + i \sin t)$$

or

$$x_1^{(1)} = e^{-6t}\cos t + ie^{-6t}\sin t,$$
$$x_2^{(1)} = e^{-6t}(\cos t - \sin t) + ie^{-6t}(\cos t + \sin t).$$

Rewrite the solution (8'):

$$x_1^{(2)} = e^{-6t}\cos t - ie^{-6t}\sin t,$$
$$x_2^{(2)} = e^{-6t}(\cos t - \sin t) - ie^{-6t}(\cos t + \sin t).$$

For systems of particular solutions we can take the real parts and the imaginary parts separately:

$$\left.\begin{aligned} \bar{x}_1^{(1)} &= e^{-6t}\cos t, & \bar{x}_2^{(1)} &= e^{-6t}(\cos t - \sin t), \\ \bar{x}_1^{(2)} &= e^{-6t}\sin t, & \bar{x}_2^{(2)} &= e^{-6t}(\cos t + \sin t), \end{aligned}\right\} \qquad (9')$$

The general solution of the system is

$$x_1 = C_1 e^{-6t}\cos t + C_2 e^{-6t}\sin t,$$
$$x_2 = C_1 e^{-6t}(\cos t - \sin t) + C_2 e^{-6t}(\cos t + \sin t).$$

By a similar method it is possible to find the solution of a system of linear differential equations of higher order with constant coefficients.

For instance, in mechanics and electric-circuit theory a study is made of the solution of a system of second-order differential equations:

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=a_{11}x+a_{12}y,\\ \frac{d^2y}{dt^2}&=a_{21}x+a_{22}y.\end{aligned}\right\}\tag{10}$$

Again we seek the solution in the form

$$x=\alpha e^{kt},\quad y=\beta e^{kt}.$$

Putting these expressions into system (10) and cancelling out e^{kt} we get a system of equations for determining α, β and k:

$$\left.\begin{aligned}(a_{11}-k^2)\alpha+a_{12}\beta&=0,\\ a_{21}\alpha+(a_{22}-k^2)\beta&=0.\end{aligned}\right\}\tag{11}$$

Nonzero α and β are determined only when the determinant of the system is equal to zero:

$$\begin{vmatrix}a_{11}-k^2 & a_{12}\\ a_{21} & a_{22}-k^2\end{vmatrix}=0.\tag{12}$$

This is the characteristic equation of system (10); it is a fourth-order equation in k. Let k_1, k_2, k_3, and k_4 be its roots (we assume that the roots are distinct). For each root k_i of system (11) we find the values of α and β. The general solution, like (6), will have the form

$$x=C_1\alpha^{(1)}e^{k_1t}+C_2\alpha^{(2)}e^{k_2t}+C_3\alpha^{(3)}e^{k_3t}+C_4\alpha^{(4)}e^{k_4t},$$
$$y=C_1\beta^{(1)}e^{k_1t}+C_2\beta^{(2)}e^{k_2t}+C_3\beta^{(3)}e^{k_3t}+C_4\beta^{(4)}e^{k_4t}.$$

If there are complex roots, then to each pair of complex roots in the general solution there will correspond expressions of the form (9).

Example 3. Find the general solution of the following system of differential equations

$$\frac{d^2x}{dt^2}=x-4y,$$
$$\frac{d^2y}{dt^2}=-x+y.$$

Solution. Write the characteristic equation (12) and find its roots:

$$\begin{vmatrix} 1-k^2 & -4 \\ -1 & 1-k^2 \end{vmatrix}=0,$$

$$k_1=i, \quad k_2=-i, \quad k_3=\sqrt{3}, \quad k_4=-\sqrt{3}.$$

We shall seek the solution in the form

$$\begin{aligned} x^{(1)}&=\alpha^{(1)}e^{it}, & y^{(1)}&=\beta^{(1)}e^{it}, \\ x^{(2)}&=\alpha^{(2)}e^{-it}, & y^{(2)}&=\beta^{(2)}e^{-it}, \\ x^{(3)}&=\alpha^{(3)}e^{\sqrt{3}\,t}, & y^{(3)}&=\beta^{(3)}e^{\sqrt{3}\,t}, \\ x^{(4)}&=\alpha^{(4)}e^{-\sqrt{3}\,t}, & y^{(4)}&=\beta^{(4)}e^{-\sqrt{3}\,t}. \end{aligned}$$

From system (11) we find $\alpha^{(j)}$ and $\beta^{(j)}$:

$$\begin{aligned} \alpha^{(1)}&=1, & \beta^{(1)}&=\frac{1}{2}, \\ \alpha^{(2)}&=1, & \beta^{(2)}&=\frac{1}{2}, \\ \alpha^{(3)}&=1, & \beta^{(3)}&=-\frac{1}{2}, \\ \alpha^{(4)}&=1, & \beta^{(4)}&=-\frac{1}{2}. \end{aligned}$$

We write out the complex solutions:

$$x^{(1)}=e^{-it}=\cos t+i\sin t, \quad y^{(1)}=\frac{1}{2}(\cos t+i\sin t),$$

$$x^{(2)}=e^{-it}=\cos t-i\sin t, \quad y^{(2)}=\frac{1}{2}(\cos t-i\sin t).$$

The real and imaginary parts separately form the solution:

$$\bar{x}^{(1)}=\cos t, \quad \bar{y}^{(1)}=\frac{1}{2}\cos t,$$

$$\bar{x}^{(2)}=\sin t, \quad \bar{y}^{(2)}=\frac{1}{2}\sin t.$$

We can now write the general solution:

$$x=C_1\cos t+C_2\sin t+C_3e^{\sqrt{3}\,t}+C_4e^{-\sqrt{3}\,t},$$

$$y=C_1\frac{1}{2}\cos t+\frac{1}{2}C_2\sin t-C_3\frac{1}{2}e^{\sqrt{3}\,t}-C_4\frac{1}{2}e^{-\sqrt{3}\,t}.$$

Note. In this section we did not consider the case of multiple roots of the characteristic equation. This question is dealt with in detail in "Lectures on the Theory of Ordinary Differential Equations" by I. G. Petrovsky.

SEC. 31. ON LYAPUNOV'S THEORY OF STABILITY

Since the solutions of most differential equations and systems of equations are not expressible in terms of elementary functions or quadratures, use is made (in these cases when solving concrete differential equations) of approximate methods of integration. The elements of these methods were given in Sec. 3; in addition, some of these methods will be considered in Secs. 32 through 34 and in Chapter XVI.

The drawback of these methods lies in the fact that they yield only one particular solution; to obtain other particular solutions, one has to carry out all the calculations again. Knowing one particular solution does not permit us to draw conclusions about the character of the other solutions.

In many problems of mechanics and engineering it is sometimes important to know not the specific values of a solution for some concrete value of the argument, but the type of behaviour for changes in the argument and, in particular, for a boundless increase of the argument. For example, it is sometimes important to know whether the solutions that satisfy the given initial conditions are periodic, whether they approach some known function asymptotically, etc. These are the questions with which the qualitative theory of differential equations deals.

One of the basic problems of the qualitative theory is that of the stability of the solution or of the stability of motion; this problem was investigated in detail by the noted Russian mathematician A. M. Lyapunov (1857-1918).

Let there be given a system of differential equations:

$$\left.\begin{aligned} \frac{dx}{dt} &= f_1(t, x, y), \\ \frac{dy}{dt} &= f_2(t, x, y). \end{aligned}\right\} \tag{1}$$

Let $x = x(t)$ and $y = y(t)$ be the solutions of this system that satisfy the initial conditions

$$\left.\begin{aligned} x_{t=0} &= x_0, \\ y_{t=0} &= y_0. \end{aligned}\right\} \tag{1'}$$

Further, let $\bar{x} = \bar{x}(t)$ and $\bar{y} = \bar{y}(t)$ be the solutions of equation (1) that satisfy the initial conditions

$$\left.\begin{aligned} \bar{x}_{t=0} &= \bar{x}_0, \\ \bar{y}_{t=0} &= \bar{y}_0. \end{aligned}\right\} \tag{1''}$$

Definition. The solutions $x=x(t)$ and $y=y(t)$ that satisfy the equations (1) and the initial conditions (1') are called Lyapunov's *stable* as $t\to\infty$ if for every arbitrarily small $\varepsilon>0$ there is a $\delta>0$ such that for all values $t>0$ the following inequalities will be fulfilled:

$$\left.\begin{aligned} |\bar{x}(t)-x(t)|&<\varepsilon,\\ |\bar{y}(t)-y(t)|&<\varepsilon.\end{aligned}\right\}\qquad(2)$$

if the initial data satisfy the inequalities

$$\left.\begin{aligned} |\bar{x}_0-x_0|&<\delta,\\ |\bar{y}_0-y_0|&<\delta.\end{aligned}\right\}\qquad(3)$$

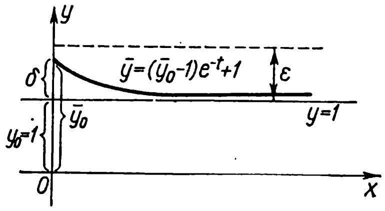


Fig. 274.

Let us figure out the meaning of this definition. From inequalities (2) and (3) it follows that for small variations in the initial conditions, the corresponding solutions differ but little for all positive values of t. If the system of differential equations is a system that describes some motion, then in the case of stability of solutions, the nature of the motions changes but slightly for small changes in the initial data.

Let us analyse an example of a first-order equation.
Let there be given a differential equation:

$$\frac{dy}{dt}=-y+1. \qquad \text{(a)}$$

The general solution of this equation is the function

$$y=Ce^{-t}+1. \qquad \text{(b)}$$

Find a particular solution that satisfies the initial condition

$$y_{t=0}=1. \qquad \text{(c)}$$

It is obvious that this solution $y=1$ results when $C=0$ (Fig. 274). Then find the particular solution that satisfies the initial condition

$$\bar{y}_{t=0}=\bar{y}_0.$$

Find the value of C from equation (b):

$$\bar{y}_0=C+1,$$

whence

$$C=\bar{y}_0-1.$$

Putting this value of C into equation (b), we get

$$\bar{y}=(\bar{y}_0-1)e^{-t}+1.$$

The solution $y=1$ is obviously stable.

Indeed,

$$y-\bar{y}=[(\bar{y}_0-1)e^{-t}+1]-1=(\bar{y}_0-1)e^{-t}\to 0$$

when $t\to\infty$.

Hence, inequality (3) will be fulfilled for an arbitrary ε if the following inequality is fulfilled:

$$(y_0-1)=\delta<\varepsilon.$$

Let us further consider the system of equations

$$\left.\begin{aligned}\frac{dx}{dt}&=cx+gy,\\ \frac{dy}{dt}&=ax+by,\end{aligned}\right\}\tag{4}$$

assuming that the coefficients a, b, c, g are constant and $g\neq 0$.

Let us find out what conditions the coefficients must satisfy so that the solution $x=0$, $y=0$ of system (4) should be stable.

Differentiating the first equation and eliminating y, we get a second-order equation:

$$\frac{d^2x}{dt^2}=c\frac{dx}{dt}+g\frac{dy}{dt}=c\frac{dx}{dt}+g(ax+by)=c\frac{dx}{dt}+agx+b\left(\frac{dx}{dt}-cx\right)$$

or

$$\frac{d^2x}{dt^2}-(b+c)\frac{dx}{dt}-(ag-bc)x=0.\tag{5}$$

Its auxiliary equation is of the form

$$\lambda^2-(b+c)\lambda-(ag-bc)=0.\tag{6}$$

Let us denote the roots of the auxiliary equation by λ_1 and λ_2. The following cases are possible.

1. The roots of the auxiliary equation are real, negative and distinct:

$$\lambda_1<0,\quad \lambda_2<0,\quad \lambda_1\neq\lambda_2.$$

Then

$$x=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t},$$

$$y=[C_1(\lambda_1-c)e^{\lambda_1 t}+C_2(\lambda_2-c)e^{\lambda_2 t}]\frac{1}{g}.$$

The solution that satisfies the initial conditions

$$x|_{t=0}=x_0,\quad y|_{t=0}=y_0,$$

will be

$$\left.\begin{aligned}x&=\frac{cx_0+gy_0-x_0\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_1 t}+\frac{x_0\lambda_1-cx_0-y_0g}{\lambda_1-\lambda_2}e^{\lambda_2 t},\\ y&=\frac{1}{g}\left[\frac{cx_0+gy_0-x_0\lambda_2}{\lambda_1-\lambda_2}(\lambda_1-c)e^{\lambda_1 t}+\frac{x_0\lambda_1-cx_0-y_0g}{\lambda_1-\lambda_2}(\lambda_2-c)e^{\lambda_2 t}\right].\end{aligned}\right\}\tag{7}$$

From the latter formulas it follows that for any $\varepsilon > 0$ it is possible to choose x_0 and y_0 so small that for all $t > 0$ we will have

$$|x(t)| < \varepsilon, \quad |y(t)| < \varepsilon \text{ since } e^{\lambda_1 t} < 1 \text{ and } e^{\lambda_2 t} < 1.$$

Hence, in this case the solution $x = 0$, $y = 0$ is **stable.**

2. Let $\lambda_1 = 0$, $\lambda_2 < 0$. Then

$$x = C_1 + C_2 e^{\lambda_2 t},$$

$$y = \frac{1}{g}[C_2(\lambda - c)e^{\lambda_2 t} - cC_1],$$

and the solution, as in the preceding case, proves **stable.**

3. Let $\lambda_1 = \lambda_2 < 0$. Then

$$x = (C_1 + C_2 t)e^{\lambda_1 t},$$

$$y = \frac{1}{g}e^{\lambda_1 t}[C_1(\lambda_1 - c) + C_2(1 + \lambda_1 t - ct)].$$

Since

$$te^{\lambda_1 t} \to 0 \text{ and } e^{\lambda_1 t} \to 0 \text{ when } t \to \infty,$$

it follows that for sufficiently small C_1 and C_2 (that is, for sufficiently small x_0 and y_0) we will have $|x(t)| < \varepsilon$ and $|y(t)| < \varepsilon$ for any $t > 0$. The solution is **stable.**

4. Let $\lambda_1 = \lambda_2 = 0$. Then

$$x = C_1 + C_2 t.$$

$$y = \frac{1}{g}[-cC_1 + C_2 - cC_2 t].$$

We see that for an arbitrarily small $C_2 \neq 0$ both x and y approach infinity (as $t \to \infty$), which means that the solution in this case is **unstable.**

5. Let at least one of the roots λ_1 and λ_2 be positive; for instance, $\lambda_1 > 0$.

From formula (7) it follows that no matter how small x_0 and y_0, if

$$cx_0 + gy_0 - x_0\lambda_2 \neq 0,$$

that is, if $C_1 \neq 0$, then $|x(t)| \to \infty$ as $t \to \infty$.

Hence, in this case too the solution is **unstable.**

6. The roots of the auxiliary equation are complex with negative real part:

$$\left.\begin{array}{l}\lambda_1 = \alpha + i\beta, \\ \lambda_2 = \alpha - i\beta,\end{array}\right\} \alpha < 0.$$

In this case,

$$\left.\begin{aligned} x &= Ce^{\alpha t}\sin(\beta t+\delta), \\ y &= \frac{1}{g}Ce^{\alpha t}[(\alpha-c)\sin(\beta t+\delta)+\beta\cos(\beta t+\delta)]. \end{aligned}\right\} \quad (8)$$

It is obvious that for any $\varepsilon > 0$ it is possible to choose x_0 and y_0 such that we will have $|C| < \varepsilon$ and $\frac{|\alpha-c|+|\beta|}{|d|} < \varepsilon$ and, consequently,

$$|x(t)| < \varepsilon \text{ and } |y(t)| < \varepsilon.$$

The solution is **stable.**

7. The roots of the auxiliary equation are pure imaginaries:

$$\lambda_1 = \beta i, \quad \lambda_2 = -\beta i.$$

In this case,

$$x = C\sin(\beta t+\delta),$$
$$y = \frac{1}{g}C[\beta\cos(\beta t+\delta) - c\sin(\beta t+\delta)]$$

which means that $x(t)$ and $y(t)$ are periodic functions of t. As in the preceding case, we verify the solution and find it **stable.**

8. The roots of the auxiliary equation are complex with positive real part ($\alpha > 0$).

From formulas (8) it follows that here for arbitrarily small x_0 and y_0 (that is, for arbitrarily small $C \neq 0$) and for increasing t the quantities $|x(t)|$ and $|y(t)|$ can take on arbitrarily large values, since $e^{\alpha t} \to \infty$ as $t \to \infty$. The solution is **unstable.**

To give a general criterion of the stability of solution of the system (1), we do as follows.

We write the roots of the auxiliary equation in the form of complex numbers:

$$\lambda_1 = \lambda_1^* + i\lambda_1^{**},$$
$$\lambda_2 = \lambda_2^* + i\lambda_2^{**}$$

(in the case of real roots, $\lambda_1^{**} = 0$ and $\lambda_2^{**} = 0$).

Let us take the plane of a complex variable $\lambda^* \lambda^{**}$ and display the roots of the auxiliary equation by points in this plane. Then, on the basis of the eight cases that have been considered, the condition of stability of solution of the system (4) may be formulated as follows.

If not a single one of the roots λ_1, λ_2 of the auxiliary equation (6) *lies to the right of the axis of imaginaries, and at least one root is nonzero, then the solution is stable; if at least one root*

lies to the right of the axis of imaginaries, or both roots are equal to zero, then the solution is unstable.

Let us now consider a more general system of equations:

$$\left.\begin{aligned}\frac{dx}{dt}&=cx+gy+P(x,\ y),\\ \frac{dy}{dt}&=ax+by+Q(x,\ y).\end{aligned}\right\}\tag{4'}$$

But for exceptional cases, the solution of this system is not expressible in terms of elementary functions and quadratures.

To establish whether the solutions of this system are stable or unstable, the system is compared with the solutions of a linear system. Suppose that for $x\to 0$ and $y\to 0$, the functions $P(x,\ y)$ and $Q(x,\ y)$ also approach zero and approach it faster than ϱ, where $\varrho=\sqrt{x^2+y^2}$; in other words,

$$\lim_{\varrho\to 0}\frac{P(x,\ y)}{\varrho}=0;\quad \lim_{\varrho\to 0}\frac{Q(x,\ y)}{\varrho}=0.$$

Then it may be proved that, save for the exceptional case, the solution of the system (4′) will be stable when the solution of the system

$$\left.\begin{aligned}\frac{dx}{dt}&=cx+gy,\\ \frac{dy}{dt}&=ax+by,\end{aligned}\right\}\tag{4}$$

is stable, and unstable when the solution of the system (4) is unstable. The exception is that case when both roots of the auxiliary equation lie on the axis of imaginaries; in this case, the question of the stability or instability of solution of the system (4′) is considerably more involved.

Lyapunov *) investigated the question of the stability of solutions of systems of equations for rather general assumptions concerning the form of these equations.

SEC. 32. EULER'S METHOD OF APPROXIMATE SOLUTION OF FIRST-ORDER DIFFERENTIAL EQUATIONS

We shall consider two methods of numerical solution of a first-order differential equation. In this section, we consider **Euler's method.**

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*) A. M. Lyapunov, The General Problem of Stability of Motion, ONTI, 1935 (Russian edition).

Find (approximately) the solution of the equation

$$\frac{dy}{dx}=f(x,\ y) \tag{1}$$

on the interval $[x_0,\ b]$ that satisfies the initial condition at $x=x_0$ $y=y_0$. Divide the interval $[x_0,\ b]$ by the points $x_0,\ x_1,\ x_2,\ldots,\ x_n=b$ into n equal parts (here $x_0<x_1<x_2<\ldots<x_n$). Denote $x_1-x_0=$ $=x_2-x_1=\ldots=b-x_{n-1}=\Delta x=h$; hence,

$$h=\frac{b-x_0}{n}.$$

Let $y=\varphi(x)$ be some approximate solution of equation (1) and

$$y_0=\varphi(x_0),\quad y_1=\varphi(x_1),\ldots,\ y_n=\varphi(x_n).$$

Denote

$$\Delta y_0=y_1-y_0,\ \Delta y_1=y_2-y_1,\ \ldots,\ \Delta y_{n-1}=y_n-y_{n-1}.$$

At each of the points $x_0,\ x_1,\ \ldots,\ x_n$ in equation (1) we replace the derivative with the ratio of finite differences:

$$\frac{\Delta y}{\Delta x}=f(x,\ y), \tag{2}$$

$$\Delta y=f(x,\ y)\,\Delta x. \tag{2'}$$

When $x=x_0$ we have

$$\frac{\Delta y_0}{\Delta x}=f(x_0\,y_0),\ \ \Delta y_0=f(x_0,\ y_0)\,\Delta x$$

or

$$y_1-y_0=f(x_0,\ y_0)\,h.$$

In this equation, $x_0,\ y_0,\ h$ are known; thus we find

$$y_1=y_0+f(x_0,\ y_0)\,h.$$

When $x=x_1$, equation (2′) takes the form

$$\Delta y_1=f(x_1,\ y_1)\,h$$

or

$$y_2-y_1=f(x_1,\ y_1)\,h,$$

$$y_2=y_1+f(x_1,\ y_1)\,h.$$

Here, $x_1,\ y_1,\ h$ are known and y_2 is determined.

Similarly, we find

$$y_3=y_2+f(x_2,\ y_2)\,h,$$

$$\cdots\cdots\cdots$$

$$y_{k+1}=y_k+f(x_k,\ y_k)\,h,$$

$$\cdots\cdots\cdots$$

$$y_n=y_{n-1}+f(x_{n-1},\ y_{n-1})\,h.$$

We have thus found the approximate values of the solution at the points $x_0, x_1, \ldots, x_n$. Connecting, in a coordinate plane, the points $(x_0, y_0), (x_1, y_1) \ldots, (x_n, y_n)$ by straight-line segments, we get a **broken line**—an approximate integral curve (Fig. 275). This broken line is called *Euler's broken line*.

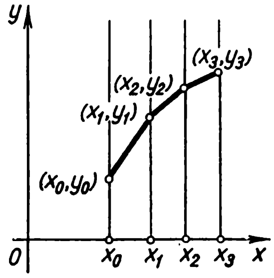


Fig. 275.

Note. We denote by $y=\varphi_h(x)$ an approximate solution of equation (1), which corresponds to Euler's broken line when $\Delta x = h$. It may be proved*) that if there exists a unique solution $y=\varphi^*(x)$ of equation (1) that satisfies the initial conditions and is defined on the interval $[x_0, b]$, then $\lim\limits_{h\to 0}|\varphi_h(x)-\varphi^*(x)|=0$ for any x of the interval $[x_0, b]$.

Example. Find the approximate value (for $x=1$) of the solution of the equation

$$y' = y + x$$

that satisfies the initial condition $y_0 = 1$ for $x_0 = 0$.

Solution. Divide the interval [0, 1] into 10 parts by the points $x_0 = 0$, 0.1, 0.2, ..., 1.0. Hence, $h = 0.1$. We seek the values $y_1, y_2, \ldots, y_n$ by formula (2'):

$$\Delta y_k = (y_k + x_k)\,h$$

or

$$y_{k+1} = y_k + (y_k + x_k)\,h.$$

We thus get
$$y_1 = 1 + (1+0)\cdot 0.1 = 1 + 0.1 = 1.1,$$
$$y_2 = 1.1 + (1.1 + 0.1)\cdot 0.1 = 1.22,$$
. . . .

Tabulating the results as we solve, we get:

x_k	y_k	$y_k + x_k$	$\Delta y_k = (y_k + x_k)\,h$
$x_0 = 0$	1.000	1.000	0.100
$x_1 = 0.1$	1.100	1.200	0.120
$x_2 = 0.2$	1.220	1.420	0.142
$x_3 = 0.3$	1.362	1.620	0.162
$x_4 = 0.4$	1.524	1.924	0.1924
$x_5 = 0.5$	1.7164	2.2164	0.2216
$x_6 = 0.6$	1.9380	2.5380	0.2538
$x_7 = 0.7$	2.1918	2.8918	0.2812
$x_8 = 0.8$	2.4730	3.2730	0.3273
$x_9 = 0.9$	2.8003	3.7003	0.3700
$x_{10} = 1.0$	3.1703		

*) For the proof see, for example, I. G. Petrovsky's "Lectures on the Theory of Ordinary Differential Equations".

We have found the approximate value $y|_{x=1}=3.1703$. The exact solution of this equation that satisfies the indicated initial conditions is

$$y=2e^x-x-1.$$

Hence,

$$y|_{x=1}=2(e-1)=3.4365.$$

The absolute error is 0.2662; the relative error is $\frac{0.2662}{3.4365}=0.077\approx 8\%$.

SEC. 33. A DIFFERENCE METHOD FOR APPROXIMATE SOLUTION OF DIFFERENTIAL EQUATIONS BASED ON TAYLOR'S FORMULA. ADAMS METHOD

We once again seek the solution of the equation

$$y'=f(x,\ y) \tag{1}$$

on the interval $[x_0,\ b]$, which solution satisfies the initial condition $y=y_0$ when $x=x_0$. We introduce notation that will be needed later on. The approximate values of the solution at the points

$$x_0,\ x_1,\ x_2,\ \ldots,\ x_n$$

will be

$$y_0,\ y_1,\ y_2,\ \ldots,\ y_n.$$

The first differences, or differences of the first order, are

$$\Delta y_0=y_1-y_0,\ \Delta y_1=y_2-y_1,\ \ldots,\ \Delta y_{n-1}=y_n-y_{n-1}.$$

The second differences, or differences of the second order, are

$$\Delta^2 y_0=\Delta y_1-\Delta y_0=y_2-2y_1+y_0,$$
$$\Delta^2 y_1=\Delta y_2-\Delta y_1=y_3-2y_2+y_1,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\Delta^2 y_{n-2}=\Delta y_{n-1}-\Delta y_{n-2}=y_n-2y_{n-1}+y_{n-2}.$$

Differences of the second differences are called differences of the third order, and so forth. We denote by $y'_0,\ y'_1,\ \ldots,\ y'_n$ the approximate values of the derivatives, and by $y''_0,\ y''_1,\ \ldots,\ y''_n$ the approximate values of the second derivatives, etc. Similarly we determine the first differences of the derivatives:

$$\Delta y'_0=y'_1-y'_0,\ \Delta y'_1=y'_2-y'_1,\ \ldots,\ \Delta y'_{n-1}=y'_n-y'_{n-1},$$

the second differences of the derivatives:

$$\Delta^2 y'_0=\Delta y'_1-\Delta y'_0,\ \Delta^2 y'_1=\Delta y'_2-\Delta y'_1,\ \ldots,\Delta^2 y'_{n-2}=\Delta y'_{n-1}-\Delta y'_{n-2},$$

and so on.

Write Taylor's formula for solving an equation in the neighbourhood of the point $x=x_0$ [Ch. IV, Sec. 6, formula (6)]:

$$y=y_0+\frac{x-x_0}{1}y_0'+\frac{(x-x_0)^2}{1\cdot 2}y_0''+\ldots+\frac{(x-x_0)^m}{1\cdot 2\ldots m}y_0^{(m)}+R_m. \qquad (2)$$

In this formula y_0 is known, and the values of y_0', y_0'', ... of the derivatives are found from equation (1) as follows. Putting the initial values x_0 and y_0 into the right side of equation (1), we find y_0':

$$y_0'=f(x_0, y_0).$$

Differentiating the terms of (1) with respect to x, we get

$$y''=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'. \qquad (3)$$

Substituting into the right side the values x_0, y_0, y_0', we find

$$y_0''=\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'\right)_{x=x_0,\ y=y_0,\ y'=y_0'}.$$

Once more differentiating (3) with respect to x and substituting the values x_0, y_0, y_0', y_0'', we find y_0'''. Continuing in this fashion,*. we can find the values of the derivatives of **any order** for $x=x_0$) All terms are known, except the remainder R_m on the right side of (2). Thus, neglecting the remainder, we can obtain an approximation of the solution for any value of x; their accuracy will depend upon the quantity $|x-x_0|$ and the number of terms in the expansion.

In the method given below, we determine by formula (2) only the first few values of y when $|x-x_0|$ is small. We determine the values y_1 and y_2 for $x_1=x_0+h$ and for $x_2=x_0+2h$, taking **four** terms of the expansion (y_0 is known from the initial data):

$$y_1=y_0+\frac{h}{1}y_0'+\frac{h^2}{1\cdot 2}y_0''+\frac{h^3}{3!}y_0''', \qquad (4)$$

$$y_2=y_0+\frac{2h}{1}y_0'+\frac{(2h)^2}{1\cdot 2}y_0''+\frac{(2h)^3}{3!}y_0'''. \qquad (4')$$

We thus consider known three values**) of the function: y_0, y_1, y_2. On the basis of these values and using equation (1),

*) From now on we shall assume that the function $f(x, y)$ is differentiable with respect to x and y as many times as is required by the reasoning.

**) If we were to seek the solution with greater accuracy, we would have to compute more than the first three values of y. This is dealt with in detail by Ya. S. Bezikovich in "Approximate Calculations" (Gostekhizdat, 1949) (Russian edition).

we find

$$y'_0=f(x_0, y_0),\ y'_1=f(x_1, y_1),\ y'_2=f(x_2, y_2).$$

Knowing y'_0, y'_1, y'_2, it is possible to determine $\Delta y'_0$, $\Delta y'_1$, $\Delta^2 y'_0$. Tabulate the results of the computations:

x	y	y'	$\Delta y'$	$\Delta^2 y'$
x_0	y_0	y'_0		
			$\Delta y'_0$	
$x_1=x_0+h$	y_1	y'_1		$\Delta^2 y'_0$
			$\Delta y'_1$	
$x_2=x_0+2h$	y_2	y'_2		
...	...	...	...	...
$x_{k-2}=x_0+(k-2)h$	y_{k-2}	y'_{k-2}		
			$\Delta y'_{k-2}$	
$x_{k-1}=x_0+(k-1)h$	y_{k-1}	y'_{k-1}		$\Delta^2 y'_{k-2}$
			$\Delta y'_{k-1}$	
$x_k=x_0+kh$	y_k	y'_k		

Now suppose that we know the values of the solution

$$y_0,\ y_1,\ y_2,\ \ldots,\ y_k.$$

From these values we can compute [using equation (1)] the values of the derivatives

$$y'_0,\ y'_1,\ y'_2,\ \ldots,\ y'_k$$

and, hence,

$$\Delta y'_0,\ \Delta y'_1,\ \ldots,\ \Delta y'_{k-1}$$

and

$$\Delta^2 y'_0,\ \Delta^2 y'_1,\ \ldots,\ \Delta^2 y'_{k-2}.$$

Let us determine the value of y_{k+1} from Taylor's formula (see Ch. V, Sec. 6), setting $a=x_k$, $x=x_{k+1}=x_k+h$:

$$y_{k+1}=y_k+\frac{h}{1}y'_k+\frac{h^2}{1\cdot 2}y''_k+\frac{h^3}{1\cdot 2\cdot 3}y'''_k+\ldots+\frac{h^m}{m!}y_k^{(m)}+R_m.$$

In our case we shall confine ourselves to four terms of the expansion:

$$y_{k+1}=y_k+\frac{h}{1}y'_k+\frac{h^2}{1\cdot 2}y''_k+\frac{h^3}{1\cdot 2\cdot 3}y'''_k. \tag{5}$$

The unknowns in this formula are y''_k and y'''_k, which we shall try to determine by using the known first-order and second-order differences.

First, represent y'_{k-1} in Taylor's formula, putting $a=x_k$, $x-a=-h$:

$$y'_{k-1}=y'_k+\frac{(-h)}{1}y''_k+\frac{(-h)^2}{1\cdot 2}y'''_k, \tag{6}$$

and y'_{k-2}, putting $a=x_k$, $x-a=-2h$:

$$y'_{k-2}=y'_k+\frac{(-2h)}{1}y''_k+\frac{(-2h)^2}{1\cdot 2}y'''_k. \tag{7}$$

From (6) we find

$$y'_k-y'_{k-1}=\Delta y'_{k-1}=\frac{h}{1}y''_k-\frac{h^2}{1\cdot 2}y'''_k. \tag{8}$$

Subtracting the terms of (7) from those of (6), we get

$$y'_{k-1}-y'_{k-2}=\Delta y'_{k-2}=\frac{h}{1}y''_k-\frac{3h^2}{2}y'''_k. \tag{9}$$

From (8) and (9) we obtain

$$\Delta y'_{k-1}-\Delta y'_{k-2}=\Delta^2 y'_{k-2}=h^2 y'''_k$$

or

$$y'''_k=\frac{1}{h^2}\Delta^2 y'_{k-1}. \tag{10}$$

Putting the expression y'''_k into (8), we get

$$y''_k=\frac{\Delta y'_{k-1}}{h}+\frac{\Delta^2 y'_{k-1}}{2h}. \tag{11}$$

Thus, y''_k and y'''_k have been found. Putting expressions (10) and (11) into the expansion (5), we obtain

$$y_{k+1}=y_k+\frac{h}{1}y'_k+\frac{h}{2}\Delta y'_{k-1}+\frac{5h}{12}\Delta^2 y'_{k-2}. \tag{12}$$

This is the so-called *Adams formula* with four terms. Formula (12) enables one to compute y_{k+1} when y_k, y_{k-1}, y_{k-2} are known. Thus, knowing y_0, y_1 and y_2 we can find y_3 and, further, $y_4, y_5, \ldots$

Note 1. We state without proof that if there exists a unique solution of equation (1) on the interval $[x_0, b]$, which solution satisfies the initial conditions, then the error of the approximate

values determined from formula (12) do not exceed, in absolute value, Mh^4, where M is a constant dependent on the length of the interval and the form of the function $f(x, y)$ and independent of the magnitude of h.

Note 2. If we want to obtain greater accuracy in our computations, we must take more terms than in expansion (5), and formula (12) will change accordingly. For instance, if in place of formula (5) we take a formula containing five terms to the right, that is, if we complete it with a term of order h^4, then in place of formula (12) we, in similar fashion, get the formula

$$y_{k+1} = y_k + \frac{h}{1} y'_k + \frac{h}{2} \Delta y'_{k-1} + \frac{5h}{12} \Delta^2 y'_{k-2} + \frac{3h}{8} \Delta^3 y'_{k-3}.$$

Here, y_{k+1} is determined by means of the values y_k, y_{k-1}, y_{k-2} and y_{k-3}. Thus, in order to begin computation using this formula we must know the first four values of the solution: y_0, y_1, y_2, y_3. When calculating these values from formulas of type (4), one should take five terms of the expansion.

Example 1. Approximate the solution of the equation

$$y' = y + x$$

that satisfies the initial condition

$$y_0 = 1 \text{ when } x_0 = 0.$$

Determine the values of the solution for $x = 0.1$, 0.2, 0.3, 0.4.

Solution. First we find y_1 and y_2 using formulas (4) and (4′). From the equation and the initial data we get

$$y'_0 = (y + x)_{x=0} = y_0 + 0 = 1 + 0 = 1.$$

Differentiating the given equation, we have

$$y'' = y' + 1.$$

Hence,

$$y''_0 = (y' + 1)_{x=0} = 1 + 1 = 2.$$

Differentiating once again, we get

$$y''' = y''.$$

Hence,

$$y'''_0 = y''_0 = 2.$$

Substituting into (4) the values y_0, y'_0, y''_0 and $h = 0.1$, we get

$$y_1 = 1 + \frac{0.1}{1} \cdot 1 + \frac{(0.1)^2}{1 \cdot 2} \cdot 2 + \frac{(0.1)^3}{1 \cdot 2 \cdot 3} \cdot 2 = 1.1103.$$

Similarly, for $h = 0.2$ we have

$$y_2 = 1 + \frac{0.2}{1} \cdot 1 + \frac{(0.2)^2}{1 \cdot 2} \cdot 2 + \frac{(0.2)^3}{1 \cdot 2 \cdot 3} \cdot 2 = 1.2426.$$

Knowing y_0, y_1, y_2, we find (on the basis of the equation)

$$y_0' = y_0 + 0 = 1,$$

$$y_1' = y_1 + 0.1 = 1.1103 + 0.1 = 1.2103,$$

$$y_2' = y_2 + 0.2 = 1.2426 + 0.2 = 1.4426.$$

$$\Delta y_0' = 0.2103,$$

$$\Delta y_1' = 0.2323,$$

$$\Delta^2 y_0' = 0.0220.$$

Tabulating the values obtained, we have

x	y	y'	$\Delta y'$	$\Delta^2 y'$
$x_0 = 0$	$y_0 = 1.0000$	$y_0' = 1$		
			$\Delta y_0' = 0.2103$	
$x_1 = 0.1$	$y_1 = 1.1103$	$y_1' = 1.2103$		$\Delta^2 y_0' = 0.0220$
			$\Delta y_1' = 0.2323$	
$x_2 = 0.2$	$y_2 = 1.2426$	$y_2' = 1.4426$		$\Delta^2 y_1' = 0.0228$
			$\Delta y_2' = 0.2551$	
$x_3 = 0.3$	$y_3 = 1.3977$	$y_3' = 1.6977$		
$x_4 = 0.4$	$y_4 = 1.5812$			

From formula (12) we find y_3:

$$y_3 = 1.2426 + \frac{0.1}{1} \cdot 1.4426 + \frac{0.1}{2} \cdot 0.2323 + \frac{5 \cdot (0.1)}{12} \cdot 0.0220 = 1.3977.$$

We then find the values of y_3', $\Delta y_2'$, $\Delta^2 y_1'$. Again using formula (12) we find y^4:

$$y_4 = 1.3977 + \frac{0.1}{1} \cdot 1.6977 + \frac{0.1}{2} \cdot 0.2551 + \frac{5}{12} \cdot 0.1 \cdot 0.0228 = 1.5812.$$

The exact expression of the solution of the given equation is

$$y = 2e^x - x - 1.$$

Hence, $y_{x=0.4}=2e^{0.4}-0.4-1=1.5836$. The absolute error is 0.0024; the relative error, $\frac{0.0024}{1.5836}=0.0015\approx 0.15\%$. (In Euler's method, the absolute error of y_4 is 0.06, the relative error, $0.038\approx 3.8\%$.)

Example 2. Approximate the solution of the equation

$$y'=y^2+x^2$$

that satisfies the initial condition $y_0=0$ for $x_0=0$. Determine the values of the solution for $x=0.1$, 0.2, 0.3, 0.4.

Solution. We find

$$y_0'=0^2+0^2=0,$$

$$y''_{x=0}=(2yy'+2x)_{x=0}=0,$$

$$y'''_{x=0}=(2y'^2+2yy''+2)_{x=0}=2.$$

By formulas (4) and (4') we have

$$y_1=\frac{(0.1)^3}{3!}\cdot 2=0.0003,\qquad y_2=\frac{(0.2)^3}{3!}\cdot 2=0.0026.$$

From the equation we find

$$y_0'=0,\quad y_1'=0.0100,\quad y_2'=0.0400.$$

Using these data, we construct the first rows of the table, and then determine the values of y_3 and y_4 from formula (12).

x	y	y'	$\Delta y'$	$\Delta^2 y'$
$x_0=0$	$y_0=0$	$y_0'=0$		
			$\Delta y_0'=0.0100$	
$x_1=0.1$	$y_1=0.0003$	$y_1'=0.0100$		$\Delta^2 y_0'=0.0200$
			$\Delta y_1'=0.0300$	
$x_2=0.2$	$y_2=0.0026$	$y_2'=0.0400$		$\Delta^2 y_1'=0.0201$
			$\Delta y_2'=0.0501$	
$x_3=0.3$	$y_3=0.0089$	$y_3'=0.0901$		
$x_4=0.4$	$y_4=0.0204$			

Thus,

$$y_3=0.0026+\frac{0.1}{1}\cdot 0.0400+\frac{0.1}{2}\cdot 0.0300+\frac{5}{12}\cdot 0.1\cdot 0.0200=0.0089,$$

$$y_4=0.0089+\frac{0.1}{1}\cdot 0.0901+\frac{0.1}{2}\cdot 0.0501+\frac{5}{12}\cdot 0.1\cdot 0.0201=0.0204.$$

We note that the first four correct decimals in y_4 are $y_4=0.0213$. (This may be obtained by other, more accurate, methods with error evaluation.)

SEC. 34. AN APPROXIMATE METHOD FOR INTEGRATING SYSTEMS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

The methods of approximate integration of differential equations considered in Secs. 32 and 33 are also applicable for solving systems of first-order differential equations. Here, we consider the difference method for solving systems of equations. Our reasoning will deal with systems of two equations in two unknown functions.

It is required to find the solutions of a system of equations

$$\frac{dy}{dx}=f_1(x, y, z), \tag{1}$$

$$\frac{dz}{dx}=f_2(x, y, z) \tag{2}$$

that satisfy the initial conditions $y=y_0$, $z=z_0$ when $x=x_0$.

We determine the values of the function y and z for values of the argument $x_0, x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n$. Once more, let

$$x_{k+1}-x_k=\Delta x=h\,(k=0, 1, 2, \ldots, n-1). \tag{3}$$

We denote the approximate values of the function as

$$y_0, y_1, \ldots, y_k, y_{k+1}, \ldots, y_n$$

and

$$z_0, z_1, \ldots, z_k, z_{k+1}, \ldots, z_n.$$

Write the recurrence formulas of type (12), Sec. 33:

$$y_{k+1}=y_k+\frac{h}{1}y'_k+\frac{h}{2}\Delta y'_{k-1}+\frac{5}{12}h\Delta^2 y'_{k-2}, \tag{4}$$

$$z_{k+1}=z_k+\frac{h}{1}z'_k+\frac{h}{2}\Delta z'_{k-1}+\frac{5}{12}h\Delta^2 z'_{k-2}. \tag{5}$$

To begin computations using these formulas we must know y_1, y_2; z_1, z_2 in addition to y_0 and z_0; we find these values from formulas of type (4) and (4′), Sec. 32:

$$y_1=y_0+\frac{h}{1}y'_0+\frac{h^2}{2}y''_0+\frac{h^3}{3!}y'''_0,$$

$$y_2=y_0+\frac{2h}{1}y'_0+\frac{(2h)^2}{2!}y''_0+\frac{(2h)^3}{3!}y'''_0,$$

$$z_1 = z_0 + \frac{h}{1} z_0' + \frac{h^2}{2} z_0'' + \frac{h^3}{3!} z_0''',$$
$$z_2 = z_0 + \frac{2h}{1} z_0' + \frac{(2h)^2}{2} z_0'' + \frac{(2h)^3}{3!} z_0'''.$$

To apply these formulas one has to know y_0', y_0'', y_0''', z_0', z_0'', z_0''', which we shall now determine. From (1) and (2) we find

$$y_0' = f_1(x_0, y_0, z_0),$$
$$z_0' = f_2(x_0, y_0, z_0).$$

Differentiating (1) and (2) and substituting the values of x_0, y_0, z_0, y_0' and z_0', we find

$$y_0'' = (y'')_{x=x_0} = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y} y' + \frac{\partial f_1}{\partial z} z'\right)_{x=x_0},$$
$$z_0'' = (z'')_{x=x_0} = \left(\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial y} y' + \frac{\partial f_2}{\partial z} z'\right)_{x=x_0}.$$

Differentiating once again, we find y_0''' and z_0'''. Knowing y_1, y_2, z_1, z_2, we find from the given equations (1) and (2),

$$y_1',\ y_2',\ z_1',\ z_2',\ \Delta y_0',\ \Delta y_1',\ \Delta^2 y_0',\ \Delta z_0',\ \Delta z_1',\ \Delta^2 z_0',$$

after which we can fill in the first five rows of the table:

x	y	y'	$\Delta y'$	$\Delta^2 y'$	z	z'	$\Delta z'$	$\Delta^2 z'$
x_0	y_0	y_0'			z_0	z_0'		
			$\Delta y_0'$				$\Delta z_0'$	
x_1	y_1	y_1'		$\Delta^2 y'$	z_1	z_1'		$\Delta^2 z_0'$
			$\Delta y_1'$				$\Delta z_1'$	
x_2	y_2	y_2'		$\Delta^2 y_1'$	z_2	z_2'		$\Delta^2 z_1'$
			$\Delta y_2'$				$\Delta z_2'$	
x_3	y_3	y_3'			z_3	z_3'		

From formulas (4) and (5) we find y_3 and z_3, and from equations (1) and (2) we find y_3' and x_3'. Computing $\Delta y_2'$, $\Delta^2 y_1'$, $\Delta z_2'$, $\Delta^2 z_1'$, we find y_4 and y_5, etc., by applying formulas (4) and (5) once again.

Example. Approximate the solutions of the system

$$y' = z, \quad z' = y$$

with initial conditions $y_0 = 0$ and $z_0 = 1$ for $x = 0$. Compute the values of the solutions for $x = 0,\ 0.1,\ 0.2,\ 0.3,\ 0.4$.

Solution. From the given equations, we find

$$y_0' = z_{x=0} = 1,$$

$$z_0' = y_{x=0} = 0.$$

Differentiating the given equations, we find

$$y_0'' = (y'')_{x=0} = (z')_{x=0} = 0,$$

$$z_0'' = (z'')_{x=0} = (y')_{x=0} = 1,$$

$$y_0''' = (y''')_{x=0} = (z'')_{x=0} = 1,$$

$$z_0''' = (z''')_{x=0} = (y'')_{x=0} = 0.$$

Using formulas of type (4) and (5), we find

$$y_1 = 0 + \frac{0.1}{1}\cdot 1 + \frac{(0.1)^2}{1\cdot 2}\cdot 0 + \frac{(0.1)^3}{3!}\cdot 1 = 0.1002,$$

$$y_2 = 0 + \frac{0.2}{1}\cdot 1 + \frac{(0.2)^2}{1\cdot 2}\cdot 0 + \frac{(0.2)^3}{3!}\cdot 1 = 0.2016,$$

$$z_1 = 1 + \frac{0.1}{1}\cdot 0 + \frac{(0.1)^2}{1\cdot 2}\cdot 1 + \frac{(0.1)^3}{3!}\cdot 0 = 1.0050,$$

$$z_2 = 1 + \frac{0.2}{1}\cdot 0 + \frac{(0.2)^2}{2!}\cdot i + \frac{(0.2)^3}{3!}\cdot 0 = 1.0200.$$

Using the given equations, we find

$$y_1' = 1.0050, \qquad y_2' = 1.0200,$$

$$z_1' = 0.1002, \qquad z_2' = 0.2016,$$

$$\Delta y_0' = 0.0050, \qquad \Delta z_0' = 0.1002,$$

$$\Delta y_1' = 0.0150, \qquad \Delta z_1' = 0.1014,$$

$$\Delta^2 y_0' = 0.0100, \qquad \Delta^2 z_0' = 0.0012.$$

Filling in the first five rows of the table, we have

x	y	y'	$\Delta y'$	$\Delta^2 y'$
$x_0=0$	$y_0=0$	$y'_0=1$		
			$\Delta y'_0=0.0050$	
$x_1=0.1$	$y_1=0.1002$	$y'_1=1.0050$		$\Delta^2 y'_0=0.0100$
			$\Delta y'_1=0.0150$	
$x_2=0.2$	$y_2=0.2016$	$y'_2=1.0200$		$\Delta^2 y'_1=0.0109$
			$\Delta y'_2=0.0259$	
$x_3=0.3$	$y_3=0.3049$	$y'_3=1.0459$		
$x_4=0.4$	$y_4=0.4117$			
x	z	z'	$\Delta z'$	$\Delta^2 z'$
$x_0=0$	$z_0=1$	$z'_0=0$		
			$\Delta z'_0=0.1002$	
$x_1=0.1$	$z_1=1.0050$	$z'_1=0.1002$		$\Delta^2 z'_0=0.0012$
			$\Delta z'_1=0.1014$	
$x_2=0.2$	$z_2=1.0200$	$z'_2=0.2016$		$\Delta^2 z'_1=0.0019$
			$\Delta z'_2=0.1033$	
$x_3=0.3$	$z_3=1.0459$	$z'_3=0.3049$		
$x_4=0.4$	$z_4=1.0817$			

From formulas (4) and (5) we find

$$y_3=0.2016+\frac{0.1}{1}\cdot 1.0200+\frac{0.1}{2!}\cdot 0.0150+\frac{5}{12}\cdot 0.1\cdot 0.0100=0.3049,$$

$$z_3=1.0200+\frac{0.1}{1}\cdot 1.2016+\frac{0.1}{2}\cdot 0.1014+\frac{5}{12}\cdot 0.1\cdot 0.0012=1.0459$$

and similarly

$$y_4=0.3049+\frac{0.1}{1}\cdot 1.0459+\frac{0.1}{2}\cdot 0.0259+\frac{5}{12}\cdot 0.1\cdot 0.0109=0.4117,$$

$$z_4=1.0459+\frac{0.1}{1}\cdot 0.3049+\frac{0.1}{2}\cdot 0.1033+\frac{5}{12}\cdot 0.1\cdot 0.0019=1.0817.$$

It is obvious that the exact solutions of the system of equations (the solutions satisfying the initial conditions) will be

$$y=\frac{1}{2}(e^{x}-e^{-x}),\quad z=\frac{1}{2}(e^{x}+e^{-x}).$$

And so, solutions correct to the fourth decimal place are

$$y_4=\frac{1}{2}(e^{0.4}-e^{-0.4})=0.4107,\quad z_4=\frac{1}{2}(e^{0.4}+e^{-0.4})=1.0811.$$

Note. Since equations of higher order and systems of equations of higher order in many cases reduce to a system of first-order equations, the method given above is applicable to the solution of such problems.

Exercises on Chapter XIII

Show that the indicated functions, which depend on arbitrary constants, satisfy the corresponding differential equations:

Functions	Differential Equations
1. $y=\sin x-1+Ce^{-\sin x}$.	$\frac{dy}{dx}+y\cos x=\frac{1}{2}\sin 2x$.
2. $y=Cx+C-C^2$.	$\left(\frac{dy}{dx}\right)^2-\frac{dy}{dx}-x\frac{dy}{dx}+y=0$.
3. $y^2=2Cx+C^2$.	$y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}-y=0$.
4. $y^2=Cx^2-\frac{a^2C}{1+C}$.	$xy\left[1-\left(\frac{dy}{dx}\right)^2\right]=(x^2-y^2-a^2)\frac{dy}{dx}$.
5. $y=C_1x+\frac{C_2}{x}+C_3$.	$\frac{d^3y}{dx^3}+\frac{3}{x}\frac{d^2y}{dx^2}=0$.
6. $y=(C_1+C_2x)e^{kx}+\frac{e^x}{(k-1)^2}$.	$\frac{d^2y}{dx^2}-2k\frac{dy}{dx}+k^2y=e^x$.
7. $y=C_1e^{a\arcsin x}+C_2e^{-a\arcsin x}$.	$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$.
8. $y=\frac{C_1}{x}+C_2$.	$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}=0$.

Integrate the differential equations with variables separable

9. $y\,dx-x\,dy=0$. *Ans.* $y=Cx$. 10. $(1+u)\,v\,du+(1-v)\,u\,dv=0$. *Ans.* $\ln uv+u-v=C$. 11. $(1+y)\,dx-(1-x)\,dy=0$. *Ans.* $(1+y)(1-x)=C$. 12. $(t^2-xt^2)\dfrac{dx}{dt}+x^2+tx^2=0$. *Ans.* $\dfrac{t+x}{tx}+\ln\dfrac{x}{t}=C$. 13. $(y-a)\,dx+x^2\,dy=0$. *Ans.* $(y-a)=Ce^{\frac{1}{x}}$. 14. $z\,dt-(t^2-a^2)\,dz=0$. *Ans.* $z^{2a}=C\dfrac{t-a}{t+a}$. 15. $\dfrac{dx}{dy}=\dfrac{1+x^2}{1+y^2}$. *Ans.* $x=\dfrac{y+C}{1-Cy}$. 16. $(1+s^2)\,dt-\sqrt{t}\,ds=0$. *Ans.* $2\sqrt{t}-\arctan s=C$. 17. $d\varrho+\varrho\tan\theta\,d\theta=0$. *Ans.* $\varrho=C\cos\theta$. 18. $\sin\theta\cos\varphi\,d\theta-\cos\theta\sin\varphi\,d\varphi=0$. *Ans.* $\cos\varphi=C\cos\theta$. 19. $\sec^2\theta\tan\varphi\,d\theta+\sec^2\varphi\tan\theta\,d\varphi=0$. *Ans.* $\tan\theta\tan\varphi=C$. 20. $\sec^2\theta\tan\varphi\,d\varphi+\sec^2\varphi\tan\theta\,d\theta=0$. *Ans.* $\sin^2\theta+\sin^2\varphi=C$. 21. $(1+x^2)\,dy-\sqrt{1-y^2}\,dx=0$. *Ans.* $\arcsin y-\arctan x=C$. 22. $\sqrt{1-x^2}\,dy-\sqrt{1-y^2}\,dx=0$. *Ans.* $y\sqrt{1-x^2}-x\sqrt{1-y^2}=C$. 23. $3e^x\tan y\times dx+(1-e^x)\sec^2 y\,dy=0$. *Ans.* $\tan y=C(1-e^x)^3$. 24. $(x-y^2x)\,dx+(y-x^2y)\,dy=0$. *Ans.* $x^2+y^2=x^2y^2+C$.

Problems in Forming Differential Equations

25. Prove that a curve having the slope of the tangent to any point proportional to the abscissa of the point of tangency is a parabola. *Ans.* $y=ax^2+C$.

26. Find a curve passing through the point $(0, -2)$ such that the slope of the tangent at any point of it is equal to the ordinate of this point increased by three units. *Ans.* $y=e^x-3$.

27. Find a curve passing through the point $(1, 1)$ so that the slope of the tangent to the curve at any point is proportional to the square of the ordinate of this point. *Ans.* $k(x-1)\,y-y+1=0$.

28. Find a curve for which the slope of the tangent at any point is n times the slope of a straight line connecting this point with the origin. *Ans.* $y=Cx^n$.

29. Through the point $(2, 1)$ draw a curve for which the tangent at any point coincides with the direction of the radius vector drawn from the origin to the same point. *Ans.* $y=\dfrac{1}{2}x$.

30. In polar coordinates, find the equation of a curve at each point of which the tangent of the angle between the radius vector and the tangent line is equal to the reciprocal of the radius vector with sign reversed. *Ans.* $r(\theta+C)=1$.

31. In polar coordinates, find the equation of a curve at each point of which the tangent of the angle formed by the radius vector and the tangent line is equal to the square of the radius vector. *Ans.* $r^2=2(\theta+C)$.

32. Prove that a curve with the property that all its normals pass through a constant point is a circle.

33. Find a curve such that at each point of it the length of the subtangent is equal to the doubled abscissa. *Ans.* $y=C\sqrt{x}$.

34. Find a curve for which the radius vector is equal to the length of the tangent between the point of tangency and the x-axis.

Solution. By hypothesis, $\dfrac{y}{y'}\sqrt{1+y'^2}=\sqrt{x^2+y^2}$, whence $\dfrac{dy}{y}=\pm\dfrac{dx}{x}$. Integrating, we get two families of curves: $y=Cx$ and $y=\dfrac{C}{x}$.

35. By Newton's law, the rate of cooling of some body in air is proportional to the difference between the temperature of the body and the temperature of the air. If the temperature of the air is 20° C and the body cools for 20 minutes from 100° to 60° C, how long will it take for its temperature to drop to 30° C?

Solution. The differential equation of the problem is $\frac{dT}{dt}=k(T-20)$. Integrating we find: $T-20=Ce^{kt}$; $T=100$ when $t=0$; $T=60$ when $t=20$; therefore, $C=80$; $40=Ce^{20k}$, $e^k=\left(\frac{1}{2}\right)^{\frac{1}{20}}$; consequently, $T=20+80\left(\frac{1}{2}\right)^{\frac{t}{20}}$. Assuming $T=30$, we find $t=60$ min.

36. During what time T will the water flow out of an opening 0.5 cm² at the bottom of a conic funnel 10 cm high with the vertex angle $\alpha=60°$?

Solution. In two ways we calculate the volume of water that will flow out during the time between the instants t and $t+\Delta t$. Given a constant rate v, during 1 sec a cylinder of water with base 0.5 cm² and altitude h flows out, and during time Δt the outflow is the volume of water dv equal to

$$-dv=-0.5\,v\,dt=-0.3\sqrt{2gh}\,dt.^{*})$$

On the other hand, due to the outflow, the height of the water receives a negative "increment" dh, and the differential of the volume of water outflow is

$$-dv=\pi r^2\,dh=\frac{\pi}{3}(h+0.7)^2\,dh.$$

Thus,

$$\frac{\pi}{3}(h+0.7)^2\,dh=-0.3\sqrt{2gh}\,dt,$$

whence

$$t=0.0315\,(10^{5/2}-h^{5/2})+0.0732\,(10^{3/2}-h^{3/2})+0.078\,(\sqrt{10}-\sqrt{h}).$$

Setting $h=0$, we get the time of outflow $T=12.5$ sec.

37. The retarding action of friction on a disk rotating in a liquid is proportional to the angular velocity of rotation ω. Find the dependence of this angular velocity on the time if it is known that the disk begins rotating at 100 revolutions per minute and, after the elapse of one minute, rotates at 60 revolutions per minute. *Ans.* $\omega=100\left(\frac{3}{5}\right)^t$ rpm.

38. Suppose that in a vertical column of air their pressure at each level is due to the pressure of the above-lying layers. Find the dependence of the pressure on the height if it is known that at sea level this pressure is 1 kg per cm², while at 500 m above sea level, 0.92 kg per cm².

Hint. Take advantage of the Boyle-Mariotte law, by virtue of which the density of the gas is proportional to the pressure. The differential equation of the problem is $dp=-kp\,dh$, whence $p=e^{-0.00017h}$. *Ans.* $p=e^{-0.00017h}$.

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*) The rate of outflow v of water from an opening a distance h from the free surface is given by the formula $v=0.6\sqrt{2gh}$, where g is the acceleration of gravity.

Integrate the following homogeneous differential equations:
39. $(y-x)\,dx+(y+x)\,dy=0$. *Ans.* $y^2+2xy-x^2=C$. 40. $(x+y)\,dx+x\,dy=0$. *Ans.* $x^2+2xy=C$. 41. $(x+y)\,dx+(y-x)\,dy=0$. *Ans.* $\ln(x^2+y^2)^{1/2}-\arctan\frac{y}{x}=C$. 42. $x\,dy-y\,dx=\sqrt{x^2+y^2}\,dx$. *Ans.* $1+2Cy-C^2x^2=0$. 43. $(8y+10x)\,dx+(5y+7x)\,dy=0$. *Ans.* $(x+y)^2(2x+y)^3=C$. 44. $(2\sqrt{st}-s)\times dt+t\,ds=0$. *Ans.* $te^{\sqrt{\frac{s}{t}}}=C$ or $s=t\ln^2\frac{C}{t}$. 45. $(t-s)\,dt+t\,ds=0$. *Ans.* $te^{\frac{s}{t}}=C$ or $s=t\ln\frac{C}{t}$. 46. $xy^2\,dy=(x^3+y^3)\,dx$. *Ans.* $y=x\sqrt[3]{3\ln Cx}$.
47. $x\cos\frac{y}{x}(y\,dx+x\,dy)=y\sin\frac{y}{x}(x\,dy-y\,dx)$. *Ans.* $xy\cos\frac{y}{x}=C$.

Integrate the differential equations that lead to homogeneous equations:
48. $(3y-7x+7)\,dx-(3x-7y-3)\,dy=0$. *Ans.* $(x+y-1)^5(x-y-1)^2=C$.
49. $(x+2y+1)\,dx-(2x+4y+3)\,dy=0$. *Ans.* $\ln(4x+8y+5)+8y-4x=C$.
50. $(x+2y+1)\,dx-(2x-3)\,dy=0$. *Ans.* $\ln(2x-3)-\frac{4y+5}{2x-3}=C$.

51. Determine the curve whose subnormal is the arithmetical mean between the abscissa and the ordinate. *Ans.* $(x-y)^2(x+2y)=C$.

52. Determine the curve in which the ratio of the segment cut off by a tangent on the y-axis to the radius vector is equal to a constant.

Solution. By hypothesis, $\dfrac{y-x\frac{dy}{dx}}{\sqrt{x^2+y^2}}=m$ whence $\left(\frac{x}{C}\right)^m-\left(\frac{C}{x}\right)^m=\frac{2y}{x}$.

53. Determine the curve in which the ratio of the segment cut off by the normal on the x-axis to the radius vector is equal to a constant.

Solution. It is given that $\dfrac{x+y\frac{dy}{dx}}{\sqrt{x^2+y^2}}=m$, whence $x^2+y^2=m^2(x-C)^2$.

54. Determine the curve in which the segment cut off by a tangent on the y-axis is equal to $a\sec\theta$, where θ is the angle between the radius vector and the x-axis.

Solution. Since $\tan\theta=\frac{y}{x}$ and by hypothesis

$$y-x\frac{dy}{dx}=a\sec\theta,$$

we obtain

$$y-x\frac{dy}{dx}=a\frac{\sqrt{x^2+y^2}}{x},$$

whence

$$y=\frac{x}{2}\left[e^{\frac{a}{x}+b}-e^{-\left(\frac{a}{b}+b\right)}\right].$$

55. Determine the curve for which the segment cut off on the y-axis by a normal drawn to some point of the curve is equal to the distance of this point from the origin.

Solution. The segment cut off by the normal on the y-axis is $y+\frac{x}{y'}$; therefore, by hypothesis, we have

$$y+\frac{x}{y'}=\sqrt{x^2+y^2},$$

whence

$$x^2=C(2y+C).$$

56. Find the shape of a mirror such that all rays emerging from a single point O would be reflected parallel to the given direction.

Solution. For the x-axis we take the given direction, and O as the origin. Let OM be the incident ray, MP the reflected ray, and MQ the normal to the desired curve.

$$\alpha=\beta;\quad OM=OQ,\quad NM=y,$$

$$NQ=NO+OQ=-x+\sqrt{x^2+y^2}=y\cot\beta=y\frac{dy}{dx},$$

whence

$$y\,dy=(-x+\sqrt{x^2+y^2})\,dx;$$

integrating, we have

$$y^2=C^2+2Cx.$$

Integrate the following linear differential equations:

57. $y'-\frac{2y}{x+1}=(x+1)^3$. *Ans.* $2y=(x+1)^4+C(x+1)^2$. 58. $y'-a\frac{y}{x}=\frac{x+1}{x}$. *Ans.* $y=Cx^a+\frac{x}{1-a}-\frac{1}{a}$. 59. $(x-x^3)\,y'+(2x^2-1)\,y-ax^3=0$. *Ans.* $y=ax+Cx\sqrt{1-x^2}$. 60. $\frac{ds}{dt}\cos t+s\sin t=1$. *Ans.* $s=\sin t+C\cos t$. 61. $\frac{ds}{dt}+s\cos t=\frac{1}{2}\sin 2t$. *Ans.* $s=\sin t-1+Ce^{-\sin t}$. 62. $y'-\frac{n}{x}y=e^xx^n$. *Ans.* $y=x^n(e^x+C)$. 63. $y'+\frac{n}{x}y=\frac{a}{x^n}$. *Ans.* $x^ny=ax+C$. 64. $y'+y=\frac{1}{e^x}$. *Ans.* $e^xy=x+C$. 65. $y'+\frac{1-2x}{x^2}y-1=0$. *Ans.* $y=x^2\left(1+Ce^{\frac{1}{x}}\right)$.

Integrate the Bernoulli equations:

66. $y'+xy=x^3y^3$. *Ans.* $y^2(x^2+1+Ce^{x^2})=1$. 67. $(1-x^2)\,y'-xy-axy^2=0$. *Ans.* $(C\sqrt{1-x^2}-a)\,y=1$. 68. $3y^2y'-ay^3-x-1=0$. *Ans.* $a^2y^3=Ce^{ax}-a(x+1)-1$. 69. $y'(x^2y^3+xy)=1$. *Ans.* $x\left[(2-y^2)\,e^{\frac{1}{2}y^2}+C\right]=e^{\frac{1}{2}y^2}$. 70. $(y\ln x-2)\,y\,dx=x\,dy$. *Ans.* $y(Cx+\ln x+1)=1$. 71. $y-y'\cos x=y^2\cos x\times(1-\sin x)$. *Ans.* $y=\frac{\tan x+\sec x}{\sin x+C}$.

Integrate the following exact differential equations:

72. $(x^2+y)\,dx+(x-2y)\,dy=0$. *Ans.* $\left[\frac{x^3}{3}+yx-y^2=C\right]$. 73. $(y-3x^2)\,dx-(4y-x)\,dy=0$. *Ans.* $2y^2-xy+x^3=C$. 74. $(y^3-x)\,y'=y$. *Ans.* $y^4=4xy+C$.
75. $\left[\frac{y^2}{(x-y)^2}-\frac{1}{x}\right]dx+\left[\frac{1}{y}-\frac{x^2}{(x-y)^2}\right]dy=0$. *Ans.* $\ln\frac{y}{x}-\frac{xy}{x-y}=C$.
76. $2(3xy^2+2x^3)\,dx+3(2x^2y+y^2)\,dy=0$. *Ans.* $x^4+3x^2y^2+y^3=C$.
77. $\frac{x\,dx+(2x+y)\,dy}{(x+y)^2}=0$. *Ans.* $\ln(x+y)-\frac{x}{x+y}=C$. 78. $\left(\frac{1}{x^2}+\frac{3y^2}{x^4}\right)dx=\frac{2y\,dy}{x^3}$. *Ans.* $x^2+y^2=Cx^3$. 79. $\frac{x^2\,dy-y^2\,dx}{(x-y)^2}=0$. *Ans.* $\frac{xy}{x-y}=C$. 80. $x\,dx+y\,dy=\frac{y\,dx-x\,dy}{x^2+y^2}$. *Ans.* $x^2+y^2-2\arctan\frac{x}{y}=C$.

81. Determine the curve that has the property that the product of the square of the distance of any point of it from the origin into the segment cut off on the x-axis by the normal at this point is equal to the cube of the abscissa of this point.
Ans. $y^2(2x^2+y^2)=C$.

82. Find the envelope of the following families of lines: a) $y=Cx+C^2$. *Ans.* $x^2+4y=0$. b) $y=\frac{x}{C}+C^2$. *Ans.* $27x^2=4y^3$. c) $\frac{x}{C}-\frac{y}{C^3}=2$. *Ans.* $27y=x^3$. d) $C^2x+Cy-1=0$. *Ans.* $y^2+4x=0$. e) $(x-C)^2+(y-C)^2=C^2$. *Ans.* $x=0$; $y=0$. f) $(x-C)^2+y^2=4C$. *Ans.* $y^2=4x+4$. g) $(x-C)^2+(y-C)^2=4$. *Ans.* $(x-y)^2=8$. h) $Cx^2+C^2y=1$. *Ans.* $x^4+4y=0$.

83. A straight line is in motion so that the sum of the segments it cuts off on the axes is a constant a. Form the equation of the envelope of all positions of the straight line. *Ans.* $x^{1/2}+y^{1/2}=a^{1/2}$ (parabola).

84. Find the envelope of a family of straight lines on which the coordinate axes cut off a segment of constant length a. *Ans.* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

85. Find the envelope of a family of circles whose diameters are the doubled ordinates of the parabola $y^2=2px$. *Ans.* $y^2=2p\left(x+\frac{p}{2}\right)$.

86. Find the envelope of a family of circles whose centres lie on the parabola $y^2=2px$; all the circles of the family pass through the vertex of this parabola. *Ans.* The cissoid $x^3+y^2(x+2p)=0$.

87. Find the envelope of a family of circles whose diameters are chords of the ellipse $b^2x^2+a^2y^2=a^2b^2$ perpendicular to the x-axis. *Ans.* $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$.

88. Find the evolute of the ellipse $b^2x^2+a^2y^2=a^2b^2$ as the envelope of its normals. *Ans.* $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

Integrate the following equations (Lagrange equations):

89. $y=2xy'+y'^2$. *Ans.* $x=\frac{C}{3p^2}-\frac{2}{3}p$; $y=\frac{2C-p^3}{3p}$. 90. $y=xy'^2+y'^2$. *Ans.* $y=(\sqrt{x+1}+C)^2$. Singular solution: $y=0$. 91. $y=x(1+y')+(y')^2$. *Ans.* $x=Ce^{-p}-2p+2$; $y=C(p+1)\,c^{-p}-p^2+2$. 92. $y=yy'^2+2xy'$. *Ans.*

$4Cx=4C^2-y^2$. **93.** Find a curve with constant normal. *Ans.* $(x-C)^2+y^2=a^2$. Singular solution: $y=\pm a$.

Integrate the given Clairaut equations:

94. $y=xy'+y'-y'^2$. *Ans.* $y=Cx+C-C^2$. Singular solution: $4y=(x+1)^2$. **95.** $y=xy'+\sqrt{1-y'^2}$. *Ans.* $y=Cx+\sqrt{1-C^2}$. Singular solution: $y^2-x^2=1$. **96.** $y=xy'+y'$. *Ans.* $y=Cx+C$. **97.** $y=xy'+\frac{1}{y'}$. *Ans.* $y=Cx+\frac{1}{C}$. Singular solution: $y^2=4x$. **98.** $y=xy'-\frac{1}{y'^2}$. *Ans.* $y=Cx-\frac{1}{C^2}$. Singular solution: $y^3=-\frac{27}{4}x^2$.

99. The area of a triangle formed by the tangent to the sought-for curve and the coordinate axes is a constant. Find the curve. *Ans.* The equilateral hyperbola $4xy=\pm a^2$. Also, any straight line of the family $y=Cx\pm a\sqrt{C}$.

100. Find a curve such that the segment of its tangent between the coordinate axes is of constant length a. *Ans.* $y=Cx\pm\frac{aC}{\sqrt{1+C^2}}$. Singular solution: $x^{2/3}+y^{2/3}=a^{2/3}$.

101. Find a curve the tangents to which form, on the axes, segments whose sum is $2a$. *Ans.* $y=Cx-\frac{2aC}{1-C}$. Singular solution: $(y-x-2a)^2=8ax$.

102. Find curves for which the product of the distance of any tangent line to two given points is constant. *Ans.* Ellipses and hyperbolas. (Orthogonal and isogonal trajectories.)

103. Find the orthogonal trajectories of the family of curves $y=ax^n$. *Ans.* $x^2+ny^2=C$.

104. Find the orthogonal trajectories of the family of parabolas $y^2=2p(x-\alpha)$ (α is the parameter of the family). *Ans.* $y=Ce^{-\frac{x}{p}}$.

105. Find the orthogonal trajectories of the family of curves $x^2-y^2=\alpha$ (α is the parameter). *Ans.* $y=\frac{C}{x}$.

106. Find the orthogonal trajectories of the family of circles $x^2+y^2=2ax$. *Ans.* Circles: $y=C(x^2+y^2)$.

107. Find the orthogonal trajectories of equal parabolas tangent at the vertex of the given straight line. *Ans.* If $2p$ is the parameter of the parabolas, and the given straight line is on the y-axis, then the equation of the trajectory will be $y+C=\frac{2}{3}\sqrt{\frac{2}{p}}x^{\frac{3}{2}}$.

108. Find the orthogonal trajectories of the cissoids $y^2=\frac{x^3}{2a-x}$. *Ans.* $(x^2+y^2)^2=C(y^2+2x^2)$.

109. Find the orthogonal trajectories of the lemniscates $(x^2+y^2)^2=(x^2-y^2)a^2$. *Ans.* $(x^2+y^2)^2=Cxy$.

110. Find the isogonal trajectories of the family of curves: $x^2=2a(y-x\sqrt{3})$, where a is a variable parameter if the constant angle formed by the trajectories and the lines of the family is $\omega=60°$.

Solution. We find the differential equation of the family $y'=\frac{2y}{x}-\sqrt{3}$ and for y' substitute the expression $q=\frac{y'-\tan\omega}{1+y'\tan\omega}$. If $\omega=60°$, then

$q=\frac{y'-\sqrt{3}}{1+\sqrt{3}\,y'}$ and we get the differential equation $\frac{y'-\sqrt{3}}{1+y'\sqrt{3}}=\frac{2y}{x}-\sqrt{3}$. The complete integral $y^2=C(x-y\sqrt{3})$ yields the desired family of trajectories.

111. Find the isogonal trajectories of the family of parabolas $y^2=4Cx$ when $\omega=45°$. *Ans.* $y^2-xy+2x^2=Ce^{\frac{6}{\sqrt{7}}\arctan\frac{2y-x}{x\sqrt{7}}}$.

112. Find the isogonal trajectories of the family of straight lines $y=Cx$ for the case $\omega=30°$, $45°$. *Ans.* The logarithmic spirals $\begin{cases} x^2+y^2=e^{2\sqrt{3}\arctan\frac{y}{x}}, \\ x^2+y^2=e^{2\arctan\frac{y}{x}}. \end{cases}$

113. $y=C_1e^x+C_2e^{-x}$. Eliminate C_1 and C_2. *Ans.* $y''-y=0$.

114. Write the differential equation of all circles lying in one plane. *Ans.* $(1+y'^2)\,y'''-3y'y''^2=0$.

115. Write the differential equation of all second-order central curves whose principal axes coincide with the x- and y-axes. *Ans.* $x(yy''+y'^2)-y'y=0$.

116. Given the differential equation $y'''-2y''-y'+2y=0$ and its general solution $y=C_1e^x+C_2e^{-x}+C_3e^{2x}$.

It is required to: 1) verify that the given family of curves is indeed the general solution; 2) find a particular solution if for $x=0$ we have $y=1$, $y'=0$, $y''=-1$. *Ans.* $y=\frac{1}{6}(9e^x+e^{-x}-4e^{2x})$.

117. Given the differential equation $y''=\frac{1}{2y'}$ and its general solution $y=\pm\frac{2}{3}(x+C_1)^{\frac{3}{2}}+C_2$.

It is required to: 1) verify that the given family of curves is indeed the general solution; 2) find the integral curve passing through the point (1, 2) if the tangent at this point forms with the positive x-direction an angle of 45°. *Ans.* $y=\frac{2}{3}\sqrt{x^3}+\frac{4}{3}$.

Integrate some of the simpler types of differential equations of the second order that lead to first-order equations.

118. $xy'''=2$. *Ans.* $y=x^2\ln x+C_1x^2+C_2x+C_3$; pick out a particular solution that satisfies the following initial conditions: $x=1$; $y=1$; $y'=1$; $y''=3$.

119. $y^{(n)}=x^m$. *Ans.* $y=\frac{m!\,x^{m+n}}{(m+n)!}+C_1x^{n-1}+\ldots+C_{n-1}x+C_n$. **120.** $y''=a^2y$. *Ans.* $ax=\ln(ay+\sqrt{a^2y^2+C_1})+C_2$ or $y=C_1e^{ax}+C_2e^{-ax}$. **121.** $y''=\frac{a}{y^3}$. *Ans.* $(C_1x+C_2)^2=C_1y^2-a$.

In Nos. 122-125 pick out a particular solution that satisfies the following initial conditions: $x=0$, $y=-1$; $y'=0$. **122.** $xy''-y'=x^2e^x$. *Ans.* $y=e^x(x-1)+C_1x^2+C_2$. Particular solution: $y=e^x(x-1)$. **123.** $yy''-(y')^2+(y')^3=0$. *Ans.* $y+C_1\ln y=x+C_2$. Particular solution: $y=-1$. **124.** $y''+y'\tan x=\sin 2x$. *Ans.* $y=C_2+C_1\sin x-x-\frac{1}{2}\sin 2x$. Particular solution: $y=2\sin x-\sin x\cos x-x-1$. **125.** $(y'')^2+(y')^2=a^2$. *Ans.* $y=C_2-a\cos(x+C_1)$. Particular solutions: $y=a-1-a\cos x$; $y=a\cos x-(a+1)$.

(**Hint.** Parametric form: $y''=a\cos t$, $y'=a\sin t$). **126.** $y''=\frac{1}{2y'}$. *Ans.* $y=\pm\frac{2}{3}(x+C_1)^{3/2}+C_2$. **127.** $y'''=y''^2$. *Ans.* $y=(C_1-x)[\ln(C_1-x)-1]+C_2x+C_3$. **128.** $y'y'''-3y''^2=0$. *Ans.* $x=C_1y^2+C_2y+C_3$.

Integrate the following linear differential equations with constant coefficients:

129. $y''=9y$. *Ans.* $y=C_1e^{3x}+C_2e^{-3x}$. **130.** $y''+y=0$. *Ans.* $y=A\cos x+B\sin x$. **131.** $y''-y'=0$. *Ans.* $y=C_1+C_2e^x$. **132.** $y''+12y=7y'$. *Ans.* $y=C_1e^{3x}+C_2e^{4x}$. **133.** $y''-4y'+4y=0$. *Ans.* $y=(C_1+C_2x)e^{2x}$. **134.** $y''+2y'+10y=0$. *Ans.* $y=e^x(A\cos 3x+B\sin 3x)$. **135.** $y''+3y'-2y=0$. *Ans.* $y=C_1e^{\frac{-3+\sqrt{17}}{2}x}+C_2e^{\frac{-3-\sqrt{17}}{2}x}$. **136.** $4y''-12y'+9y=0$. *Ans.* $y=(C_1+C_2x)e^{3/2x}$. **137.** $y''+y'+y=0$. *Ans.* $y=e^{-\frac{1}{2}x}\times\left[A\cos\left(\frac{\sqrt{3}}{2}x\right)+B\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$.

138. Two identical loads are suspended from the end of a spring. Find the motion imparted to one load if the other breaks loose. *Ans.* $x=a\cos\left(\sqrt{\frac{g}{a}}t\right)$, where a is the increase in length of the spring under the action of one load at rest.

139. A material point of mass m is attracted by each of two centres with a force proportional to the distance. The factor of proportionality is k. The distance between the centres is $2c$. At the initial instant the point lies on the line connecting the centres at a distance a from the middle. The initial velocity is zero. Find the law of motion of the point. *Ans.* $x=a\cos\left(\sqrt{\frac{2k}{m}}t\right)$.

140. $y^{IV}-5y''+4y=0$. *Ans.* $y=C_1e^x+C_2e^{-x}+C_3e^{2x}+C_4e^{-2x}$. **141.** $y'''-2y''-y'+2y=0$. *Ans.* $y=C_1e^{2x}+C_2e^x+C_3e^{-x}$. **142.** $y'''-3ay''+3a^2y'-a^3y=0$. *Ans.* $y=(C_1+C_2x+C_3x^2)e^{ax}$. **143.** $y^V-4y'''=0$. *Ans.* $y=C_1+C_2x+C_3x^2+C_4e^{2x}+C_5e^{-2x}$. **144.** $y^{IV}+2y''+9y=0$. *Ans.* $y=(C_1\cos\sqrt{2}x+C_2\sin\sqrt{2}x)e^{-x}+(C_3\cos\sqrt{2}x+C_4\sin\sqrt{2}x)e^x$. **145.** $y^{IV}-8y''+16y=0$. *Ans.* $y=C_1e^{2x}+C_2e^{-2x}+C_3xe^{2x}+C_4xe^{-2x}$. **146.** $y^{IV}+y=0$. *Ans.* $y=e^{\frac{x}{\sqrt{2}}}\left(C_1\cos\frac{x}{\sqrt{2}}+C_2\sin\frac{x}{\sqrt{2}}\right)+e^{-\frac{x}{\sqrt{2}}}\left(C_3\cos\frac{x}{\sqrt{2}}+C_4\sin\frac{x}{\sqrt{2}}\right)$.

147. $y^{IV}-a^4y=0$. Find the general solution and pick out a particular solution that satisfies the initial conditions for $x_0=0$, $y=1$, $y'=0$, $y''=-a^2$, $y'''=0$. *Ans.* General solution: $y=C_1e^{ax}+C_2e^{-ax}+C_3\cos ax+C_4\sin ax$. Particular solution: $y_0=\cos ax$.

Integrate the following nonhomogeneous linear differential equations (find the general solution):

148. $y''-7y'+12y=x$. *Ans.* $y=C_1e^{3x}+C_2e^{4x}+\frac{12x+7}{144}$. **149.** $s''-a^2s=t+1$. *Ans.* $s=C_1e^{at}+C_2e^{-at}\frac{t+1}{a^2}$. **150.** $y''+y'-2y=8\sin 2x$. *Ans.* $y=C_1e^x+$

$+C_2e^{-2x}-\frac{1}{5}(6\sin 2x+2\cos 2x)$. **151.** $y''-y=5x+2$. *Ans.* $y=C_1e^x+C_2e^{-x}-5x-2$. **152.** $s''-2as'+a^2s=e^t$ $(a\neq 1)$. *Ans.* $s=C_1e^{at}+C_2te^{at}+\frac{e^t}{(a-1)^2}$. **153.** $y''+6y'+5y=e^{2x}$. *Ans.* $y=C_1e^{-x}+C_2e^{-5x}+\frac{1}{21}e^{2x}$. **154.** $y''+9y=6e^{3x}$. *Ans.* $y=C_1\cos 3x+C_2\sin 3x+\frac{1}{3}e^{3x}$. **155.** $y''-3y'=2-6x$. *Ans.* $y=C_1+C_2e^{3x}+x^2$. **156.** $y''-2y'+3y=e^{-x}\cos x$. *Ans.* $y=e^x(A\cos\sqrt{2}x+B\sin\sqrt{2}x)+\frac{e^{-x}}{41}(5\cos x-4\sin x)$. **157.** $y''+4y=2\sin 2x$. *Ans.* $y=A\sin 2x+B\cos 2x-\frac{x}{2}\cos 2x$. **158.** $y'''-4y''+5y'-2y=2x+3$. *Ans.* $y=(C_1+C_2x)e^x+C_3e^{2x}-x-4$. **159.** $y^{IV}-a^4y=5a^4e^{ax}\sin ax$. *Ans.* $y=(C_1-\sin ax)e^{ax}+C_2e^{-ax}+C_3\cos ax+C_4\sin ax$. **160.** $y^{IV}+2a^2y''+a^4y=8\cos ax$. *Ans.* $y=(C_1+C_2x)\cos ax+(C_3+C_4x)\sin ax-\frac{x^2}{a^2}\cos ax$.

161. Find the integral curve of the equation $y''+k^2y=0$ that passes through the point $M(x_0, y_0)$ and is tangent at the point of the curve $y=ax$. *Ans.* $y=y_0\cos k(x-x_0)+\frac{a}{k}\sin k(x-x_0)$.

162. Find a solution of the equation $y''+2hy'+n^2y=0$ that satisfies the conditions $y=a$, $y'=C$ when $x=0$. *Ans.* For $h<n$ $y=e^{-hx}\left(a\cos\sqrt{n^2-h^2}\,x+\frac{C+ah}{\sqrt{n^2-h^2}}\sin\sqrt{n^2-h^2}\,x\right)$; for $h=n$ $y=e^{-hx}[(C+ah)x+a]$; for $h>n$ $y=\frac{C+a(h+\sqrt{h^2-n^2}}{2\sqrt{h^2-n^2}}e^{-(h-\sqrt{h^2-n^2})x}-\frac{C+a(h-\sqrt{h^2-n^2})}{2\sqrt{h^2-n^2}}e^{-(h+\sqrt{h^2-n^2})x}$.

163. Find solutions of the equation $y''+n^2y=h\sin px$ $(p\neq n)$ that satisfy the conditions: $y=a$, $y'=C$ for $x=0$. *Ans.* $y=a\cos nx+\frac{C(n^2-p^2)-hp}{n(n^2-p^2)}\sin nx+\frac{h^2}{n^2-p^2}\sin px$.

164. A load weighing 4 kg is suspended from a spring and increases its length by 1 cm. Find the law of motion of this load if we assume that the upper end of the spring performs harmonic oscillations under the law $y=\sin\sqrt{100g}\,t$, where y is measured vertically.

Solution. Denoting by x the vertical coordinate of the load reckoned from the position of rest, we have

$$\frac{4}{g}\frac{d^2x}{dt^2}=-k(x-y-l),$$

where l is the length of the spring in the free state and $k=400$, as is evident from the initial conditions. Whence $\frac{d^2x}{dt^2}+100gx=100g\sin\sqrt{100g}\,t+100\,lg$. We must seek the particular integral of this equation in the form

$$t(C_1\cos\sqrt{100g}\,t+C_2\sin\sqrt{100g}\,t)+g,$$

since the first term on the right enters into the solution of the homogeneous equation.

165. In Problem 139, the initial velocity is v_0 and the direction is perpendicular to the straight line connecting the centres. Find the trajectories.

Solution. If for the origin we take the mid-point between the centres, the differential equations of motion will be $m\frac{d^2x}{dt^2}=k(C-x)-k(C+x)=-2kx$, $m\frac{d^2y}{dt^2}=-2ky$. The initial data for $t=0$ are

$$x=a;\quad \frac{dx}{dt}=0;\quad y=0;\quad \frac{dy}{dt}=v_0.$$

Integrating, we find

$$x=a\cos\left(\sqrt{\frac{2k}{m}}\,t\right),\quad y=v_0\sqrt{\frac{m}{2k}}\sin\left(\sqrt{\frac{2k}{m}}\,t\right).$$

Whence $\frac{x^2}{a^2}+\frac{y^2 2k}{mv_0^2}=1$ (ellipse).

166. A horizontal tube is in rotation about a vertical axis with constant angular velocity ω. A sphere inside the tube slides along it without friction. Find the law of motion of the sphere if at the initial instant it lies on the axis of rotation and has velocity v_0 (along the tube).

Hint. The differential equation of motion is $\frac{d^2r}{dt^2}=\omega^2 r$. The initial data are: $r=0$, $\frac{dr}{dt}=v_0$ for $t=0$. Integrating, we find

$$r=\frac{v_0}{2\omega}\left[e^{\omega t}+e^{-\omega t}\right].$$

Applying the method of variation of parameters, integrate the following differential equations:

167. $y''-7y'+6y=\sin x$. *Ans.* $y=C_1e^x+C_2e^{6x}+\frac{5\sin x+7\cos x}{74}$. **168.** $y''+y=\sec x$. *Ans.* $y=C_1\cos x+C_2\sin x+x\sin x+\cos x\ln\cos x$. **169.** $y''+y=\frac{1}{\cos 2x\sqrt{\cos 2x}}$. *Ans.* $y=C_1\cos x+C_2\sin x-\sqrt{\cos 2x}$.

Integrate the following systems of equations:

170. $\frac{dx}{dt}=y+1$, $\frac{dy}{dt}=x+1$. Pick out the particular solutions that satisfy the initial conditions $x=-2$, $y=0$ for $t=0$. *Ans.* $y=C_1\cos t+C_2\sin t$, $x=(C_1+C_2)\cos t+(C_2-C_1)\sin t$. Particular solution: $x^*=\cos t-\sin t$, $y^*=\cos t$.

171. $\frac{dx}{dt}=x-2y$, $\frac{dy}{dt}=x-y$. Pick out the particular solutions that satisfy the initial conditions: $x=1$, $y=1$ for $t=0$. *Ans.* $y=C_1\cos t+C_2\sin t$, $x=(C_1+C_2)\cos t+(C_2-C_1)\sin t$. Particular solution: $x^*=\cos t-\sin t$, $y^*=\cos t$.

172. $\begin{cases} 4\frac{dx}{dt}-\frac{dy}{dt}+3x=\sin t, \\ \frac{dx}{dt}+y=\cos t. \end{cases}$ *Ans.* $x=C_1e^{-t}+C_2e^{-3t}$, $y=C_1e^{-t}+3C_2e^{-3t}+\cos t$.

173. $\begin{cases} \dfrac{d^2y}{dt^2}=x, \\ \dfrac{d^2x}{dt^2}=y. \end{cases}$ *Ans.* $x=C_1e^t+C_2e^{-t}+C_3\cos t+C_4\sin t,$ $y=C_1e^t+C_2e^{-t}-C_3\cos t-C_4\sin t.$

174. $\begin{cases} \dfrac{d^2x}{dt^2}+\dfrac{dy}{dt}+x=e^t, \\ \dfrac{dx}{dt}+\dfrac{d^2y}{dt^2}=1. \end{cases}$ *Ans.* $x=C_1+C_2t+C_3t^2-\frac{1}{6}t^3+e^t,$ $y=C_4-(C_1+2C_3)t-\frac{1}{2}(C_2-1)t^2-$ $-\frac{1}{3}C_3t^3+\frac{1}{24}t^4-e^t.$

175. $\begin{cases} \dfrac{dy}{dx}=z-y, \\ \dfrac{dz}{dx}=-y-3z. \end{cases}$ *Ans.* $y=(C_1+C_2x)e^{-2x},$ $z=(C_2-C_1-C_2x)e^{-2x}.$

176. $\begin{cases} \dfrac{dy}{dx}+z=0, \\ \dfrac{dz}{dx}+4y=0. \end{cases}$ *Ans.* $y=C_1e^{2x}+C_2e^{-2x},$ $z=-2(C_1e^{2x}-C_2e^{-2x}).$

177. $\begin{cases} \dfrac{dy}{dx}+2y+z=\sin x, \\ \dfrac{dz}{dx}-4y-2z=\cos x. \end{cases}$ *Ans.* $y=C_1+C_2x+2\sin x,$ $z=-2C_1-C_2(2x+1)-3\sin x-2\cos x.$

178. $\begin{cases} \dfrac{dx}{dt}=y+z, \\ \dfrac{dy}{dt}=x+z, \\ \dfrac{dz}{dt}=x+y. \end{cases}$ *Ans.* $x=C_1e^{-t}+C_2e^{2t},$ $y=C_3e^{-t}+C_2e^{2t},$ $z=-(C_1+C_3)e^{-t}+C_2e^{2t}.$

179. $\begin{cases} \dfrac{dy}{dx}=1-\dfrac{1}{z}, \\ \dfrac{dz}{dx}=\dfrac{1}{y-x}. \end{cases}$ *Ans.* $z=C_2e^{C_1x},$ $y=x+\dfrac{1}{C_1C_2}e^{-C_1x}.$

180. $\begin{cases} \dfrac{dy}{dx}=\dfrac{x}{yz}, \\ \dfrac{dz}{dx}=\dfrac{x}{y^2}. \end{cases}$ *Ans.* $\dfrac{z}{y}=C_1,$ $zy^2-\dfrac{3}{2}x^2=C_2.$

Integrate the following different types of equations:

181. $yy''=y'^2+1$. *Ans.* $y=\dfrac{1}{2C_1}[e^{C_1(x-C_2)}+e^{-C_1(x-C_2)}]$. 182. $\dfrac{x^2dy-y^2dx}{(x-y)^2}=0.$

Ans. $\frac{xy}{x-y}=C$. **183.** $y=xy'^2+y'^2$. *Ans.* $y=(\sqrt{x+1}+C)^2$. Singular solutions: $y=0$; $x+1=0$. **184.** $y''+y=\sec x$. *Ans.* $y=C_1\cos x+C_2\sin x+x\sin x+\cos x\ln\cos x$. **185.** $(1+x^2)y'-xy-a=0$. *Ans.* $y=ax+C\sqrt{1+x^2}$. **186.** $x\cos\frac{y}{x}\frac{dy}{dx}=y\cos\frac{y}{x}-x$. *Ans.* $xe^{\sin\frac{y}{x}}=C$. **187.** $y''-4y=e^{2x}\sin 2x$. *Ans.* $y=C_1e^{-2x}+C_2e^{2x}-\frac{e^{2x}}{20}(\sin 2x+2\cos 2x)$. **188.** $xy'+y-y^2\ln x=0$. *Ans.* $(\ln x+1+Cx)y=1$. **189.** $(2x+2y-1)dx+(x+y-2)dy=0$. *Ans.* $2x+y-3\ln(x+y+1)=C$. **190.** $3e^x\tan y\,dx+(1-e^x)\sec^2 y\,dy=0$. *Ans.* $\tan y=C(1-e^x)^3$.

Investigate and determine whether the solution $x=0$, $y=0$ is stable for the following systems of differential equations:

191. $\begin{cases}\frac{dx}{dt}=2x-3y,\\ \frac{dy}{dt}=5x+6y.\end{cases}$ *Ans.* Unstable.

192. $\begin{cases}\frac{dx}{dt}=-4x-10y,\\ \frac{dy}{dt}=x-2y.\end{cases}$ *Ans.* Stable.

193. $\begin{cases}\frac{dx}{dt}=12x+18y,\\ \frac{dy}{dt}=-8x-12y.\end{cases}$ *Ans.* Unstable.

194. Approximate the solution of the equation $y'=y^2+x$ that satisfies the initial condition $y=1$ when $x=0$. Find the values of the solution for x equal to 0.1, 0.2, 0.3, 0.4, 0.5. *Ans.* $y_{x=0.5}=2.114$.

195. Approximate the value of $y_{x=1.4}$ of a solution of the equation $y'+\frac{1}{x}y=e^x$ that satisfies the initial conditions $y=1$ when $x=1$. Compare the result obtained with the exact solution.

196. Find the approximate values of $x_{t=1.4}$ and $y_{t=1.4}$ of the solutions of a a system of equations $\frac{dx}{dt}=y-x$, $\frac{dy}{dt}=-x-3y$ that satisfy the initial conditions $x=0$, $y=1$ when $t=1$. Compare the values obtained with the exact values.

CHAPTER XIV

MULTIPLE INTEGRALS

SEC. 1. DOUBLE INTEGRALS

In an xy-plane we consider a closed*) region D bounded by a line L.

In this region D let there be given a continuous function

$$z=f(x,\ y).$$

Using arbitrary lines we divide the region D into n parts

$$\Delta s_1,\ \Delta s_2,\ \Delta s_3,\ \ldots,\ \Delta s_n$$

(Fig. 276) which we shall call subregions. So as not to introduce new symbols we will denote by $\Delta s_1, \ldots, \Delta s_n$ both the subregions and their areas. In each subregion Δs_i (it is immaterial whether in the interior or on the boundary) take a point P_i; we will then have n points:

$$P_1,\ P_2,\ \ldots,\ P_n.$$

We denote by $f(P_1),\ f(P_2),\ \ldots,\ f(P_n)$ the values of the functions at the chosen points and then form the sum of the products $f(P_i)\,\Delta s_i$:

$$V_n=f(P_1)\Delta s_1+f(P_2)\,\Delta s_2+\ldots+f(P_n)\Delta s_n=\sum_{i=1}^{n} f(P_i)\Delta s_i. \qquad (1)$$

This is the *integral sum* of the function $f(x,\ y)$ in the region D.

If $f\geqslant 0$ in D, then each term $f(P_i)\,\Delta s_i$ may be represented geometrically as the volume of a small cylinder with base Δs_i and altitude $f(P_i)$.

The sum V_n is the sum of the volumes of the indicated elementary cylinders, that is, the volume of a certain "step-like" solid (Fig. 277).

Consider an arbitrary sequence of integral sums formed by means of the function $f(x,\ y)$ for the given region D,

$$V_{n_1},\ V_{n_2},\ \ldots,\ V_{n_k},\ \ldots \qquad (2)$$

*) A region D is called closed if it is bounded by a closed line, and the points lying on the boundary are considered as belonging to the region D.

for different ways of partitioning D into subregions Δs_i. We shall assume that the maximum diameter of the subregions Δs_i approaches zero as $n_k \to \infty$, and the following proposition, which we give without proof, holds true.

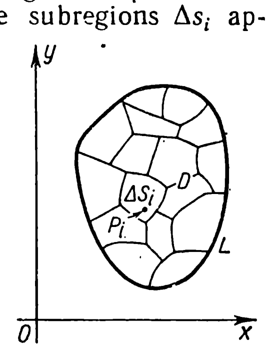


Fig. 276.

Theorem 1. *If a function $f(x, y)$ is continuous in a closed region D, then there is a limit of the sequence (2) of integral sums (1) if the maximum diameter of the subregions Δs_i approaches zero as $n \to \infty$. This limit is the same for any sequence of type (2), that is, it is independent either of the way the region D is partitioned into subregions Δs_i or of the choice of the point P_i inside the subregion Δs_i.*

This limit is called the *double integral* of the function $f(x, y)$ over the region D and is denoted by

$$\iint_D f(P)\,ds \quad \text{or} \quad \iint_D f(x, y)\,dx\,dy,$$

that is,

$$\lim_{\operatorname{diam} \Delta s_i \to 0} \sum_{i=1}^{n} f(P_i)\,\Delta s_i = \iint_D f(x, y)\,dx\,dy.$$

This region D is called the *domain* (*region*) *of integration*.

If $f(x, y) \geqslant 0$, then the double integral of $f(x, y)$ over D is equal to the volume of the solid Q bounded by a surface $z = f(x, y)$,

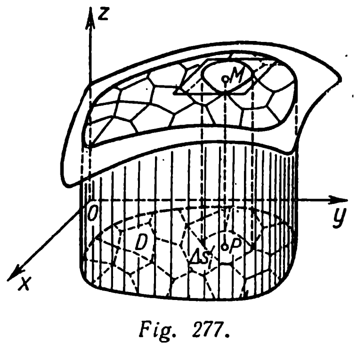


Fig. 277.

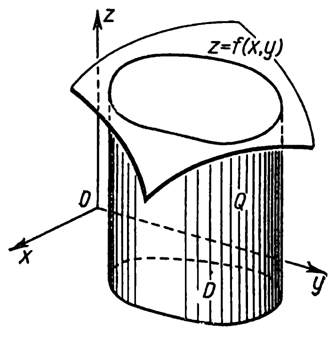


Fig. 278.

the plane $z = 0$, and a cylindrical surface whose generators are parallel to the z-axis, while the directrix is the boundary of the region D (Fig. 278).

Now consider the following theorems about the double integral.

Theorem 2. *The double integral of a sum of two functions* $\varphi(x, y)+\psi(x, y)$ *over the region D is equal to the sum of the double integrals over D of each of the functions taken separately*:

$$\iint_D [\varphi(x, y)+\psi(x, y)]\,ds=\iint_D \varphi(x, y)\,ds+\iint_D \psi(x, y)\,ds.$$

Theorem 3. *A constant factor may be taken outside the double integral sign*:

if a-const, *then*

$$\iint_D a\varphi(x, y)\,ds=a\iint_D \varphi(x, y)\,ds.$$

The proof of both theorems is exactly the same as that of the corresponding theorems for the definite integral (see. Sec. 3, Ch. XI).

Theorem 4. *If a region D is divided into two regions* D_1 *and* D_2 *without common interior points, and the function* $f(x, y)$ *is continuous at all points of D, then*

$$\iint_D f(x, y)\,dx\,dy=\iint_{D_1} f(x, y)\,dx\,dy+\iint_{D_2} f(x, y)\,dx\,dy. \qquad (3)$$

Proof. The integral sum over D may be given in the form (Fig. 279)

$$\sum_D f(P_i)\,\Delta s_i=\sum_{D_1} f(P_i)\,\Delta s_i+\sum_{D_2} f(P_i)\,\Delta s_i, \qquad (4)$$

where the first sum contains terms that correspond to the subregions of D_1, the second, those corresponding to the subregions of D_2. Indeed, since the double integral does not depend on the manner of partition, we divide the region D so that the common boundary of the regions D_1 and D_2 is a boundary of the subregions Δs_i. Passing to the limit in (4) as $\Delta s_i \to 0$, we get (3). This theorem is obviously true for any number of terms.

SEC. 2. CALCULATING DOUBLE INTEGRALS

Let a region D lying in the xy-plane be such that any straight line parallel to one of the coordinate axes (for example, the y-axis) and passing through an **interior***) point of the region, cuts the boundary of the region at two points N_1 and N_2 (Fig. 280).

*) An interior point of a region is one that does not lie on the boundary of the region.

In this case we assume that the region D is bounded by the lines: $y=\varphi_1(x)$, $y=\varphi_2(x)$, $x=a$, $x=b$ and that

$$\varphi_1(x) \leqslant \varphi_2(x), \quad a < b$$

while the functions $\varphi_1(x)$ and $\varphi_2(x)$ are continuous on the interval $[a, b]$. We shall call such a region *regular in the y-direction*. The definition is similar for a region *regular in the x-direction.*

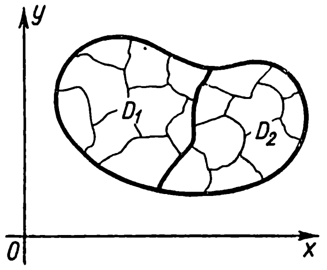


Fig. 279.

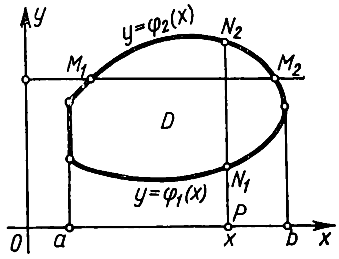


Fig. 280.

A region that is regular in both x- and y-directions we shall simply call a *regular region*. In Fig. 280 we have a regular region D.

Let the function $f(x, y)$ be continuous in D.

Consider the expression

$$I_D=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy\right)dx$$

which we shall call an *iterated* integral of $f(x, y)$ over D. In this expression we first calculate the integral in the parentheses (the integration is performed with respect to y) while x is considered to be constant. The integration yields a continuous *) function of x:

$$\Phi(x)=\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy.$$

We integrate this function with respect to x from a to b:

$$I_D=\int_a^b \Phi(x)\,dx.$$

This yields a certain constant.

*) We do not here prove that the function $\Phi(x)$ is continuous.

Example 1. To calculate the iterated integral

$$I_D=\int_0^1\left(\int_0^{x^2}(x^2+y^2)\,dy\right)dx.$$

Solution. First calculate the inner integral (in brackets):

$$\Phi(x)=\int_0^{x_2}(x^2+y^2)\,dy=\left[x^2y+\frac{y^3}{3}\right]_0^{x^2}=x^2x^2+\frac{(x^2)^3}{3}=x^4+\frac{x^6}{3}.$$

Integrating the function obtained from 0 to 1, we find

$$\int_0^1\left(x^4+\frac{x^6}{3}\right)dx=\left[\frac{x^5}{5}+\frac{x^7}{3\cdot 7}\right]_0^1=\frac{1}{5}+\frac{1}{21}=\frac{26}{105}.$$

Determine the region D. Here, D is considered the region bounded by the lines (Fig. 281)

$$y=0,\quad x=0,\quad y=x^2,\quad x=1.$$

It may happen that the region D is such that one of the functions $y=\varphi_1(x)$, $y=\varphi_2(x)$ cannot be represented by a single

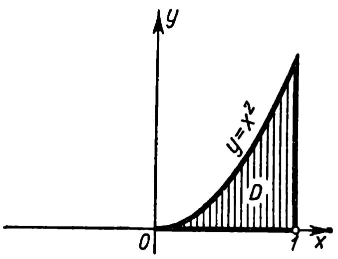


Fig. 281.

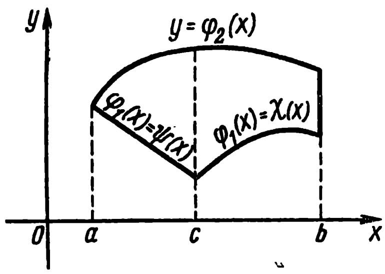


Fig. 282.

analytic expression over the entire range of x (from $x=a$ to $x=b$). For example, let $a<c<b$, and

$$\varphi_1(x)=\psi(x) \text{ on the interval } [a, c],$$
$$\varphi_1(x)=\chi(x) \text{ on the interval } [c, b].$$

where $\psi(x)$ and $\chi(x)$ are analytic functions (Fig. 282). Then the

iterated integral will be written as follows:

$$\int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy\right] dx =$$

$$= \int_a^c \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy\right] dx + \int_c^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy\right] dx =$$

$$= \int_a^c \left[\int_{\psi(x)}^{\varphi_2(x)} f(x, y)\,dy\right] dx + \int_c^b \left[\int_{\chi(x)}^{\varphi_2(x)} f(x, y)\,dy\right] dx.$$

The first of these equations is written on the basis of a familiar property of the definite integral, the second, due to the fact that on the interval $[a, c]$ we have $\varphi_1(x) \equiv \psi(x)$, and on the interval $[c, b]$ we have $\varphi_1(x) \equiv \chi(x)$.

We would also have a similar notation for the iterated integral if the function $\varphi_2(x)$ were defined by different analytic expressions on different subintervals of the interval $[a, b]$.

Let us establish some properties of an iterated integral.

Property 1. *If a regular y-direction region D is divided into two regions D_1 and D_2 by a straight line parallel to the y-axis or the x-axis, then the iterated integral I_D over D will be equal to the sum of such integrals over D_1 and D_2; that is,*

$$I_D = I_{D_1} + I_{D_2}. \tag{1}$$

Proof. a) Let the straight line $x = c$ $(a < c < b)$ divide the region D into two regular y-direction regions*) D_1 and D_2. Then

$$I_D = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy\right) dx = \int_a^b \Phi(x)\,dx = \int_a^c \Phi(x)\,dx + \int_c^b \Phi(x)\,dx =$$

$$= \int_a^c \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy\right) dx + \int_c^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy\right) dx = I_{D_1} + I_{D_2}.$$

---

*) The fact that a part of the boundary of the region D_1 (and of D_2) is a portion of the vertical straight line does not stop this region from being regular in the y-direction: for a region to be regular, it is only necessary that any vertical straight line passing through an interior point of the region should have no more than two common points with the boundary (see footnote on page 610).

b) Let the straight line $y=h$ divide the region D into two regular y-direction regions D_1 and D_2 as shown in Fig. 283. Denote by M_1 and M_2 the points of intersection of the straight line $y=h$ with the boundary L of D. Denote the abscissas of these points by a_1 and b_1.

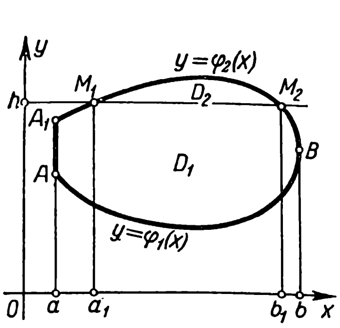


Fig. 283.

The region D_1 is bounded by continuous lines:

1) $y=\varphi_1(x)$;

2) the curve $A_1M_1M_2B$, whose equation we shall conditionally write in the form

$$y=\varphi_1^*(x),$$

having in view that $\varphi_1^*(x)=\varphi_2(x)$ when $a\leqslant x\leqslant a_1$ and when $b_1\leqslant x\leqslant b$ and that

$$\varphi_1^*(x)=h \quad \text{when } a_1\leqslant x\leqslant b_1;$$

3) by the straight lines $x=a$, $x=b$.

The region D_2 is bounded by the lines

$$y=\varphi_1^*(x),\ y=\varphi_2(x),\ \text{where } a_1\leqslant x\leqslant b_1.$$

We write the identity by applying to the inner integral the theorem for partitioning the interval of integration:

$$I_D=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy\right)dx=$$

$$=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_1^*(x)} f(x,\ y)\,dy+\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x,\ y)\,dy\right]dx=$$

$$=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_1^*(x)} f(x,\ y)\,dy\right)dx+\int_a^b\left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x,\ y)\,dy\right)dx.$$

We break up the latter integral into three integrals and apply to the outer integral the theorem for dividing the interval of

integration:

$$\int_a^b \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy \right) dx = \int_a^{a_1} \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy \right) dx +$$

$$+ \int_{a_1}^{b_1} \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy \right) dx + \int_{b_1}^{b} \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy \right) dx;$$

since $\varphi_1^*(x) = \varphi_2(x)$ on the interval $[a, a_1]$ and on $[b_1, b]$, it follows that the first and third integrals are identically zero. Therefore,

$$I_D = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_1^*(x)} f(x, y)\,dy \right) dx + \int_{a_1}^{b_1} \left(\int_{\varphi_1^*(x)}^{\varphi_2(x)} f(x, y)\,dy \right) dx.$$

Here, the first integral is an iterated integral over D_1, the second, over D_2. Consequently,

$$I_D = I_{D_1} + I_{D_2}.$$

The proof will be similar for any position of the cutting straight line M_1M_2. If M_1M_2 divides D into three or a larger number of regions, we get a relation similar to (1), in the first part of which we will have the appropriate number of terms.

Corollary. We can again divide each of the regions obtained (using a straight line parallel to the y-axis or x-axis) into regular y-direction regions, and we can apply to them equation (2). Thus, D may be divided by straight lines parallel to the coordinate axes into any number of regular regions

$$D_1, D_2, D_3, \ldots, D_i,$$

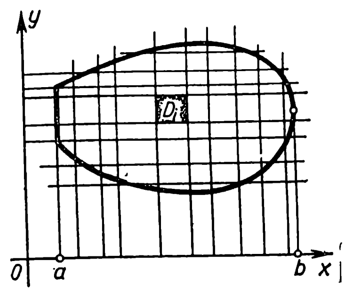


Fig. 284.

and the assertion that the iterated integral over D is equal to the sum of iterated integrals over subregions holds; that is (Fig. 284),

$$I_D = I_{D_1} + I_{D_2} + I_{D_3} + \ldots + I_{D_i}. \tag{2}$$

Property 2 (Evaluation of an iterated integral). *Let m and M be the least and greatest values of the function $f(x, y)$ in the*

region D. Denote by S the area of D. Then we have the relation

$$mS \leqslant \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \right) dx \leqslant MS. \tag{3}$$

Proof. Evaluate the inner integral denoting it by $\Phi(x)$:

$$\Phi(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \leqslant \int_{\varphi_1(x)}^{\varphi_2(x)} M\, dy = M[\varphi_2(x) - \varphi_1(x)].$$

We then have

$$I_D = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \right) dx \leqslant \int_a^b M[\varphi_2(x) - \varphi_1(x)]\, dx = MS,$$

that is,

$$I_D \leqslant MS. \tag{3'}$$

Similarly

$$\Phi(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \geqslant \int_{\varphi_1(x)}^{\varphi_2(x)} m\, dx = m[\varphi_2(x) - \varphi_1(x)],$$

$$I_D = \int_a^b \Phi(x)\, dx \geqslant \int_a^b m[\varphi_2(x) - \varphi_1(x)]\, dx = mS,$$

that is,

$$I_D \geqslant mS. \tag{3''}$$

From the inequalities (3′) and (3″) follows the relation (3):

$$mS \leqslant I_D \leqslant MS.$$

In the next section we will determine the geometric meaning of this theorem.

Property 3. (Mean-Value Theorem). *An iterated integral I_D of a continuous function $f(x, y)$ over a region D with area S is equal to the product of the area S by the value of the function at some point P in the region D; that is,*

$$\int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy \right) dx = f(P) S. \tag{4}$$

Proof. From (3) we obtain

$$m \leqslant \frac{1}{S} I_D \leqslant M.$$

The number $\frac{1}{S} I_D$ lies between the greatest and least values of $f(x, y)$ in D. Due to the continuity of the function $f(x, y)$, at some point P of D it takes on a value equal to the number $\frac{1}{S} I_D$; that is,

$$\frac{1}{S} I_D = f(P),$$

whence

$$I_D = f(P) S. \qquad (5)$$

SEC. 3. CALCULATING DOUBLE INTEGRALS (CONTINUED)

Theorem. *The double integral of a continuous function $f(x, y)$ over a regular region D is equal to the iterated integral of this function over D; that is,**)

$$\iint_D f(x, y)\,dx\,dy = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy \right) dx.$$

Proof. Partition the region D with straight lines parallel to the coordinate axes into n regular (rectangular) subregions:

$$\Delta s_1,\ \Delta s_2,\ \ldots,\ \Delta s_n.$$

By Property 1 [formula (2)] of the preceding section we have

$$I_D = I_{\Delta s_1} + I_{\Delta s_2} + \ldots + I_{\Delta s_n} = \sum_{i=1}^{n} I_{\Delta s_i}. \qquad (1)$$

Each of the terms on the right we transform by the mean-value theorem for an iterated integral:

$$I_{\Delta s_i} = f(P_i)\,\Delta s_i.$$

Then (1) takes the form

$$I_D = f(P_1)\,\Delta s_1 + f(P_2)\,\Delta s_2 + \ldots + f(P_n)\,\Delta s_n = \sum_{i=1}^{n} f(P_i)\,\Delta s_i, \qquad (2)$$

where P_i is some point of the subregion Δs_i. On the right is the integral sum of the function $f(x, y)$ over the region D. From the existence theorem of a double integral it follows that the limit of this sum, as $n \to \infty$ and as the greatest diameter of the subregions Δs_i approach zero, exists and is equal to the double integral of $f(x, y)$ over D. The value of the double integral I_D on

*) Here, we again assume that the region D is regular in the y-direction and bounded by the lines $y = \varphi_1(x)$, $y = \varphi_2(x)$, $x = a$, $x = b$.

the right side of (2) does not depend on n. Thus, passing to the limit in (2), we obtain

$$I_D = \lim_{\text{diam}\,\Delta s_i \to 0} \sum f(P_i)\,\Delta s_i = \iint_D f(x, y)\,dx\,dy$$

or

$$\iint_D f(x, y)\,dx\,dy = I_D. \tag{3}$$

Writing out in full the expression of the iterated integral I_D, we finally get

$$\iint_D f(x, y)\,dx\,dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy\right] dx. \tag{4}$$

Note 1. For the case when $f(x, y) \geqslant 0$, formula (4) has a pictorial geometric interpretation. Consider a solid bounded by the surface $z = f(x, y)$, the plane $z = 0$, and a cylindrical surface whose generators are parallel to the z-axis and the directrix of which is the boundary of the region D (Fig. 285). Calculate the volume of this solid V. It has already been shown that the volume of this solid is equal to the double integral of the function $f(x, y)$ over the region D:

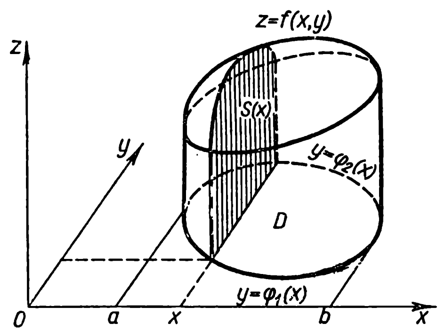


Fig. 285.

$$V = \iint_D f(x, y)\,dx\,dy. \tag{5}$$

Now let us calculate the volume of this solid using the results of Sec. 4, Ch. XII, on the evaluation of the volume of a solid from the areas of parallel sections (slices). Draw the plane $x = \text{const}$ $(a < x < b)$ that cuts the solid. Calculate the area $S(x)$ of the figure obtained by cutting $x = \text{const}$. This figure is a curvilinear trapezoid bounded by the lines $z = f(x, y)$ $(x = \text{const})$, $z = 0$, $y = \varphi_1(x)$, $y = \varphi_2(x)$. Hence, this area can be expressed by the integral

$$S(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\,dy. \tag{6}$$

Knowing the areas of parallel sections, it is easy to find the

volume of the solid:

$$V=\int_a^b S(x)\,dx;$$

or, substituting expression (6), we get for the area $S(x)$

$$V=\int_a^b\Big[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy\Big]\,dx. \tag{7}$$

In formulas (5) and (7) the left sides are equal; and so the right sides are equal too:

$$\iint_D f(x,\ y)\,dx\,dy=\int_a^b\Big[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy\Big]\,dx.$$

It is now easy to figure out the geometric meaning of the evaluation theorem of an iterated integral (Property 2, Sec. 2): the volume V of a solid bounded by the surface $z=f(x,\ y)$, the

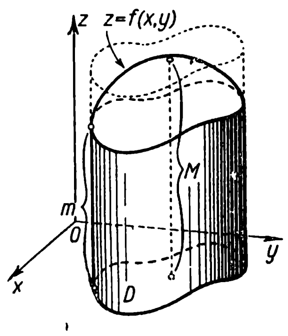


Fig 286.

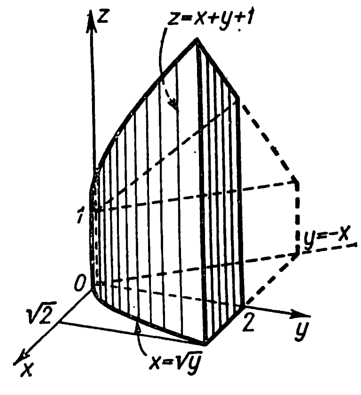


Fig. 287.

plane $z=0$, and a cylindrical surface whose directrix is the boundary of the region D, exceeds the volume of a cylinder with base area S and altitude m, but is less than the volume of a cylinder with base area S and altitude M [where m and M are the least and greatest values of the function $z=f(x,\ y)$ in the region D (Fig. 286)]. This follows from the fact that iterated integral I_D is equal to the volume V of this solid.

Example 1. Evaluate the double integral $\iint\limits_D (4-x^2-y^2)\,dx\,dy$ if the region D is bounded by the straight lines $x=0$, $x=1$, $y=0$, and $y=\frac{3}{2}$.

Solution. By the formula

$$v=\int\limits_0^{3/2}\left[\int\limits_0^1 (4-x^2-y^2)\,dx\right]dy=\int\limits_0^{3/2}\left[4x-y^2x-\frac{x^3}{3}\right]_0^1 dx=$$

$$=\int\limits_0^{3/2}\left(4-y^2-\frac{1}{3}\right)dy=\left(4y-\frac{y^3}{3}-\frac{1}{3}y\right)\Bigg|_0^{3/2}=\frac{35}{8}.$$

Example 2. Evaluate the double integral of the function $f(x, y)=1+x+y$ over a region bounded by the lines $y=-x$, $x=\sqrt{y}$, $y=2$, $z=0$ (Fig. 287).

Solution.

$$v=\int\limits_0^2\left[\int\limits_{-y}^{\sqrt{y}}(1+x+y)\,dx\right]dy=\int\limits_0^2\left[x+xy+\frac{x^2}{2}\right]_{-y}^{\sqrt{y}}dy=$$

$$=\int\limits_0^2\left[\left(\sqrt{y}+y\sqrt{y}+\frac{y}{2}\right)-\left(-y-y^2+\frac{y^2}{2}\right)\right]dy=$$

$$=\int\limits_0^2\left[\sqrt{y}+\frac{3y}{2}+y\sqrt{y}-\frac{y^2}{2}\right]dy=$$

$$=\left[\frac{2y^{\frac{3}{2}}}{3}+\frac{3y^2}{4}+\frac{2y^{\frac{5}{2}}}{5}-\frac{y^3}{6}\right]_0^2=\frac{44}{15}\sqrt{2}+\frac{5}{3}.$$

Note 2. Let a regular x-direction region D be bounded by the lines

$$x=\psi_1(y),\quad x=\psi_2(y),\quad y=c,\quad y=d,$$

and let $\psi_1(y)\leqslant\psi_2(y)$ (Fig. 288).

In this case, obviously,

$$\iint\limits_D (x,\ y)\,dx\,dy=\int\limits_c^d\left(\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,\ y)\,dx\right)dy. \tag{8}$$

To evaluate the double integral we must represent it as an iterated integral. As we have already seen, this may be done in two different ways: either by formula (4) or by formula (8). Depending upon the type of the region D or the integrand in each specific case, we choose one of the formulas to calculate the double integral.

Example 3. Change the order of integration in the integral

$$I=\int_0^1\left(\int_x^{\sqrt{x}} f(x,\ y)\,dy\right)dx.$$

Solution. The region of integration is bounded by the straight line $y=x$ and the parabola $y=\sqrt{x}$. (Fig. 289).

Every straight line parallel to the x-axis cuts the boundary of the region at no more than two points; hence, we can compute the integral by formula (8), setting

$$\psi_1(y)=y^2,\quad \psi_2(y)=y,\quad 0\leqslant y\leqslant 1;$$

then

$$I=\int_0^1\left(\int_{y^2}^{y} f(x,\ y)\,dx\right)dy.$$

Example 4. Evaluate $\iint\limits_D e^{\frac{y}{x}}\,ds$ if the region D is a triangle bounded by the straight lines $y=x$, $y=0$, and $x=1$ (Fig. 290).

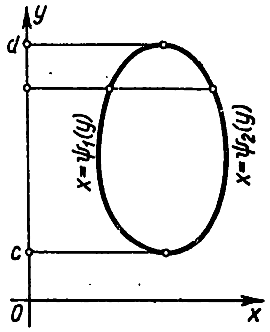


Fig. 288.

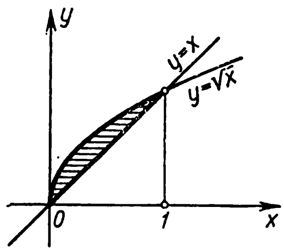


Fig. 289.

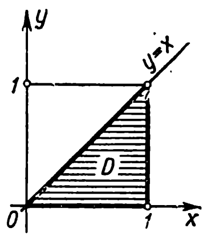


Fig. 290.

Solution. Replace this double integral by an iterated integral using formula (4). [If we used formula (8), we would have to integrate the function $e^{\frac{y}{x}}$ with respect to x; but this integral is not expressible in terms of elementary functions]:

$$\iint\limits_D e^{\frac{y}{x}}\,dx=\int_0^1\left[\int_0^x e^{\frac{y}{x}}\,dy\right]dx=\int_0^1\left[xe^{\frac{y}{x}}\right]_0^x dx=$$

$$=\int_0^1 x(e-1)\,dx=(e-1)\left.\frac{x^2}{2}\right|_0^1=\frac{e-1}{2}=0{,}859\ldots$$

Note 3. If the region D is not regular either in the x-direction or the y-direction (that is, there exist vertical and horizontal straight lines which, while passing through interior points of the region, cut the boundary of the region at more than two points), then we cannot represent the double integral over this region in

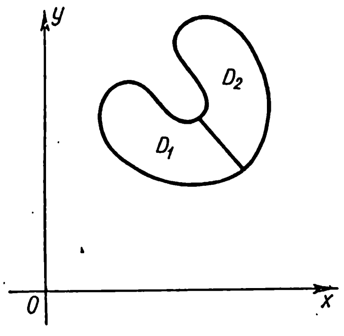


Fig. 291.

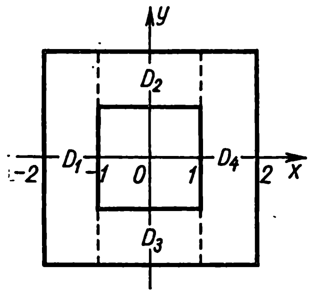


Fig. 292.

the form of an iterated integral. If we manage to partition the irregular region D into a finite number of regular x-direction or y-direction regions $D_1, D_2, \ldots, D_n$, then, by evaluating the double integral over each of these subregions by means of the iterated integral and adding the results obtained, we get the sought-for integral over D.

Fig. 291 is an example of how an irregular region D may be divided into two regular subregions D_1 and D_2.

Example 5. Evaluate the double integral

$$\iint_D e^{x+y}\, ds$$

over the region D which lies between two squares with centre at the origin and with sides parallel to the axes of coordinates, if each side of the inner square is equal to 2 and that of the outer square is 4 (Fig. 292).

Solution. The region D is irregular. However, the straight lines $x=-1$ and $x=1$ divide it into four regular subregions D_1, D_2, D_3, D_4. Therefore,

$$\iint_D e^{x+y}\, dx = \iint_{D_1} e^{x+y}\, ds + \iint_{D_2} e^{x+y}\, ds + \iint_{D_3} e^{x+y}\, ds + \iint_{D_4} e^{x+y}\, ds.$$

Representing each of these integrals in the form of an iterated integral, we find

$$\iint\limits_D e^{x+y}\,ds = \int\limits_{-2}^{-1}\Big[\int\limits_{-2}^{2} e^{x+y}\,dy\Big]\,dx + \int\limits_{-1}^{1}\Big[\int\limits_{1}^{2} e^{x+y}\,dy\Big]\,dx +$$

$$+\int\limits_{-1}^{1}\Big[\int\limits_{-2}^{-1} e^{x+y}\,dy\Big]\,dx + \int\limits_{1}^{2}\Big[\int\limits_{-2}^{2} e^{x+y}\,dy\Big]\,dx =$$

$$=(e^2-e^{-2})(e^{-1}-e^{-2})+(e^2-e)(e-e^{-1})+(e^{-1}-e^{-2})(e-e^{-1})+$$
$$+(e^2-e^{-2})(e^2-e)=(e^3-e^{-3})(e-e^{-1})=4\sinh 3\sinh 1.$$

Note 4. From now on, when writing the iterated integral

$$I_D=\int\limits_a^b\Big[\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy\Big]\,dx,$$

we will drop the brackets containing the inner integral and will write

$$I_D=\int\limits_a^b\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,\ y)\,dy\,dx.$$

Here, just as in the case when we have brackets, we will consider that the first integration is performed with respect to the variable whose differential is written first, and then with respect to the variable whose differential is written second. [We note, however, that this is not the generally accepted practice; in some books the reverse is done: integration is performed first with respect to the variable whose differential is last.*)]

SEC. 4. CALCULATING AREAS AND VOLUMES BY MEANS OF DOUBLE INTEGRALS

1. **Volume.** As we saw in Sec. 1, the volume V of a solid bounded by the surface $z=f(x,\ y)$, where $f(x, y)$ is a nonnegative function, by the plane $z=0$ and by a cylindrical surface whose directrix is the boundary of the region D and the generators are parallel to the z-axis, is equal to the double integral of the function $f(x,\ y)$ over the region D:

$$V=\iint\limits_D f(x,\ y)\,ds.$$

*) The following notation is also sometimes used:

$$I_D=\int\limits_a^b\Big[\int\limits_{\varphi_1}^{\varphi_2} f(x,\ y)\,dy\Big]\,dx=\int\limits_a^b dx\int\limits_{\varphi_1}^{\varphi_2} f(x,\ y)\,dy.$$

Example 1. Calculate the volume of a solid bounded by the surfaces $x=0$, $y=0$, $x+y+z=1$, $z=0$ (Fig. 293).

Solution.

$$V=\iint_D (1-x-y)\,dy\,dx,$$

where D is (in Fig. 293) the shaded triangular region in the xy-plane bounded by the straight lines $x=0$, $y=0$, and $x+y=1$. Putting the limits in the double integral, we calculate the volume:

$$V=\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx=\int_0^1\left[(1-x)\,y-\frac{y^2}{2}\right]_0^{1-x}dx=\int_0^1\frac{1}{2}(1-x)^2\,dx=\frac{1}{6}.$$

Thus. $V=\frac{1}{6}$ cubic units.

Note 1. If a solid, the volume of which is being sought, is bounded above by the surface $z=\Phi_2(x, y)\geqslant 0$, and below by the surface $z=\Phi_1(x, y)\geqslant 0$, and the region D is the projection of both surfaces on the xy-plane, then the volume V of this solid

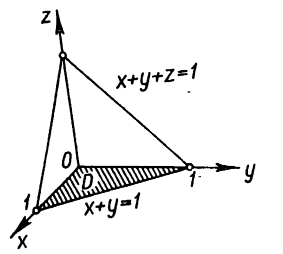


Fig. 293.

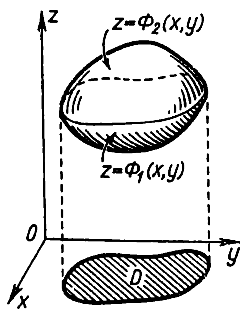


Fig. 294.

is equal to the difference between the volumes of the two "cylindrical" bodies; the first of these cylindrical bodies has the region D as its lower base, and the surface $z=\Phi_2(x, y)$ for its upper base; the second body also has D as its lower base, and the surface $z=\Phi_1(x, y)$ for its upper base (Fig. 294).

Therefore, the volume V is equal to the difference between the two double integrals

$$V=\iint_D \Phi_2(x, y)\,ds-\iint_D \Phi_1(x, y)\,ds,$$

or

$$V=\iint_D [\Phi_2(x, y)-\Phi_1(x, y)]\,ds. \qquad (1)$$

Further, it is easy to prove that formula (1) holds true not only for the case when $\Phi_1(x, y)$ and $\Phi_2(x, y)$ are nonnegative, but also when $\Phi_1(x, y)$ and $\Phi_2(x, y)$ are any continuous functions that satisfy the relationship

$$\Phi_2(x, y) \geqslant \Phi_1(x, y).$$

Note 2. If in the region D the function $f(x, y)$ changes sign, then we divide the region into two parts: 1) the subregion D_1 where $f(x, y) \geqslant 0$; 2) the subregion D_2 where $f(x, y) \leqslant 0$. Suppose the subregions D_1 and D_2 are such that the double integrals over them exist. Then the integral over D_1 will be positive and equal to the volume of the solid lying above the xy-plane. The integral over D_2 will be negative and equal, in absolute value, to the volume of the solid lying below the xy-plane. Thus, the integral over D will be expressed as the difference between the corresponding volumes.

2. **Calculating the area of a plane region.** If we form the integral sum of the function $f(x, y) \equiv 1$ over the region D, then this sum will be equal to the area S,

$$S = \sum_{i=1}^{n} 1 \cdot \Delta s_i,$$

for any method of partition. Passing to the limit on the right side of the equation, we get

$$S = \iint_D dx\,dy.$$

If D is regular (see, for instance, Fig. 280), then the area will be expressed by the double integral

$$S = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} dy \right] dx.$$

Performing the integration in the brackets, we obviously have

$$S = \int_a^b [\varphi_2(x) - \varphi_1(x)]\, dx$$

(cf. Sec. 1, Ch. XII).

Example 2. Calculate the area of a region bounded by the curves

$$y = 2 - x^2, \quad y = x.$$

Solution. Determine the points of intersection of the given curves (Fig. 295). At the point of intersection the ordinates are equal; that is,

$$x = 2 - x^2,$$

whence

$$x^2+x-2=0,$$
$$x_1=-2$$
$$x_2=1.$$

We get two points of intersection: $M_1(-2, -2)$, $M_2(1, 1)$. Hence, the required area is

$$S=\int_{-2}^{1}\left(\int_{x}^{2-x^2} dy\right)dx=\int_{-2}^{1}(2-x^2-x)\,dx=\left[2x-\frac{x^3}{3}-\frac{x^2}{2}\right]_{-2}^{1}=\frac{27}{6}.$$

SEC. 5. THE DOUBLE INTEGRAL IN POLAR COORDINATES

Suppose that in a polar coordinate system θ, ϱ, a region D is given such that each ray*) passing through an interior point of the region cuts the boundary of D at no more than two points.

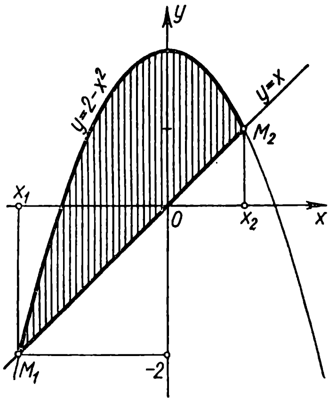


Fig. 295.

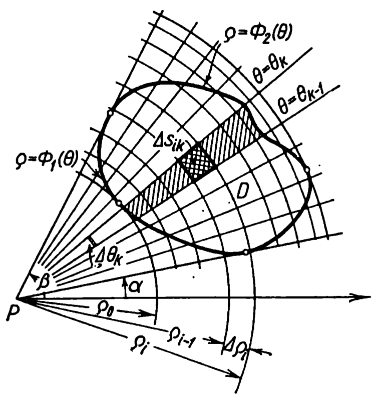


Fig. 296.

Suppose that the region D is bounded by the curves $\varrho=\Phi_1(\theta)$, $\varrho=\Phi_2(\theta)$ and the rays $\theta=\alpha$ and $\theta=\beta$, where $\Phi_1(\theta)\leqslant\Phi_2(\theta)$ and $\alpha<\beta$ (Fig. 296). Again we shall call such a region **regular.**

In the region D let there be given a continuous function of the coordinates θ and ϱ:

$$z=F(\theta, \varrho).$$

We divide D in some way into subregions $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$.

*) A ray is any half-line issuing from the coordinate origin, that is, from the pole P.

Form the integral sum

$$V_n = \sum_{k=1}^{n} F(P_k)\,\Delta s_k, \tag{1}$$

where P_k is some point in the subregion Δs_k.

From the existence theorem of a double integral it follows that as the greatest diameter of the subregion Δs_k approaches zero, there exists a limit V of the integral sum (1). By definition, this limit V is the double integral of the function $F(\theta, \varrho)$ over the region D:

$$V = \iint_D F(\theta, \varrho)\,ds. \tag{2}$$

Let us now evaluate this double integral.

Since the limit of the sum is independent of the manner of partitioning D into subregions Δs_k, we can divide the region in a way that is most convenient. This most convenient (for purposes of calculation) manner will be to partition the region by means of the rays $\theta = \theta_0$, $\theta = \theta_1$, $\theta = \theta_2$, ..., $\theta = \theta_n$ (where $\theta_0 = \alpha$, $\theta_n = \beta$, $\theta_0 < \theta_1, < \theta_2 < \ldots < \theta_n$) and the concentric circles $\varrho = \varrho_0$, $\varrho = \varrho_1$, ..., $\varrho = \varrho_m$ [where ϱ_0 is equal to the least value of the function $\Phi_1(\theta)$, and ϱ_m, to the greatest value of the function $\Phi_2(\theta)$ in the interval $\alpha \leqslant \theta \leqslant \beta$, $\varrho_0 < \varrho_1 < \ldots < \varrho_m$].

Denote by Δs_{ik} the subregion bounded by the lines $\varrho = \varrho_{i-1}$, $\varrho = \varrho_i$, $\theta = \theta_{k-1}$, $\theta = \theta_k$.

The subregions Δs_{ik} will be of three kinds:

1) those that are not cut by the boundary and lie in D;
2) those that are not cut by the boundary and lie outside D;
3) those that are cut by the boundary of D.

The sum of the terms corresponding to the cut subregions have zero as their limit when $\Delta\theta_k \to 0$ and $\Delta\varrho_i \to 0$ and for this reason these terms will be disregarded. The subregions Δs_{ik} that lie outside D do not interest us since they do not enter into the sum. Thus, the integral sum may be written as follows:

$$V_n = \sum_{k=1}^{n} \left[\sum_i F(P_{ik})\,\Delta s_{ik}\right],$$

where P_{ik} is an arbitrary point of the subregion Δs_{ik}.

The double summation sign here should be understood as meaning that we first perform the summation with respect to the index i, holding k fast (that is, we pick out all terms that correspond to the subregions lying between two adjacent rays *).

*) We note that in summing over the index i this index will not run through all values from 1 to m, because not all of the subregions lying between the rays $\theta = \theta_k$ and $\theta = \theta_{k+1}$, belong to D.

The outer summation sign signifies that we take together all the sums obtained in the first summation (that is, we sum with respect to the index k).

Let us find the expression of the area of the subregion Δs_{ik} that is not cut by the boundary of the region. It will be equal to the difference of the areas of the two sectors:

$$\Delta s_{ik}=\frac{1}{2}(\varrho_i+\Delta\varrho_i)^2\Delta\theta_k-\frac{1}{2}\varrho_i^2\Delta\theta_k=\left(\varrho_i+\frac{\Delta\varrho_i}{2}\right)\Delta\varrho_i\Delta\theta_k$$

or

$$\Delta s_{ik}=\varrho_i^*\Delta\varrho_i\Delta\theta_k, \text{ where } \varrho_i<\varrho_i^*<\varrho_i+\Delta\varrho_i.$$

Thus, the integral sum will have the form*)

$$V_n=\sum_{k=1}^{n}\left[\sum_i F(\theta_k^*, \varrho_i^*)\,\varrho_i^*\Delta\varrho_i\Delta\theta_k\right],$$

where $P(\theta_k^*, \varrho_i^*)$ is a point of the subregion Δs_{ik}.

Now take the factor $\Delta\theta_k$ outside the sign of the inner sum (this is permissible since it is a common factor for all the terms of this sum):

$$V_n=\sum_{k=1}^{n}\left[\sum_i F(\theta_k^*\varrho_i^*)\,\varrho_i^*\Delta\varrho_i\right]\Delta\theta_k.$$

Suppose that $\Delta\varrho_i\to 0$ and $\Delta\theta_k$ remains constant. Then the expression in the brackets will tend to the integral

$$\int_{\Phi_1(\theta_k^*)}^{\Phi_2(\theta_k^*)} F(\theta_k^*, \varrho)\,\varrho\,d\varrho.$$

Now, assuming that $\Delta\theta_k\to 0$, we finally get**)

$$V=\int_{\alpha}^{\beta}\left(\int_{\Phi_1(\theta)}^{\Phi_2(\theta)} F(\theta, \varrho)\,\varrho\,d\varrho\right)d\theta. \qquad (3)$$

*) We can consider the integral sum in this form because the limit of the sum does not depend on the position of the point inside the subregion.

**) Our derivation of formula (3) is not rigorous; in deriving this formula we first let $\Delta\varrho_i$ approach zero, leaving $\Delta\theta_k$ constant, and only then made $\Delta\theta_k$ approach zero. This does not exactly correspond to the definition of a double integral, which we regard as the limit of an integral sum as the diameters of the subregions approach zero (i. e., in the simultaneous approach to zero of $\Delta\theta_k$ and $\Delta\varrho_i$). However, though the proof lacks rigour, the result is true [i. e., formula (3) is true]. This formula could be rigorously derived by the same method used when considering the double integral in rectangular coordinates. We also note that this formula will be derived once again in Sec. 6 with different reasoning (as a particular case of the more general formula for transforming coordinates in the double integral).

Formula (3) is used to compute double integrals in polar coordinates.

If the first integration is performed over θ and the second one over ϱ, then we get the formula (Fig. 297)

$$V=\int_{\varrho_1}^{\varrho_2}\left(\int_{\omega_1(\varrho)}^{\omega_2(\varrho)} F(\theta,\ \varrho)\,d\theta\right)\varrho\,d\varrho. \qquad (3')$$

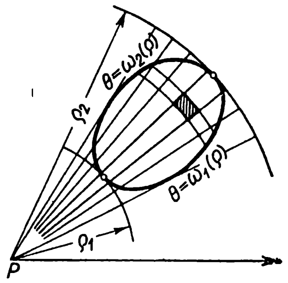


Fig. 297.

Let it be required to compute the double integral of a function $f(x,\ y)$ over a region D given in rectangular coordinates:

$$\iint_D f(x,\ y)\,dx\,dy.$$

If D is regular in the polar coordinates θ, ϱ then the computation of the given integral can be reduced to computing the iterated integral in polar coordinates.

Indeed, since

$$x=\varrho\cos\theta,\quad y=\varrho\sin\theta,$$

$$f(x,\ y)=f[\varrho\cos\theta,\ \varrho\sin\theta]=F(\theta,\ \varrho),$$

it follows that

$$\iint_D f(x,\ y)\,dx\,dy=\int_{\alpha}^{\beta}\left(\int_{\Phi_1(\theta)}^{\Phi_2(\theta)} f[\varrho\cos\theta,\ \varrho\sin\theta]\,\varrho\,d\varrho\right)d\theta. \qquad (4)$$

Example 1. Compute the volume V of a solid bounded by the spherical surface

$$x^2+y^2+z^2=4a^2$$

an.. the cylinder

$$x^2+y^2-2ay=0.$$

Solution. For the region of integration here we can take the base of the cylinder $x^2+y^2-2ay=0$, that is, a circle with centre at (0, a) and radius a. The equation of this circle may be written in the form $x^2+(y-a)^2=a^2$ (Fig. 298).

We calculate $\frac{1}{4}$ of the required volume V, namely that part which is situated in the first octant. Then for the region of integration we will have to take the semicircle whose boundaries are defined by the equations

$$x=\varphi_1(y)=0,\quad x=\varphi_2(y)=\sqrt{2ay-y^2},$$

$$y=0,\ y=2a.$$

The integrand is

$$z=f(x,\ y)=\sqrt{4a^2-x^2-y^2}.$$

Consequently,

$$\frac{1}{4}V=\int_0^{2a}\left(\int_0^{\sqrt{2ay-y^2}}\sqrt{4a^2-x^2-y^2}\,dx\right)dy.$$

Transform the integral obtained to the polar coordinates θ, ϱ:

$$x=\varrho\cos\theta,\quad y=\varrho\sin\theta.$$

Determine the limits of integration. To do so, write the equation of the given circle in polar coordinates; since

$$x^2+y^2=\varrho^2,$$
$$y=\varrho\sin\theta,$$

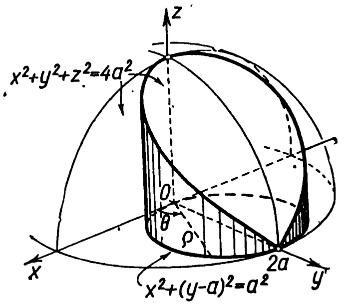


Fig. 298.

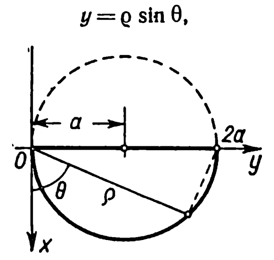


Fig. 299.

it follows that

$$\varrho^2-2a\varrho\sin\theta=0$$

or

$$\varrho=2a\sin\theta.$$

Hence, in polar coordinates (Fig. 299), the boundaries of the region are defined by the equations

$$\varrho=\Phi_1(\theta)=0,\quad \varrho=\Phi_2(\theta)=2a\sin\theta,\quad \alpha=0,\quad \beta=\frac{\pi}{2},$$

and the integrand has the form

$$F(\theta,\ \varrho)=\sqrt{4a^2-\varrho^2}.$$

Thus, we have

$$\frac{V}{4}=\int_0^{\frac{\pi}{2}}\left(\int_0^{2a\sin\theta}\sqrt{4a^2-\varrho^2}\,\varrho\,d\varrho\right)d\theta=\int_0^{\frac{\pi}{2}}\left[-\frac{(4a^2-\varrho^2)^{3/2}}{3}\right]_0^{2a\sin\theta}d\theta=$$

$$=-\frac{1}{3}\int_0^{\frac{\pi}{2}}[(4a^2-4a^2\sin^2\theta)^{3/2}-(4a^2)^{3/2}]\,d\theta=$$

$$=\frac{8a^3}{3}\int_0^{\frac{\pi}{2}}(1-\cos^3\theta)\,d\theta=\frac{4}{9}a^3(3\pi-4).$$

Example 2. Evaluate the Poisson integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx.$$

Solution. First evaluate the integral $I_R=\iint_D e^{-x^2-y^2}\,dx\,dy$, where the region of integration D is the circle

$$x^2+y^2=R^2$$

(Fig. 300).

Passing to the polar coordinates θ, ϱ, we obtain

$$I_R=\int_0^{2\pi}\left(\int_0^R e^{-\rho^2}\varrho\,d\varrho\right)d\theta=-\frac{1}{2}\int_0^{2\pi} e^{-\rho^2}\Big|_0^R d\theta=\pi\left(1-e^{-R^2}\right).$$

Now, if we increase the radius R without bound (that is, if we expand without limit the region of integration), we get the so-called *improper iterated integral:*

$$\int_0^{2\pi}\left(\int_a^{\infty} e^{-\rho^2}\varrho\,d\varrho\right)d\theta=\lim_{R\to\infty}\int_0^{2\pi}\left(\int_0^R e^{-\rho^2}\varrho\,d\varrho\right)d\theta=\lim_{R\to\infty}\pi\left(1-e^{-R^2}\right)=\pi.$$

We shall show that the integral $\iint_D e^{-x^2-y^2}\,dx\,dy$ approaches the limit π if the region D' of arbitrary form expands in such manner that finally any point of the plane gets into D' and remains there (we shall conditionally indicate such an expansion of D' by the relationship $D'\longrightarrow\infty$).

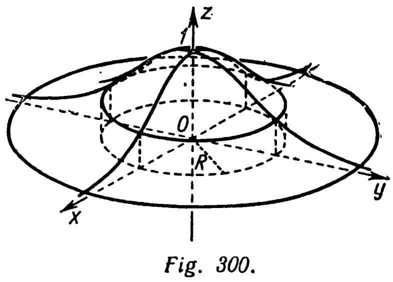


Fig. 300.

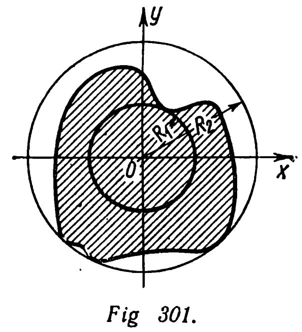


Fig 301.

Let R_1 and R_2 be the least and greatest distances of the boundary of D' from the origin (Fig. 301).

Since the function $e^{-x^2-y^2}$ is everywhere greater than zero, the following inequalities hold:

$$I_{R_1}\leqslant\iint_{D'} e^{-x^2-y^2}\,dx\,dy\leqslant I_{R_2}$$

or

$$\pi\left(1-e^{-R_1^2}\right)\leqslant\iint_{D'} e^{-x^2-y^2}\,dx\,dy\leqslant\pi\left(1-e^{-R_2^2}\right).$$

Since for $D'\longrightarrow\infty$ it is obvious that $R_1\longrightarrow\infty$ and $R_2\longrightarrow\infty$, it follows that the extreme parts of the inequality tend to one and the same limit π. Hence, the median term also approaches this limit; that is,

$$\lim_{D'\to\infty}\iint_{D'} e^{-x^2-y^2}\,dx\,dy=\pi. \tag{5}$$

As a particular instance, let D' be a square with side $2a$ and centre at the origin; then

$$\iint_{D'} e^{-x^2-y^2}\,dx\,dy=\int_{-a}^{a}\int_{-a}^{a} e^{-x^2-y^2}\,dx\,dy=$$

$$=\int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\,dy=\int_{-a}^{a}\left[\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\right]dy.$$

Now take the factor e^{-y^2} outside the sign of the inner integral (this is permissible since e^{-y^2} does not depend on the variable of integration x). Then

$$\iint_{D'} e^{-x^2-y^2}\,dx\,dy=\int_{-a}^{a} e^{-y^2}\left[\int_{-a}^{a} e^{-x^2}\,dx\right]dy.$$

Set $\int_{-a}^{a} e^{-x^2}\,dx=B_a$. This is a constant (dependent only on a); therefore,

$$\iint_{D'} e^{-x^2-y^2}\,dx\,dy=\int_{-a}^{a} e^{-y^2}B_a\,dy=B_a\int_{-a}^{a} e^{-y^2}\,dy.$$

But the latter integral is likewise equal to B_a $\left(\text{because } \int_{-a}^{a} e^{-x^2}\,dx=\right.$ $\left.=\int_{-a}^{a} e^{-y^2}\,dy\right)$;

thus,

$$\iint_{D'} e^{-x^2-y^2}\,dx\,dy=B_aB_a=B_a^2.$$

We pass to the limit in this equation, by making a approach infinity (in the process, D' expands without limit):

$$\lim_{D'\to\infty}\iint_{D'} e^{-x^2-y^2}\,dx\,dy=\lim_{a\to\infty}B_a^2=\lim_{a\to\infty}\left[\int_{-a}^{a} e^{-x^2}\,dx\right]^2=\left[\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right]^2.$$

But, as has been proved [see (5)],

$$\lim_{D' \to \infty} \iint_{D'} e^{-x^2-y^2}\,dx\,dy = \pi .$$

Hence,

$$\left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]^2 = \pi,$$

or

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

This integral is frequently encountered in probability theory and in statistics. We remark that we would not be able to compute this integral directly (by means of an indefinite integral) because the derivative of e^{-x^2} is not expressible in terms of elementary functions.

SEC. 6. CHANGING VARIABLES IN A DOUBLE INTEGRAL (GENERAL CASE)

In the xy-plane let there be a region D bounded by the line L. Suppose that the coordinates x and y are functions of new variables u and v:

$$x = \varphi(u, v), \quad y = \psi(u, v); \tag{1}$$

let the functions $\varphi(u, v)$ and $\psi(u, v)$ be single-valued and continuous, and let them have continuous derivatives in some region D', which will be defined later on. Then by formulas (1) to each pair of values u and v there corresponds a unique pair of values

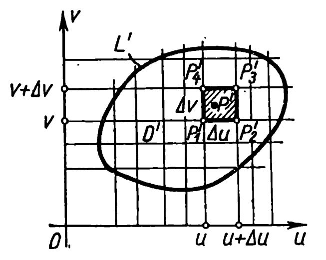


Fig. 302.

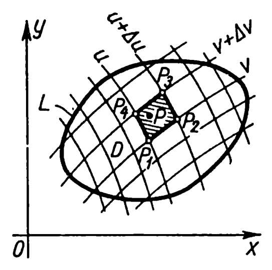


Fig. 303.

x and y. Further, suppose that the functions φ and ψ are such that if we give x and y definite values in D, then by formulas (1) we will find definite values of u and v.

Consider a rectangular coordinate system Ouv (Fig. 302). From the foregoing it follows that with each point $P(x, y)$ in the

xy-plane (Fig. 303) there is uniquely associated a point $P'(u, v)$ in the uv-plane with coordinates u, v, which are determined by formulas (1). The numbers u and v are called *curvilinear* coordinates of the point P.

If in the xy-plane a point describes a closed line L bounding the region D, then in the uv-plane a corresponding point will trace out a closed line L' bounding a certain region D'; and to each point of D' there will correspond a point of D.

Thus, the formulas (1) establish a *one-to-one correspondence between the points of the regions D and D'*, or, the mapping, by formulas (1), of the *region D onto region D' is said to be one-to-one.*

In the region D' let us consider a line $u = \text{const}$. By formulas (1) we find that in the xy-plane there will, generally speaking, be a certain curve that will correspond to it. In exactly the same way, to each straight line $v = \text{const}$ of the uv-plane there will correspond some line in the xy-plane.

Let us divide the region D' (using the straight lines $u = \text{const}$ and $v = \text{const}$) into rectangular subregions (we shall disregard subregions that overlap the boundary of the region D'). Using suitable curved lines, divide D into certain curvilinear quadrangles (Fig. 303).

Consider, in the uv-plane, the rectangular subregion $\Delta s'$ bounded by the straight lines $u = \text{const}$, $u + \Delta u = \text{const}$, $v = \text{const}$, $v + \Delta v = \text{const}$, and consider also the curvilinear subregion Δs corresponding to it in the xy-plane. We denote the areas of these subregions by $\Delta s'$ and Δs, respectively. Then, obviously,

$$\Delta s' = \Delta u \, \Delta v.$$

Generally speaking, the areas Δs and $\Delta s'$ are different.

In the region D, let there be a continuous function

$$z = f(x, y).$$

To each value of the function $z = f(x, y)$ in the region D there corresponds the very same value of the function $z = F(u, v)$ in the region D', where

$$F(u, v) = f[\varphi(u, v), \psi(u, v)].$$

Consider the integral sums of the function z over D. It is obvious that we have the following equation:

$$\sum f(x, y) \, \Delta s = \sum F(u, v) \, \Delta s. \tag{2}$$

Let us compute Δs, which is the area of the curvilinear quadrangle $P_1P_2P_3P_4$ in the xy-plane (see Fig. 303).

We determine the coordinates of its vertices:

$$\left.\begin{array}{lll} P_1(x_1, y_1), & x_1=\varphi(u, v), & y_1=\psi(u, v), \\ P_2(x_2, y_2), & x_2=\varphi(u+\Delta u, v), & y_2=\psi(u+\Delta u, v), \\ P_3(x_3, y_3), & x_3=\varphi(u+\Delta u, v+\Delta v); & y_3=\psi(u+\Delta u, v+\Delta v), \\ P_4(x_4, y_4), & x_4=\varphi(u, v+\Delta v), & y_4=\psi(u, v+\Delta v). \end{array}\right\} \quad (3)$$

When computing the area of the curvilinear quadrangle P_1, P_2, P_3, P_4 we shall consider the lines P_1P_2, P_2P_3, P_3P_4, P_4P_1 as parallel in pairs; we shall also replace the increments of the functions by corresponding differentials. We shall thus ignore infinitesimals of order higher than the infinitesimals Δu, Δv. Then formulas (3) will have the form

$$\left.\begin{array}{ll} x_1=\varphi(u, v), & y_1=\psi(u, v), \\ x_2=\varphi(u, v)+\frac{\partial\varphi}{\partial u}\Delta u, & y_2=\psi(u, v)+\frac{\partial\psi}{\partial u}\Delta u, \\ x_3=\varphi(u, v)+\frac{\partial\varphi}{\partial u}\Delta u+\frac{\partial\varphi}{\partial v}\Delta v, & y_3=\psi(u, v)+\frac{\partial\psi}{\partial u}\Delta u+\frac{\partial\psi}{\partial v}\Delta v, \\ x_4=\varphi(u, v)+\frac{\partial\varphi}{\partial v}\Delta v, & y_4=\psi(u, v)+\frac{\partial\psi}{\partial v}\Delta v. \end{array}\right\} \quad (3')$$

With these assumptions, the curvilinear quadrangle $P_1P_2P_3P_4$ may be ragarded as a parallelogram. Its area Δs is approximately equal to the doubled area of the triangle $P_1P_2P_3$ and is found by the following formula of analytic geometry:

$$\Delta s \approx |(x_3-x_1)(y_3-y_2)-(x_3-x_2)(y_3-y_1)| =$$

$$=\left|\left(\frac{\partial\varphi}{\partial u}\Delta u+\frac{\partial\varphi}{\partial v}\Delta v\right)\frac{\partial\psi}{\partial v}\Delta v-\frac{\partial\varphi}{\partial v}\Delta v\left(\frac{\partial\psi}{\partial u}\Delta u+\frac{\partial\psi}{\partial v}\Delta v\right)\right| =$$

$$=\left|\frac{\partial\varphi}{\partial u}\frac{\partial\psi}{\partial v}\Delta u\,\Delta v-\frac{\partial\varphi}{\partial v}\frac{\partial\psi}{\partial u}\Delta u\,\Delta v\right| = \left|\frac{\partial\varphi}{\partial u}\frac{\partial\psi}{\partial v}-\frac{\partial\varphi}{\partial v}\frac{\partial\psi}{\partial u}\right|\Delta u\,\Delta v =$$

$$= \left\| \begin{matrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} \end{matrix} \right\| \Delta u\,\Delta v. \text{*)}$$

*) The doubled lines in the determinant indicate that the absolute value of the determinant is taken.

We introduce the notation

$$\begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} \end{vmatrix} = I.$$

Thus,

$$\Delta s \approx |I| \Delta s'. \tag{4}$$

The determinant I is called the *functional determinant* of the functions $\varphi(u, v)$ and $\psi(u, v)$. It is also called the *Jacobian* after the German mathematician Jacobi.

The equality (4) is only approximate, because in the process of computing the area of Δs we neglected infinitesimals of higher order. However, the smaller the dimensions of the subregions Δs and $\Delta s'$, the more exact will this equality be. And it becomes absolutely exact in the limit, when the diameters of the subregions Δs and $\Delta s'$ approach zero:

$$|I| = \lim_{\text{diam}\,\Delta s \to 0} \frac{\Delta s}{\Delta s'}.$$

Let us now apply the equation obtained to an evaluation of the double integral. From (2) we can write

$$\sum f(x, y)\,\Delta s \approx \sum F(u, v)\,|I|\,\Delta s'$$

(the integral sum on the right is extended over the region D'). Passing to the limit as diam $\Delta s' \to 0$, we get the exact equation

$$\iint\limits_D f(x, y)\,dx\,dy = \iint\limits_{D'} F(u, v)\,|I|\,du\,dv. \tag{5}$$

This is the *formula for transformations of coordinates in a double integral*. It permits reducing the evaluation of a double integral over a region D to the computation of a double integral over a region D', which may simplify the problem. A rigorous proof of this formula was first given by the noted Russian mathematician M. V. Ostrogradsky.

Note. The transformation from rectangular coordinates to polar coordinates considered in the preceding section is a special case of change of variables in a double integral. Here, $u = \theta$, $v = \varrho$:

$$x = \varrho\cos\theta, \quad y = \varrho\sin\theta.$$

The curve $AB\,(\varrho = \varrho_1)$ in the xy-plane (Fig. 304) is transformed into the straight line $A'B'$ in the $\theta\varrho$-plane (Fig. 305). The curve $DC\,(\varrho = \varrho_2)$ in the xy-plane is transformed into the straight line $D'C'$ in the $\theta\varrho$-plane.

The straight lines AD and BC in the xy-plane are transformed into the straight lines $A'D'$ and $B'C'$ in the $\theta\varrho$-plane. The curves L_1 and L_2 are transformed into the curves L_1' and L_2'.

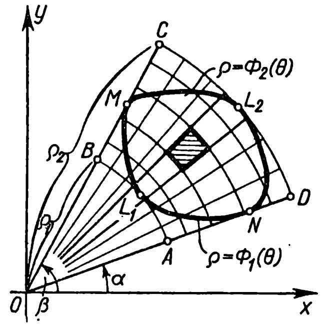


Fig 304.

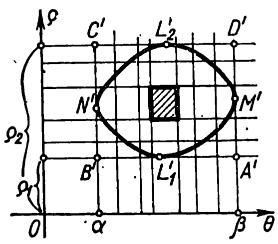


Fig 305.

Let us calculate the Jacobian of transformation of the Cartesian coordinates x and y into the polar coordinates θ and ϱ:

$$I=\begin{vmatrix}\frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varrho}\\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varrho}\end{vmatrix}=\begin{vmatrix}-\varrho\sin\theta & \cos\theta\\ \varrho\cos\theta & \sin\theta\end{vmatrix}=-\varrho\sin^2\theta-\varrho\cos^2\theta=-\varrho.$$

Hence, $|I|=\varrho$ and therefore

$$\iint\limits_D f(x, y)\,dx\,dy=\int\limits_\alpha^\beta\Big(\int\limits_{\Phi_1(\theta)}^{\Phi_2(\theta)} F(\theta, \varrho)\,\varrho\,d\varrho\Big)\,d\theta.$$

This was the formula that we derived in the preceding section.

Example. Let it be required to compute the double integral

$$\iint\limits_D (y-x)\,dx\,dy$$

over the region D in the xy-plane bounded by the straight lines

$$y=x+1,\quad y=x-3,\quad y=-\frac{1}{3}x+\frac{7}{9},\quad y=-\frac{1}{3}x+5.$$

It would be difficult to compute this double integral directly; however, a simple change of variables permits reducing this integral to one over a rectangle whose sides are parallel to the coordinate axes.

Set

$$u=y-x,\quad v=y+\frac{1}{3}x. \tag{6}$$

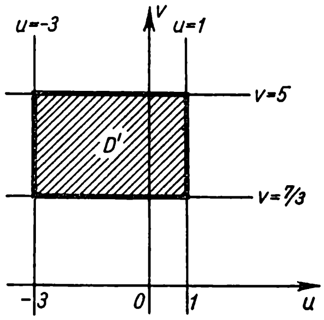


Fig. 306.

Then the straight lines $y=x+1$, $y=x-3$ will be transformed, respectively, into the straight lines $u=1$, $u=-3$ in the uv-plane; and the straight lines $y=-\frac{1}{3}x+\frac{7}{3}$, $y=-\frac{1}{3}x+5$ will be transformed into the straight lines $v=\frac{7}{3}$, $v=5$.

Consequently, the given region D is transformed into the rectangular region D' shown in Fig. 306. It remains to compute the Jacobian of transformation. To do this, express x and y in terms of u and v. Solving the system of equations (6), we obtain

$$x=-\frac{3}{4}u+\frac{3}{4}v;\quad y=\frac{1}{4}u+\frac{3}{4}v.$$

Consequently,

$$I=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}-\frac{3}{4} & \frac{3}{4}\\ \frac{1}{4} & \frac{3}{4}\end{vmatrix}=-\frac{9}{16}-\frac{3}{16}=-\frac{3}{4},$$

and the absolute value of the Jacobian is $|I|=\frac{3}{4}$. Therefore,

$$\iint\limits_D (y-x)\,dx\,dy=\iint\limits_{D'}\left[\left(+\frac{1}{4}u+\frac{3}{4}v\right)-\left(-\frac{3}{4}u+\frac{3}{4}v\right)\right]\frac{3}{4}\,du\,dv=$$

$$=\iint\limits_{D'}\frac{3}{4}u\,du\,dv=\int\limits_{\frac{7}{3}}^{5}\int\limits_{-3}^{1}\frac{3}{4}u\,du\,dv=-18.$$

SEC. 7. COMPUTING THE AREA OF A SURFACE

Let it be required to compute the area of a surface bounded by the line Γ (Fig. 307); the surface is defined by the equation $z=f(x, y)$, where the function $f(x, y)$ is continuous and has continuous partial derivatives.

Denote the projection of the line Γ on the xy-plane by L. Denote by D the region on the xy-plane bounded by the line L. In arbitrary fashion, divide D into n elementary subregions $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. In each subregion Δs_i take a point $P_i(\xi_i \eta_i)$. To the point P_i there will correspond, on the surface, a point

$$M_i[\xi_i, \eta_i, f(\xi_i, \eta_i)].$$

Through M_i draw a tangent plane to the surface. Its equation is of the form

$$z - z_i = f'_x(\xi_i, \eta_i)(x - \xi_i) + f'_y(\xi_i, \eta_i)(y - \eta_i) \quad (1)$$

(see Sec. 6, Ch. IX). In this plane, pick out a subregion $\Delta\sigma_i$ which is projected onto the xy-plane in the form of a subregion Δs_i. Consider the sum of all the subregions $\Delta\sigma_i$:

$$\sum_{i=1}^{n} \Delta\sigma_i.$$

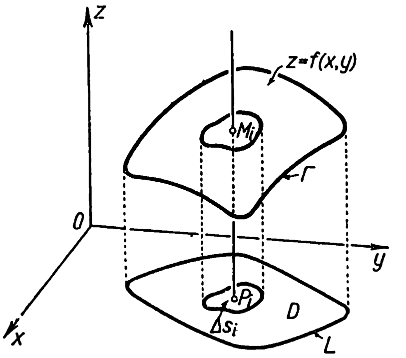


Fig. 307.

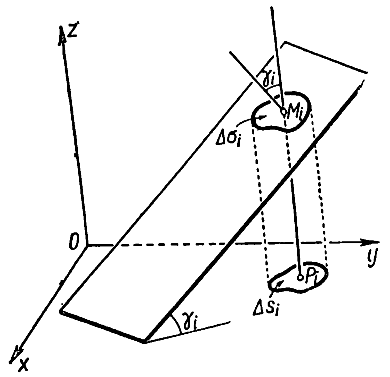


Fig. 308.

We shall call the limit σ of this sum, when the greatest of the diameters of the subregions $\Delta\sigma_i$ approaches zero, the *area of the surface*; that is, by definition we set

$$\sigma = \lim_{\operatorname{diam}\Delta\sigma_i \to 0} \sum_{i=1}^{n} \Delta\sigma_i. \quad (2)$$

Now let us calculate the area of the surface. Denote by γ_i the angle between the tangent plane and the xy-plane. Using a familiar formula of analytic geometry we can write (Fig. 308)

$$\Delta s_i = \Delta\sigma_i \cos\gamma_i$$

or

$$\Delta\sigma_i = \frac{\Delta s_i}{\cos\gamma_i}. \quad (3)$$

The angle γ_i is at the same time the angle between the z-axis and the perpendicular to the plane (1). Therefore, by equation

(1) and the formula of analytic geometry we have

$$\cos\gamma_i=\frac{1}{\sqrt{1+f_x'^2(\xi_i,\ \eta_i)+f_y'^2(\xi_i,\ \eta_i)}}.$$

Hence,

$$\Delta\sigma_i=\sqrt{1+f_x'^2(\xi_i,\ \eta_i)+f_y'^2(\xi_i,\ \eta_i)}\,\Delta s_i.$$

Putting this expression into formula (2), we get

$$\sigma=\lim_{\operatorname{diam}\Delta s_i\to 0}\sum_{i=1}^{n}\sqrt{1+f_x'^2(\xi_i,\ \eta_i)+f_y'^2(\xi_i,\ \eta_i)}\,\Delta s_i.$$

Since the limit of the integral sum on the right side of the last equation is, by definition, the double integral

$$\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy,$$ we finally get

$$\sigma=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy. \tag{4}$$

This is the formula used to compute the area of the surface $z=f(x,\ y)$.

If the equation of the surface is given in the form

$$x=\mu(y,\ z)\text{ or in the form }y=\chi(x,\ z),$$

then the corresponding formulas for calculating the surface are of the form

$$\sigma=\iint_{D'}\sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dy\,dz, \tag{3'}$$

$$\sigma=\iint_{D''}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dx\,dz, \tag{3''}$$

where D' and D'' are the regions in the xy-plane and the xz-plane in which the given surface is projected.

Example 1. Compute the surface σ of the sphere

$$x^2+y^2+z^2=R^2.$$

Solution. Compute the surface of the upper half of the sphere:

$$z=\sqrt{R^2-x^2-y^2}$$

(Fig. 309). In this case

$$\frac{\partial z}{\partial x}=-\frac{x}{\sqrt{R^2-x^2-y^2}};$$

$$\frac{\partial z}{\partial y}=-\frac{y}{\sqrt{R^2-x^2-y^2}}.$$

Hence,

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{\frac{R^2}{R^2-x^2-y^2}}=\frac{R}{\sqrt{R^2-x^2-y^2}}.$$

The region of integration is defined by the condition

$$x^2+y^2\leqslant R^2.$$

Thus, by formula (4) we will have

$$\frac{1}{2}\sigma=\int_{-R}^{R}\left(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\frac{R}{\sqrt{R^2-x^2-y^2}}\,dy\right)dx.$$

To compute the double integral obtained let us make the transformation to polar coordinates. In polar coordinates the boundary of the region of integration is determined by the equation $\varrho=R$. Hence,

$$\sigma=2\int_0^{2\pi}\left(\int_0^R\frac{R}{\sqrt{R^2-\varrho^2}}\varrho\,d\varrho\right)d\theta=2R\int_0^{2\pi}[-\sqrt{R^2-\varrho^2}]_0^R\,d\theta=$$

$$=2R\int_0^{2\pi}R\,d\theta=4\pi R^2.$$

Example 2. Find the area of that part of the surface of the cylinder

$$x^2+y^2=a^2$$

which is cut out by the cylinder

$$x^2+z^2=a^2.$$

Solution. Fig. 310 shows 1/8th of the desired surface. The equation of the surface has the form $y=\sqrt{a^2-x^2}$;

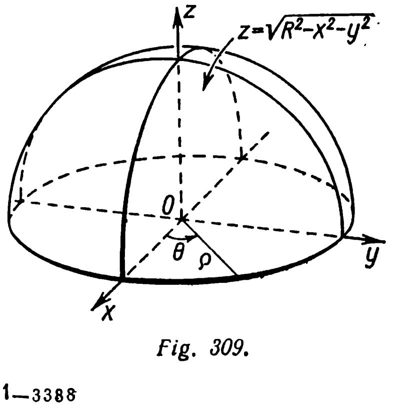


Fig. 309.

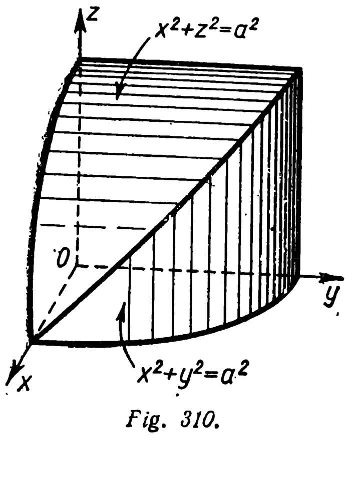


Fig. 310.

therefore,

$$\frac{\partial y}{\partial x}=-\frac{x}{\sqrt{a^2-x^2}};\ \frac{\partial y}{\partial z}=0;$$

$$\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}=\sqrt{1+\frac{x^2}{a^2-x^2}}=\frac{a}{\sqrt{a^2-x^2}}.$$

The region of integration is a quarter circle, that is, it is determined by the conditions

$$x^2+z^2\leqslant a^2,\ x\geqslant 0;\ z\geqslant 0.$$

Consequently,

$$\frac{1}{8}\sigma=\int_0^a\left(\int_0^{\sqrt{a^2-x^2}}\frac{a}{\sqrt{a^2-x^2}}dz\right)dx=a\int_0^a\frac{z}{\sqrt{a^2-x^2}}\Bigg|_0^{\sqrt{a^2-x^2}}dx=a\int_0^a dx=a^2,$$

$$\sigma=8a^2.$$

SEC. 8. THE DENSITY OF DISTRIBUTION OF MATTER AND THE DOUBLE INTEGRAL

In a region D, let a certain substance be distributed in such manner that there is a definite amount of this substance per unit area of D. We shall henceforward speak of the distribution of mass, although our reasoning will hold also for the case when speaking of the distribution of electric charge, of quantity of heat, and so forth.

We consider an arbitrary subregion Δs of the region D. Let the mass of substance associated with this given subregion be Δm. Then the ratio $\frac{\Delta m}{\Delta s}$ is called the mean surface density of the substance in the subregion Δs.

Now let the subregion Δs decrease and contract to the point $P(x, y)$. Consider the limit $\lim\limits_{\Delta s\to 0}\frac{\Delta m}{\Delta s}$. If this limit exists, then, generally speaking, it will depend on the position of the point P, that is, upon its coordinates x and y, and will be some function $f(P)$ of the point P. We shall call this limit the *surface density* of the substance at the point P:

$$\lim_{\Delta s\to 0}\frac{\Delta m}{\Delta s}=f(P)=f(x, y). \qquad (1)$$

Thus, the surface density is a function $f(x, y)$ of the coordinates of the point of the region.

Conversely, let there be given, in a region D, the surface density of some substance as some continuous function $f(P)=f(x, y)$

and let it be required to determine the total quantity of substance M contained in the region D. Divide D into subregions Δs_i ($i = 1, 2, \ldots, n$) and in each subregion take a point P_i; then $f(P_i)$ is the surface density in the point P_i.

To within higher-order infinitesimals, the product $f(P_i)\Delta s_i$ gives us the quantity of substance contained in the subregion Δs_i, and the sum

$$\sum_{i=1}^{n} f(P_i)\,\Delta s_i$$

expresses approximately the total quantity of substance distributed in the region D. But this is the integral sum of the function $f(P)$ in the region D. The exact value is obtained in the limit as $\Delta s_i \to 0$.

Thus, *)

$$M = \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(P_i)\Delta s_i = \iint_D f(P)\,ds = \iint_D f(x, y)\,dx\,dy, \qquad (2)$$

or the total quantity of substance in the region D is equal to the double integral (over D) of the density $f(P) = f(x, y)$ of this substance.

Example. Determine the mass of a circular plate of radius R if the surface density $f(x, y)$ of the material of the plate at each point $P(x, y)$ is proportional to the distance of the point (x, y) from the centre of the circle, that is, if

$$f(x, y) = k\sqrt{x^2 + y^2}.$$

Solution. By formula (2) we have

$$M = \iint_D k\sqrt{x^2 + y^2}\,dx\,dy,$$

where the region of integration D is the circle $x^2 + y^2 \leqslant R^2$.

Passing to polar coordinates, we obtain

$$M = k\int_0^{2\pi}\left(\int_0^R \varrho\varrho\,d\varrho\right)d\theta = k2\pi\frac{R^3}{3}\Bigg|_0^R = \frac{2}{3}k\pi R^3.$$

SEC. 9. THE MOMENT OF INERTIA OF THE AREA OF A PLANE FIGURE

The moment of inertia I of a material point M of mass m relative to some point O is the product of the mass m by the

*) The relationship $\Delta s_i \to 0$ is to be understood in the sense that the diameter of the subregion Δs_i approaches zero.

square of its distance r from the point O:

$$I = mr^2.$$

The moment of inertia of a system of material points m_1, m_2, $\ldots$, m_n relative to O is the sum of moments of inertia of the individual points of the system:

$$I = \sum_{i=1}^{n} m_i r_i^2.$$

Let us determine the moment of inertia of a material plane figure D.

Let D be located in an xy-coordinate plane. Let us determine the moment of inertia of this figure relative to the origin, assuming that the surface density is everywhere equal to unity.

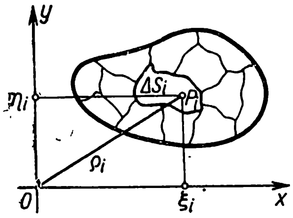


Fig. 311.

Divide the region D into elementary subregions $\Delta s_i \longrightarrow (i = 1, 2, \ldots, n)$ (Fig. 311). In each subregion take a point P_i with coordinates ξ_i, η_i. Let us call the product of the mass of the subregion Δs_i by the square of the distance $r_i^2 = \xi_i^2 + \eta_i^2$ an elementary moment of inertia ΔI_i of the subregion Δs_i:

$$\Delta I_i = (\xi_i^2 + \eta_i^2)\,\Delta s_i,$$

and let us form the sum of such moments:

$$\sum_{i=1}^{n} (\xi_i^2 + \eta_i^2)\,\Delta s_i.$$

This is the integral sum of the function $f(x, y) = x^2 + y^2$ over the region D.

We define the moment of inertia of the figure D as the limit of this integral sum when the diameter of each elementary subregion Δs_i approaches zero:

$$I_0 = \lim_{\operatorname{diam} \Delta s_i \to 0} \sum_{i=1}^{n} (\xi_i^2 + \eta_i^2)\,\Delta s_i.$$

But the double integral $\iint\limits_D (x^2 + y^2)\,dx\,dy$ is the limit of this sum. Thus, the moment of inertia of the figure D relative to the origin is

$$I_0 = \iint\limits_D (x^2 + y^2)\,dx\,dy, \tag{1}$$

where D is a region which coincides with the given plane figure

The integrals

$$I_{xx}=\iint_D y^2 dx\, dy, \qquad (2)$$

$$I_{yy}=\iint_D x^2\, dx\, dy \qquad (3)$$

are called, respectively, the moments of inertia of the figure D relative to the x-axis and y-axis.

Example 1. Compute the moment of inertia of the area of a circle D of radius R relative to the centre O.

Solution. By formula (1) we have

$$I_0=\iint_D (x^2+y^2)\, dx\, dy.$$

To evaluate this integral we transform to the polar coordinates θ, ϱ. The equation of the circle in polar coordinates is $\varrho=R$.
Therefore

$$I_0=\int_0^{2\pi}\left(\int_0^R \varrho^2\varrho\, d\varrho\right) d\theta=\frac{\pi R^4}{2}.$$

Note. If the surface density γ is not equal to unity, but is some function of x and y, i. e., $\gamma=\gamma\,(x, y)$, then the mass of the subregion ΔS_i, will, to within infinitesimals of higher order, be equal to $\gamma(\xi_i, \eta_i)\,\Delta s_i$ and, for this reason, the moment of inertia of the plane figure relative to the origin will be

$$I_0=\iint_D \gamma\,(x, y)\,(x^2+y^2)\, dx\, dy. \qquad (1')$$

Example 2. Compute the moment of inertia of a plane material figure D bounded by the lines $y^2=1-x$; $x=0$, $y=0$ relative to the y-axis if the surface density at each point is equal to y (Fig. 312).

Solution.

$$I_{yy}=\int_0^1\left(\int_0^{\sqrt{1-x}} yx^2\, dy\right) dx=\int_0^1 \frac{x^2y^2}{2}\Bigg|_0^{\sqrt{1-x}} dx=\frac{1}{2}\int_0^1 x^2\,(1-x)\, dx=\frac{1}{24}.$$

Ellipse of inertia. Let us determine the moment of inertia of the area of a plane figure D relative to some axis OL that passes through the point O, which we shall take as the coordinate origin. Denote by φ the angle formed by the straight line OL with the positive x-axis (Fig. 313).

The normal equation of OL is

$$x \sin\varphi - y\cos\varphi=0.$$

The distance r of some point $M(x, y)$ from this line is

$$r = |x \sin \varphi - y \cos \varphi|.$$

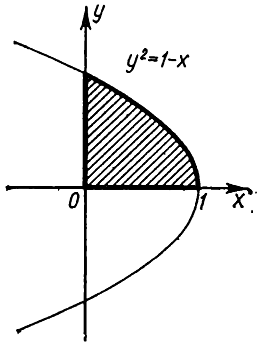


Fig. 312.

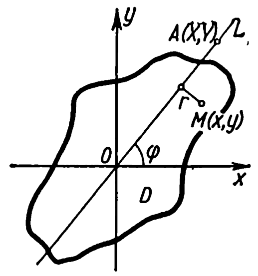


Fig. 313.

The moment of inertia I of the area of D relative to OL is expressed, by definition, by the integral

$$I = \iint_D r^2 \, dx \, dy = \iint_D (x \sin \varphi - y \cos \varphi)^2 \, dx \, dy =$$

$$= \sin^2 \varphi \iint_D x^2 \, dx \, dy - 2 \sin \varphi \cos \varphi \iint_D xy \, dx \, dy + \cos^2 \varphi \iint_D y^2 \, dx \, dy.$$

Therefore

$$I = I_{yy} \sin^2 \varphi - 2I_{xy} \sin \varphi \cos \varphi + I_{xx} \cos^2 \varphi; \tag{4}$$

here, $I_{yy} = \iint_D x^2 \, dx \, dy$ is the moment of inertia of the figure relative to the y-axis, $I_{xx} = \iint_D y^2 \, dx \, dy$ is the moment of inertia relative to the x-axis, and $I_{xy} = \iint_D xy \, dx \, dy$. Dividing all terms of the latter equation by I, we get

$$1 = I_{xx} \left(\frac{\cos \varphi}{\sqrt{I}} \right)^2 - 2I_{xy} \left(\frac{\sin \varphi}{\sqrt{I}} \right) \left(\frac{\cos \varphi}{\sqrt{I}} \right) + I_{yy} \left(\frac{\sin \varphi}{\sqrt{I}} \right)^2. \tag{5}$$

On the line OL take a point $A(X, Y)$ such that

$$OA = \frac{1}{\sqrt{I}}.$$

To the various directions of the OL-axis, that is, to various values

of the angle φ, there correspond different values I and different points A. Let us find the locus of the points A. Obviously,

$$X=\frac{1}{\sqrt{I}}\cos\varphi,\quad Y=\frac{1}{\sqrt{I}}\sin\varphi.$$

By virtue of (5), the quantities X and Y are connected by the relationship

$$1=I_{xx}X^2-2I_{xy}XY+I_{yy}Y^2. \tag{6}$$

Thus, the locus of points $A(X, Y)$ is a second-degree curve (6). We shall prove that this curve is an ellipse.

The following inequality established by the Russian mathematician Bunyakovsky *) holds true:

$$\left(\iint_D xy\,dx\,dy\right)^2<\left(\iint_D x^2\,dx\,dy\right)\left(\iint_D y^2\,dx\,dy\right)$$

or

$$I_{xx}I_{yy}-I_{xy}^2>0.$$

---

*) To prove Bunyakovsky's (also spelt Buniakowski) inequality, we consider the following obvious inequality:

$$\iint_D [f(x, y)-\lambda\varphi(x, y)]^2\,dx\,dy\geqslant 0,$$

where λ is a constant. The equality sign is possible only when $f(x, y)-\lambda\varphi(x, y)\equiv 0$; that is, if $f(x, y)=\lambda\varphi(x, y)$. If we assume that $\frac{f(x, y)}{\varphi(x, y)}\neq$ $\neq \text{const}=\lambda$, then there will always be the inequality sign. Thus, removing brackets under the integral sign, we obtain

$$\iint_D f^2(x, y)\,dx\,dy-2\lambda\iint_D f(x, y)\,\varphi(x, y)\,dx\,dy+\lambda^2\iint_D \varphi^2(x, y)\,dx\,dy>0.$$

Consider the expression on the left as a function of λ. This is a second-degree polynomial that never vanishes; hence, its roots are complex, and this will occur when the discriminant formed of the coefficients of the quadratic polynomial is negative, that is,

$$\left(\iint_D f_0\varphi\,dx\,dy\right)^2-\iint_D f^2\,dx\,dy\iint_D \varphi^2\,dx\,dy<0$$

or

$$\left(\iint_D f\varphi\,dx\,dy\right)^2<\iint_D f^2\,dx\,dy\iint_D \varphi^2\,dx\,dy.$$

This is Bunyakovsky's inequality.

In our case, $f(x, y)=x$, $\varphi(x, y)=y$, $\frac{x}{y}\neq \text{const}$.

Bunyakovsky's inequality is widely used in various fields of mathematics. In many textbooks it is incorrectly called Schwarz' inequality. Bunyakovsky published it (among other important inequalities) in 1859. Schwarz published his work 16 years later, in 1875.

Thus, the discriminant of the curve (6) is positive and, consequently, the curve is an ellipse (Fig. 314). This ellipse is called the **ellipse of inertia**. The notion of an ellipse of inertia is very important in mechanics.

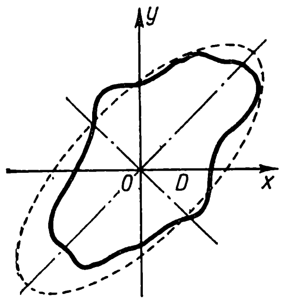


Fig. 314.

We note that the lengths of the axes of the ellipse of inertia and its position in the plane depend on the shape of the given plane figure. Since the distance from the origin to some point A of the ellipse is equal to $\frac{1}{\sqrt{I}}$, where I is the moment of inertia of the figure relative to the OA-axis, it follows that, after constructing the ellipse, we can readily calculate the moment of inertia of the figure D relative to some straight line passing through the coordinate origin. In particular, it is easy to see that the moment of inertia of the figure will be least relative to the major axis of the ellipse of inertia and greatest relative to the minor axis of this ellipse.

SEC. 10. THE COORDINATES OF THE CENTRE OF GRAVITY OF THE AREA OF A PLANE FIGURE

In Sec. 8, Ch. XII, it was stated that the coordinates of the centre of gravity of a system of material points $P_1, P_2, \ldots, P_n$ with masses $m_1, m_2, \ldots, m_n$ are defined by the formulas

$$x_c=\frac{\sum x_i m_i}{\sum m_i}\;;\quad y_c=\frac{\sum y_i m_i}{\sum m_i}. \tag{1}$$

Let us now determine the coordinates of the centre of gravity of a plane figure D. Divide this figure into very small elementary subregions ΔS_i. If the surface density is taken as equal to unity, then the mass of the subregion will be equal to its area. If it is approximately considered that the entire mass of an elementary subregion ΔS_i is concentrated in some point of it $P_i(\xi_i, \eta_i)$, the figure D may be regarded as a **system of material points**. Then, by formulas (1), the coordinates of the centre of gravity of this figure will be **approximately** determined by the equations

$$x_c\approx\frac{\sum\limits_{i=1}^{i=n}\xi_i\Delta S_i}{\sum\limits_{i=1}^{i=n}\Delta S_i}\;;\qquad y_c\approx\frac{\sum\limits_{i=1}^{i=n}\eta_i\Delta S_i}{\sum\limits_{i=1}^{i=n}\Delta S_i}.$$

In the limit, as $\Delta S_i \to 0$, the integral sums in the numerators and denominators of the fractions will pass into double integrals, and we obtain exact formulas for computing the coordinates of the centre of gravity of a plane figure:

$$x_c = \frac{\iint\limits_D x\,dx\,dy}{\iint\limits_D dx\,dy}; \quad y_c = \frac{\iint\limits_D y\,dx\,dy}{\iint\limits_D dx\,dy}; \qquad (2)$$

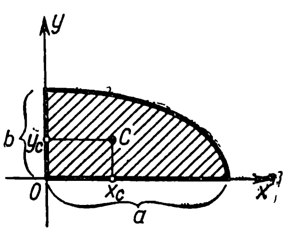


Fig. 315.

These formulas, which have been derived for a plane figure with surface density 1, obviously, hold true also for a figure with any other density γ constant at all points.

If, however, the surface density is variable,

$$\gamma = \gamma(x, y),$$

then the corresponding formulas will have the form

$$x_c = \frac{\iint\limits_D \gamma(x, y)\,x\,dx\,dy}{\iint\limits_D \gamma(x, y)\,dx\,dy}; \qquad y_c = \frac{\iint\limits_D \gamma(x, y)\,y\,dx\,dy}{\iint\limits_D \gamma(x, y)\,dx\,dy}.$$

The expressions $M_y = \iint\limits_D \gamma(x, y)\,x\,dx\,dy$ and $M_x = \iint\limits_D \gamma(x, y)$ $y\,dx\,dy$ are called *static moments* of the plane figure D relative to the y-axis and x-axis.

The integral $\iint \gamma(x, y)\,dx\,dy$ expresses the quantity of mass of the figure in question.

Example. Determine the coordinates of the centre of gravity of a quarter of the ellipse (Fig. 315)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

assuming that the surface density at all points is equal to 1.

Solution. By formulas (2) we have

$$x_c = \frac{\int\limits_0^a \left(\int\limits_0^{\frac{b}{a}\sqrt{a^2-x^2}} x\,dy \right) dx}{\int\limits_0^a \left(\int\limits_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy \right) dx} = \frac{\frac{b}{a}\int\limits_0^a \sqrt{a^2-x^2}\,x\,dx}{\frac{1}{4}\pi ab} = \frac{-\frac{b}{a}\cdot\frac{1}{3}(a^2-x^2)^{3/2}\Big|_0^a}{\frac{1}{4}\pi ab} = \frac{4a}{3\pi},$$

$$y_c = \frac{\int\limits_0^a \left(\int\limits_0^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy \right) dx}{\frac{1}{4}\pi ab} = \frac{4b}{3\pi}.$$

SEC. 11. TRIPLE INTEGRALS

Let there be given, in space, a certain region V bounded by a closed surface S. Let some continuous function $f(x, y, z)$, where x, y, z are the rectangular coordinates of a point of the region, be given in the region V and on its boundary. For clarity, if $f(x, y, z) \geqslant 0$, we can regard this function as the density of distribution of some substance in the region V.

Divide V, in arbitrary fashion, into subregions Δv_i; the symbol Δv_i will denote not only the region itself, but its volume as well. Within the limits of each subregion Δv_i, choose an arbitrary point P_i and denote by $f(P_i)$ the value of the function f at this point. Form an integral sum of the type

$$\sum f(P_i)\,\Delta v_i \tag{1}$$

and increase without bound the number of subregions Δv_i so that the largest diameter of Δv_i should approach zero.*) If the function $f(x, y, z)$ is continuous, there will be a limit of the integral sums of type (1), where the limit of integral sums is to be understood in the same sense as for the definition of the double integral.**) This limit is not dependent either on the manner of partitioning the region V or on the choice of points P_i; it is designated by the symbol $\iiint\limits_V f(P)\,dv$ and is called a *triple integral*.

Thus, by definition,

$$\lim_{\operatorname{diam} \Delta v_i \to 0} \sum f(P_i)\,\Delta v_i = \iiint\limits_V f(P)\,dv$$

or

$$\iiint\limits_V f(P)\,\Delta v = \iiint\limits_V f(x, y, z)\,dx\,dy\,dz. \tag{2}$$

---

*) The diameter of a subregion Δv_i is the maximum distance between points lying on the boundary of the subregion.

**) This theorem of the existence of a limit of integral sums (that is, of the existence of a triple integral) for any function continuous in a closed region V (including the boundary) is accepted without proof.

If $f(x, y, z)$ is considered the volume density of distribution of a substance over the region V, then the integral (2) yields the mass of the entire substance contained in V.

SEC. 12. EVALUATING A TRIPLE INTEGRAL

Suppose that the spatial (three-dimensional) region V bounded by the closed surface S possesses the following properties:

1) every straight line parallel to the z-axis and drawn through an interior (that is, not lying on the boundary S) point of the region V cuts the surface S at two points;

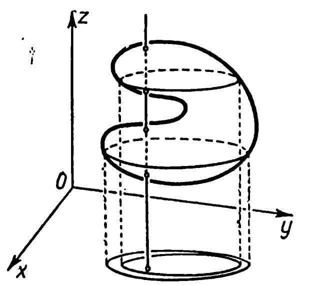


Fig. 316.

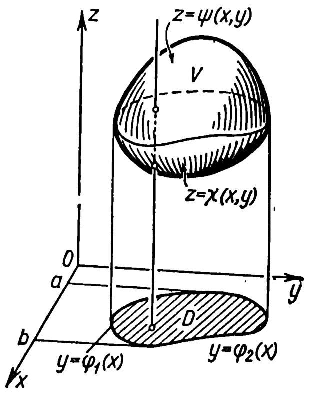


Fig. 317.

2) the entire region V is projected on the xy-plane into a regular (two-dimensional) region D;

3) any part of the region V cut off by a plane parallel to any one of the coordinate planes (Oxy, Oxz, Oyz) likewise possesses Properties 1 and 2.

We shall call the region V that possesses the indicated properties a *regular* three-dimensional region.

To illustrate, an ellipsoid, a rectangular parallelepiped, a tetrahedron, and so on are examples of regular three-dimensional regions. An instance of an irregular three-dimensional region is given in Fig. 316. In this section we will consider only regular regions.

Let the surface bounding the region V below have the equation $z=\chi(x, y)$, and the surface bounding this region above, the equation $z=\psi(x, y)$ (Fig. 317).

We introduce the concept of a **threefold iterated** integral I_V, over the region V, of a function of three variables $f(x, y, z)$ defined and continuous in V. Suppose that the region D is the projection of the region V onto the xy-plane bounded by the

lines

$$y=\varphi_1(x),\quad y=\varphi_2(x),\quad x=a,\quad y=b.$$

Then *a threefold iterated integral* of the function $f(x, y, z)$ over the region V is defined as follows:

$$I_V=\int\limits_a^b\left[\int\limits_{\varphi_1(x)}^{\varphi_2(x)}\left\{\int\limits_{\chi(x,y)}^{\psi(x,y)} f(x, y, z)\,dz\right\}dy\right]dx. \qquad (1)$$

We note that as a result of integration with respect to z and substitution of limits in the braces (inner brackets) we get a function of x and y. We then compute the double integral of this function over the region D as has already been done.

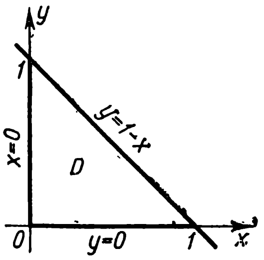


Fig. 318.

The following is an example of the evaluation of a threefold iterated integral.

Example 1. Compute the iterated integral of the function $f(x, y, z)=xyz$ over the region V bounded by the planes

$$x=0,\quad y=0,\quad z=0,\quad x+y+z=1.$$

Solution. This region is regular, it is bounded above and below by the planes $z=0$ and $z=1-x-y$ and is projected on the xy-plane into a regular plane region D, which is a triangle bounded by the straight lines $x=0$, $y=0$, $y=1-x$ (Fig. 318). Therefore, the threefold iterated integral I_V is computed as follows:

$$I_V=\iint\limits_D\left[\int\limits_0^{1-x-y} xyz\,dz\right]d\sigma.$$

Setting up the limits in the twofold iterated integral over the region D, we obtain

$$I_V=\int\limits_0^1\left\{\int\limits_0^{1-x}\left[\int\limits_0^{1-x-y} xyz\,dz\right]dy\right\}dx=\int\limits_0^1\left\{\int\limits_0^{1-x}\frac{xyz^2}{2}\Bigg|_{z=0}^{z=1-x-y}dy\right\}dx=$$

$$=\int\limits_0^1\left[\int\limits_0^{1-x} xy(1-x-y)^2\,dy\right]dx=\int\limits_0^1\frac{x}{12}(1-x)^4\,dx=\frac{1}{360}.$$

Let us now consider some of the properties of a threefold iterated integral.

Property 1. *If a region V is divided into two regions V_1 and V_2 by a plane parallel to some of the coordinate planes, then the threefold iterated integral over V is equal to the sum of the threefold iterated integrals over the regions V_1 and V_2.*

The proof of this property is exactly the same as that for twofold iterated integrals. We shall not repeat it.

Corollary. For any kind of partition of the region V into a finite number of subregions $V_1, \ldots, V_n$ by planes parallel to the coordinate planes, we have the equality

$$I_V = I_{V_1} + I_{V_2} + \ldots + I_{V_n}.$$

Property 2 (Theorem of the evaluation of a threefold iterated integral). *If m and M are, respectively, the smallest and largest values of the function $f(x, y, z)$ in the region V, we have the inequality*

$$mV \leqslant I_V \leqslant MV,$$

where V is the volume of the given region and I_V is a threefold iterated integral of the function $f(x, y, z)$ over the region V.

Proof. Let us first evaluate the inside integral in the iterated integral $I_V = \iint\limits_D \left[\int\limits_{\chi(x, y)}^{\psi(x, y)} f(x, y, z)\, dz \right] d\sigma$:

$$\int\limits_{\chi(x, y)}^{\psi(x, y)} f(x, y, z)\, dz \leqslant \int\limits_{\chi(x, y)}^{\psi(x, y)} M dz - M \int\limits_{\chi(x, y)}^{\psi(x, y)} dz = Mz \Big|_{\chi(x, y)}^{\psi(x, y)} =$$

$$= M [\psi(x, y) - \chi(x, y)].$$

Thus, the inside integral does not exceed the expression $M[\psi(x, y) - \chi(x, y)]$. Therefore, by virtue of the theorem of Sec. 1 for double integrals, we get (denoting by D the projection of the region V on the xy-plane)

$$I_V = \iint\limits_D \left[\int\limits_{\chi(x, y)}^{\psi(x, y)} f(x, y, z)\, dz \right] d\sigma \leqslant \iint\limits_D M [\psi(x, y) - \chi(x, y)]\, d\sigma =$$

$$= M \iint\limits_D [\psi(x, y) - \chi(x, y)]\, d\sigma.$$

But the latter iterated integral is equal to the double integral of the function $\psi(x, y) - \chi(x, y)$ and, consequently, is equal to the volume of the region which lies between the surface $z = \chi(x, y)$ and $z = \psi(x, y)$, that is, to the volume of the region V. Therefore,

$$I_V \leqslant MV.$$

It is similarly proved that $I_V \geqslant mV$. Property 2 is thus proved.

Property 3 (Mean-Value Theorem). *The threefold iterated integral I_V of a continuous function $f(x, y, z)$ over a region V is equal to the product of its volume V by the value of the function at*

some point P of V; that is,

$$I_V=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\left\{\int_{\chi(x,y)}^{\psi(x,y)} f(x, y, z)\,dz\right\}dy\right]dx=f(P)\,V. \tag{2}$$

The proof of this property is carried out in the same way as that for a twofold iterated integral [see Sec. 2, Property 3, formula (4)]. We can now prove the theorem for evaluating a triple integral.

Theorem. *The triple integral of a function $f(x, y, z)$ over a regular region V is equal to a threefold iterated integral over the same region; that is,*

$$\iiint_V f(x, y, z)\,dv=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\left\{\int_{\chi(x,y)}^{\psi(x,y)} f(x, y, z)\,dz\right\}dy\right]dx.$$

Proof. Divide the region V by planes parallel to the coordinate planes into n regular subregions:

$$\Delta v_1+\Delta v_2+\ldots+\Delta v_n.$$

As done above, denote by I_V the threefold iterated integral of the function $f(x, y, z)$ over the region V, and by $I_{\Delta v_i}$ the threefold iterated integral of this function over the subregion Δ_{v_i}. Then by the corollary of Property 1 we can write the equation

$$I_V=I_{\Delta v_1}+I_{\Delta v_2}+\ldots+I_{\Delta v_n}. \tag{3}$$

We transform each of the terms on the right by formula (2):

$$I_V=f(P_1)\,\Delta v_1+f(P_2)\,\Delta v_2+\ldots+f(P_n)\,\Delta v_n, \tag{4}$$

where P_i is some point of the subregion Δv_i.

On the right side of this equation is an integral sum. It is assumed that the function $f(x, y, z)$ is continuous in the region V; and for this reason the limit of this sum, as the largest diameter of Δv_i approaches zero, exists and is equal to the triple integral of the function $f(x, y, z)$ over V. Thus, passing to the limit in (4), as diam $\Delta v_i \longrightarrow 0$, we get

$$I_V=\iiint_V f(x, y, z)\,dv,$$

or, finally, interchanging the expressions on the right and left,

$$\iiint_V f(x, y, z)\,dv=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\left\{\int_{\chi(x,y)}^{\psi(x,y)} f(x, y, z)\,dz\right\}dy\right]dx.$$

Thus, the theorem is proved.

Here, $z=\chi(x, y)$ and $z=\psi(x, y)$ are the equations of the surfaces bounding the regular region V below and above. The lines $y=\varphi_1(x)$, $y=\varphi_2(x)$, $x=a$, $x=b$ bound the region D, which is the projection of V onto the xy-plane.

Note. Like in the case of the double integral, we can form a threefold iterated integral with a different order of integration with respect to the variables and with other limits, if, of course, the shape of the region V permits this.

Computing the volume of a solid by means of a threefold iterated integral. If the integrand $f(x, y, z)=1$, then the triple integral over the region V expresses the **volume of the region** V:

$$V=\iiint_V dx\,dy\,dz. \qquad (5)$$

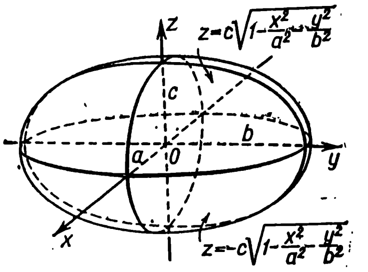


Fig. 319.

Example 2. Compute the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Solution. The ellipsoid (Fig. 319) is bounded below by the surface $z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$, and above by the surface $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$. The projection of this ellipsoid on the xy-plane (region D) is an ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Hence, reducing to a threefold iterated integral, we obtain

$$V=\int_{-a}^{a}\left[\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}\left(\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dz\right)dy\right]dx=$$

$$=2c\int_{-a}^{a}\left[\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\right]dx.$$

When computing the inside integral, x is held constant. Make the substitution:

$$y=b\sqrt{1-\frac{x^2}{a^2}}\sin t, \quad dy=b\sqrt{1-\frac{x^2}{a^2}}\cos t\,dt.$$

The variable y varies from $-b\sqrt{1-\frac{x^2}{a^2}}$ to $b\sqrt{1-\frac{x^2}{a^2}}$; therefore t

varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Putting new limits in the integral, we get

$$V=2c\int_{-a}^{a}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\left(1-\frac{x^2}{a^2}\right)-\left(1-\frac{x^2}{a^2}\right)\sin^2 t}\; b\sqrt{1-\frac{x^2}{a^2}}\cos t\,dt\right]_t dx=$$

$$=2cb\int_{-a}^{a}\left[\left(1-\frac{x^2}{a^2}\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt\right]dx=\frac{cb\pi}{a^2}\int_{-a}^{a}(a^2-x^2)\,dx=\frac{4\pi abc}{3}.$$

Hence,

$$V=\frac{4}{3}\pi abc.$$

If $a=b=c$, we get the volume of the sphere:

$$V=\frac{4}{3}\pi a^3.$$

SEC. 13. CHANGE OF VARIABLES IN A TPIPLE INTEGRAL

1. Triple integral in cylindrical coordinates. In the case of cylindrical coordinates, the position of a point P in space is determined by the three numbers θ, ϱ, z, where θ and ϱ are polar coordinates of the projection of the point P on the xy-plane and z is the z-coordinate of P, that is, the distance of the point to the xy-plane—with the plus sign if the point lies above the xy-plane, and with the minus sign if below the xy-plane (Fig. 320).

In this case, we divide the given three-dimensional region V into elementary volumes by the coordinate surfaces $\theta=\theta_i$, $\varrho=\varrho_j$, $z=z_k$ (half-planes adjoining the z-axis, circular cylinders whose axis coincides with the z-axis, planes perpendicular to the z-axis). The curvilinear "prism" shown in Fig. 321 will be a volume element. The base area of this prism is equal, to within infinitesimals of higher order, to $\varrho\Delta\theta\Delta\varrho$, the altitude is Δz (to simplify notation we drop the indices i, j, k). Thus, $\Delta v=\varrho\Delta\theta\Delta\varrho\Delta z$. Hence, the triple integral of the function $F(\theta, \varrho, z)$ over the region V has the form

$$I=\iiint_V F(\theta, \varrho, z)\,\varrho\,d\theta\,d\varrho\,dz. \qquad (1)$$

The limits of integration are determined by the shape of the region V.

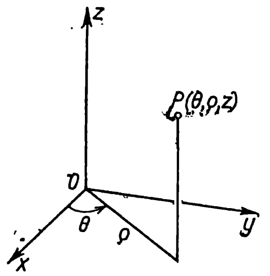


Fig. 320.

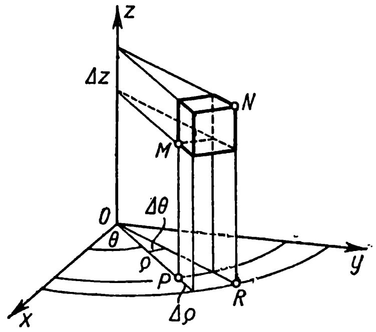


Fig. 321.

If a triple integral of the function $f(x, y, z)$ is given in rectangular coordinates, it can readily be changed to a triple integral in cylindrical coordinates. Indeed, noting that

$$x=\varrho\cos\theta; \quad y=\varrho\sin\theta; \quad z=z,$$

we have

$$\iiint\limits_V f(x, y, z)\,dx\,dy\,dz=\iiint\limits_V F(\theta, \varrho, z)\,\varrho\,d\theta\,d\varrho\,dz,$$

where

$$f(\varrho\cos\theta, \varrho\sin\theta, z)=F(\theta, \varrho, z).$$

Example. Determine the mass M of a hemisphere of radius R with centre at the origin, if the density F of its substance at each point (x, y, z) is proportional to the distance of this point from the base, that is, $F=kz$.

Solution. The equation of the upper part of the hemisphere

$$z=\sqrt{R^2-x^2-y^2}$$

in cylindrical coordinates has the form

$$z=\sqrt{R^2-\varrho^2}.$$

Hence,

$$M=\iiint\limits_V kz\varrho\,d\theta\,d\varrho\,dz=\int\limits_0^{2\pi}\left[\int\limits_0^R\left(\int\limits_0^{\sqrt{R^2-\varrho^2}} kz\,dz\right)\varrho\,d\varrho\right]d\theta=$$

$$=\int\limits_0^{2\pi}\left[\int\limits_0^R \frac{kz^2}{2}\Bigg|_0^{\sqrt{R^2-\varrho^2}}\varrho\,d\varrho\right]d\theta=\int\limits_0^{2\pi}\left[\int\limits_0^R\frac{k}{2}(R^2-\varrho^2)\,\varrho\,d\varrho\right]d\theta=$$

$$=\frac{k}{2}\int\limits_0^{2\pi}\left[\frac{R^4}{2}-\frac{R^4}{4}\right]d\theta=\frac{k}{2}\,\frac{R^4}{4}\,2\pi=\frac{k\pi R^4}{4},$$

2. A triple integral in spherical coordinates. In spherical coordinates, the position of a point P in space is determined by three numbers, θ, r, φ, where r is the distance of the point from the origin, the so-called radius vector of the point, φ is the

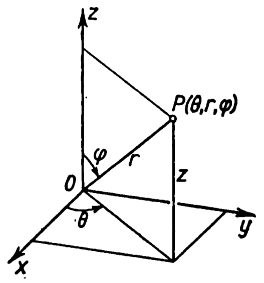


Fig. 322.

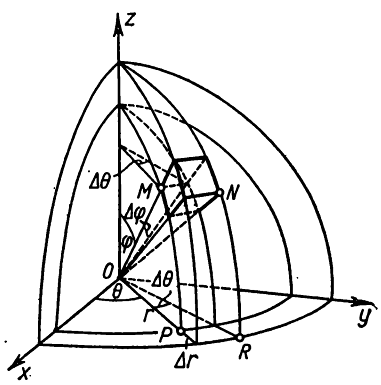


Fig. 323.

angle between the radius vector and the z-axis, θ is the angle between the projection of the radius vector on the xy-plane and the x-axis reckoned from this axis in a positive sense (counterclockwise) (Fig. 322). For any point of space we have

$$0 \leqslant r < \infty, \quad 0 \leqslant \varphi \leqslant \pi; \quad 0 \leqslant \theta \leqslant 2\pi.$$

Divide this region V into volume elements Δv by the coordinate surfaces $r = \text{const}$ (sphere), $\varphi = \text{const}$ (conic surfaces with vertices at origin), $\theta = \text{const}$ (half-planes passing through the z-axis). To within infinitesimals of higher order, the volume element Δv may be considered a parallelepiped with edges of length Δr, $r\Delta\varphi$, $r\sin\varphi\Delta\theta$. Then the volume element is equal (see Fig. 323) to

$$\Delta v = r^2 \sin\varphi \, \Delta r \, \Delta\theta \, \Delta\varphi.$$

The triple integral of a function $F(\theta, r, \varphi)$ over the region V has the form

$$I = \iiint\limits_V F(\theta, r, \varphi) \, r^2 \sin\varphi \, dr \, d\theta \, d\varphi. \tag{1'}$$

The limits of integration are determined by the shape of the region V. From Fig. 322 it is easy to establish the expressions of Cartesian coordinates in terms of spherical coordinates:

$$x = r\sin\varphi\cos\theta,$$
$$y = r\sin\varphi\sin\theta,$$
$$z = r\cos\varphi.$$

For this reason, the formula for transforming the triple integral from Cartesian coordinates to spherical coordinates has the form

$$\iiint_V f(x, y, z)\,dx\,dy\,dz =$$

$$= \iiint_V f[r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi]\,r^2\sin\varphi\,dr\,d\theta\,d\varphi.$$

3. General change of variables in the triple integral. Transformations from Cartesian coordinates to cylindrical and spherical coordinates in the triple integral represent special cases of the general transformation of coordinates in space.

Let the functions

$$x = \varphi(u, t, w),$$
$$y = \psi(u, t, w),$$
$$z = \chi(u, t, w)$$

map, in one-to-one manner, the region V in Cartesian coordinates x, y, z onto the region V' in curvilinear coordinates u, t, w. Let the volume element Δv of the region V be carried over to the volume element $\Delta v'$ of V' and let

$$\lim_{\Delta v' \to 0} \frac{\Delta v}{\Delta v'} = |I|.$$

Then

$$\iiint_V f(x, y, z)\,dx\,dy\,dz =$$

$$= \iiint_{V'} f[\varphi(u, t, w),\ \psi(u, t, w),\ \chi(u, t, w)]\,|I|\,du\,dt\,dw.$$

As in the case of the double integral, I is called the Jacobian; and as in the case of double integrals, it may be proved that the Jacobian is numerically equal to a determinant of order three:

$$I = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Thus, in the case of cylindrical coordinates we have

$$x = \varrho\cos\theta,\ y = \varrho\sin\theta,\ z = z\ (\varrho = u,\ \theta = t,\ z = w);$$

$$I = \begin{vmatrix} \cos\theta & -\varrho\sin\theta & 0 \\ \sin\theta & \varrho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \varrho.$$

In the case of spherical coordinates:

$x = r\sin\varphi\cos\theta,\ y = r\sin\varphi\sin\theta,\ z = r\cos\varphi\ (r = u,\ \varphi = t,\ \theta = w);$

$$I = \begin{vmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{vmatrix} = r^2\sin\varphi.$$

SEC. 14. THE MOMENT OF INERTIA AND THE COORDINATES OF THE CENTRE OF GRAVITY OF A SOLID

1. The moment of inertia of a solid. The moments of inertia of a **point** $M(x, y, z)$ of mass m relative to the coordinate axes Ox, Oy, and Oz (Fig. 324) are expressed, respectively, by the formulas

$$I_{xx} = (y^2 + z^2)m,$$
$$I_{yy} = (x^2 + z^2)m, \quad I_{zz} = (x^2 + y^2)m.$$

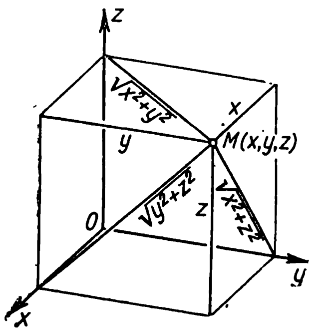


Fig. 324.

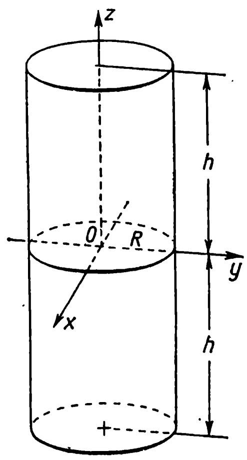


Fig. 325.

The moments of inertia of a **solid** are expressed by the corresponding integrals. For instance, the moment of inertia of a solid relative to the z-axis is expressed by the integral $I_{zz} = \iiint\limits_V (x^2+y^2)\gamma(x, y, z)\,dx\,dy\,dz$, where $\gamma(x, y, z)$ is the density of the substance.

Example 1. Compute the moment of inertia of a right circular cylinder of altitude $2h$ and radius R relative to the diameter of its median section, considering the density constant and equal to γ_0.

Solution. Choose a coordinate system as follows: direct the z-axis along the axis of the cylinder, and put the origin of coordinates at its centre of symmetry (Fig. 325).

Then the problem reduces to computing the moment of inertia of the cylinder relative to the x-axis:

$$I_{xx}=\iiint\limits_V (y^2+z^2)\,\gamma_0\,dx\,dy\,dz.$$

Changing to cylindrical coordinates, we obtain

$$I_{xx}=\gamma_0\int\limits_0^{2\pi}\left\{\int\limits_0^R\left[\int\limits_0^h (z^2+\varrho^2\sin^2\theta)\,dz\right]\varrho\,d\varrho\right\}d\theta=$$

$$=\gamma_0\int\limits_0^{2\pi}\left\{\int\limits_0^R\left[\frac{2h^3}{3}+2h\varrho^2\sin^2\theta\right]\varrho\,d\varrho\right\}d\theta=\gamma_0\int\limits_0^{2\pi}\left\{\frac{2h^3}{3}\frac{R^2}{2}+\frac{2hR^4}{4}\sin^2\theta\right\}d\theta=$$

$$=\gamma_0\left[\frac{2h^3R^2}{6}2\pi+\frac{2hR^4}{4}\pi\right]=\gamma_0\pi hR^2\left[\frac{2}{3}h^2+\frac{R^2}{2}\right].$$

2. **The coordinates of the centre of gravity of a solid.** Like what we had in Sec. 8, Ch. XII for plane figures, the coordinates of the centre of gravity of a solid are expressed by the formulas

$$x_c=\frac{\iiint\limits_V x\gamma(x,\,y,\,z)\,dx\,dy\,dz}{\iiint\limits_V \gamma(x,\,y,\,z)\,dx\,dy\,dz};\quad y_c=\frac{\iiint\limits_V y\gamma(x,\,y,\,z)\,dx\,dy\,dz}{\iiint\limits_V \gamma(x,\,y,\,z)\,dx\,dy\,dz};$$

$$z_c=\frac{\iiint\limits_V z\gamma(x,\,y,\,z)\,dx\,dy\,dz}{\iiint\limits_V \gamma(x,\,y,\,z)\,dx\,dy\,dz},$$

where $\gamma(x,\,y,\,z)$ is the density.

Example 2. Determine the coordinates of the centre of gravity of the upper half of a sphere of radius R with centre at the origin, considering the density γ_0 constant.

Solution. The hemisphere is bounded by the surfaces

$$z=\sqrt{R^2-x^2-y^2},\quad z=0.$$

The z-coordinate of its centre of gravity is given by the formula

$$z_c=\frac{\iiint\limits_V z\gamma_0\,dx\,dy\,dz}{\iiint\limits_V \gamma_0\,dx\,dy\,dz}.$$

Changing to spherical coordinates, we get

$$z_c=\frac{\gamma_0\int\limits_0^{2\pi}\left[\int\limits_0^{\frac{\pi}{2}}\left(\int\limits_0^R r\cos\varphi r^2\sin\varphi\,dr\right)d\varphi\right]d\theta}{\gamma_0\int\limits_0^{2\pi}\left[\int\limits_0^{\frac{\pi}{2}}\left(\int\limits_0^R r^2\sin\varphi\,dr\right)d\varphi\right]d\theta}=\frac{2\pi\frac{R^4}{4}\frac{1}{2}}{\frac{4}{6}\pi R^3}=\frac{3}{8}R.$$

Obviously, by virtue of the symmetry of the hemisphere, $x_c=y_c=0$.

SEC. 15. COMPUTING INTEGRALS DEPENDENT ON A PARAMETER

Consider the integral dependent on the parameter α.

$$I(\alpha)=\int\limits_a^b f(x,\ \alpha)\,dx.$$

(We examined such integrals in Sec. 10, Ch. XI.) We state without proof that if a function $f(x,\ \alpha)$ is continuous with respect to x over the interval $[a,\ b]$ and with respect to α over the interval $[\alpha_1,\ \alpha_2]$, then the function

$$I(\alpha)=\int\limits_a^b f(x,\ \alpha)\,dx$$

is a continuous function on $[\alpha_1,\ \alpha_2]$. Consequently, the function $I(\alpha)$ may be integrated with respect to α on the interval $[\alpha_1, \alpha_2]$:

$$\int\limits_{\alpha_1}^{\alpha_2} I(\alpha)\,d\alpha=\int\limits_{\alpha_1}^{\alpha_2}\left[\int\limits_a^b f(x,\ \alpha)\,dx\right]d\alpha.$$

The expression on the right is an iterated integral of the function $f(x,\ \alpha)$ with respect to a rectangle situated in the plane $xO\alpha$. We can change the order of integration in this integral:

$$\int\limits_{\alpha_1}^{\alpha_2}\left[\int\limits_a^b f(x,\ \alpha)\,dx\right]d\alpha=\int\limits_a^b\left[\int\limits_{\alpha_1}^{\alpha_2} f(x,\ \alpha)\,d\alpha\right]dx.$$

This formula shows that for integration of an integral dependent on a parameter α, it is sufficient to integrate the element of integration with respect to the parameter α. This formula is also useful when computing definite integrals.

Example. Compute the integral

$$\int_0^{\infty} \frac{e^{-ax}-e^{-bx}}{x}\,dx.$$

This integral is not expressible in terms of elementary functions. To evaluate it, we consider another integral that may be readily computed:

$$\int_0^{\infty} e^{-\alpha x}dx = \frac{1}{\alpha}\ (\alpha > 0).$$

Integrating this equation between the limits $\alpha = a$ and $\alpha = b$, we get

$$\int_a^b \left[\int_0^{\infty} e^{-\alpha x}\,dx\right] d\alpha = \int_a^b \frac{d\alpha}{\alpha} = \ln\frac{b}{a}.$$

Changing the order of integration in the first integral, we rewrite this equation in the following form:

$$\int_0^{\infty} \left[\int_a^b e^{-\alpha x}d\alpha\right] dx = \ln\frac{b}{a},$$

whence, computing the inner integral, we get

$$\int_0^{\infty} \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\frac{b}{a}.$$

Exercises on Chapter XIV

Evaluate the integrals *: **1.** $\int_0^1\int_1^2 (x^2+y^2)\,dx\,dy$. *Ans.* $\frac{8}{3}$. **2.** $\int_3^4\int_1^2 \frac{dy\,dx}{(x+y)^2}$. *Ans.* $\ln\frac{25}{24}$. **3.** $\int_1^2\int_x^{x\sqrt{3}} xy\,dx\,dy$. *Ans.* $\frac{15}{4}$. **4.** $\int_0^{2\pi}\int_{a\sin\theta}^{a} r\,dr\,d\theta$. *Ans.* $\frac{1}{2}\pi a^2$.

---

*) If the integral is written as $\int_M^N\int_K^L f(x, y)\,dx\,dy$ then, as has already been stated, we can consider that the first integration is performed with respect to the variable whose differential occupies the first place; that is,

$$\int_M^N\int_K^L f(x, y)\,dx\,dy = \int_M^N\left(\int_K^L f(x, y)\,dx\right)dy.$$

5. $\int\limits_0^a \int\limits_{\frac{x}{a}}^{x} \frac{x\,dy\,dx}{x^2+y^2}$. *Ans.* $\frac{\pi a}{4} - a \arctan \frac{1}{a}$. 6. $\int\limits_0^a \int\limits_{y-a}^{2y} xy\,dx\,dy$. *Ans.* $\frac{11a^4}{24}$.

7. $\int\limits_{\frac{b}{2}}^{b} \int\limits_0^{\frac{\pi}{2}} \rho\,d\theta\,d\rho$. *Ans.* $\frac{3}{16}\pi b^2$.

Determine the limits of integration for the integral $\iint\limits_D f(x, y)\,dx\,dy$ where the region of integration is bounded by the lines: 8. $x=2$, $x=3$, $y=-1$, $y=5$. *Ans.* $\int\limits_2^3 \int\limits_{-1}^5 f(x, y)\,dy\,dx$. 9. $y=0$, $y=1-x^2$. *Ans.* $\int\limits_{-1}^1 \int\limits_0^{1-x^2} f(x, y)\,dy\,dx$.
10. $x^2+y^2=a^2$. *Ans.* $\int\limits_{-a}^{a} \int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\,dy\,dx$. 11. $y=\frac{2}{1+x^2}$, $y=x^2$. *Ans.* $\int\limits_{-1}^{1} \int\limits_{x^2}^{\frac{2}{1+x^2}} f(x, y)\,dy\,dx$. 12. $y=0$, $y=a$, $y=x$, $y=x-2a$. *Ans.* $\int\limits_0^a \int\limits_y^{y+2a} f(x, y)\,dx\,dy$.

Change the order of integration in the integrals: 13. $\int\limits_1^2 \int\limits_3^4 f(x, y)\,dy\,dx$. *Ans.* $\int\limits_3^4 \int\limits_1^2 f(x, y)\,dx\,dy$. 14. $\int\limits_0^1 \int\limits_{x^3}^{\sqrt{x}} f(x, y)\,dy\,dx$. *Ans.* $\int\limits_0^1 \int\limits_{y^2}^{\sqrt[3]{y}} f(x, y)\,dx\,dy$.
15. $\int\limits_0^a \int\limits_0^{\sqrt{2ay-y^2}} f(x, y)\,dx\,dy$. *Ans.* $\int\limits_0^a \int\limits_{a-\sqrt{a^2-x^2}}^{a} f(x, y)\,dy\,dx$. 16. $\int\limits_{-1}^1 \int\limits_0^{\sqrt{1-x^2}} f(x, y)\,dy\,dx$. *Ans.* $\int\limits_0^1 \int\limits_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y)\,dx\,dy$. 17. $\int\limits_0^1 \int\limits_{-\sqrt{1-y^2}}^{1-y} f(x, y)\,dx\,dy$. *Ans.* $\int\limits_{-1}^0 \int\limits_0^{\sqrt{1-x^2}} f(x, y)\,dy\,dx + \int\limits_0^1 \int\limits_0^{1-x} f(x, y)\,dy\,dx$.

Compute the following integrals by changing to polar coordinates:

18. $\int\limits_0^a \int\limits_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$. *Ans.* $\int\limits_0^{\frac{\pi}{2}} \int\limits_0^a \sqrt{a^2-\rho^2}\,\rho\,d\rho\,d\theta = \frac{\pi}{6}a^3$.

19. $\int\limits_0^a \int\limits_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$. *Ans.* $\int\limits_0^{\frac{\pi}{2}}\int\limits_0^a \varrho^3\,d\varrho\,d\theta=\frac{\pi a^4}{8}$. **20.** $\int\limits_0^\infty\int\limits_0^\infty e^{-(x^2+y^2)}\,dy\,dx$. *Ans.* $\int\limits_0^{\frac{\pi}{2}}\int\limits_0^\infty e^{-\varrho^2}\varrho\,d\varrho\,d\theta=\frac{\pi}{4}$. **21.** $\int\limits_0^{2a}\int\limits_0^{\sqrt{2ax-x^2}} dy\,dx$. *Ans.* $\int\limits_0^{\frac{\pi}{2}}\int\limits_0^{2a\cos\theta} \varrho\,d\varrho\,d\theta=$ $=\frac{\pi a^2}{2}$.

Transform the double integrals by introducing new variables u and v connected with x and y by the formulas $x=u-uv$, $y=uv$: **22.** $\int\limits_0^e\int\limits_{\alpha x}^{\beta x} f(x, y)\,dy\,dx$.

Ans. $\int\limits_{\frac{\alpha}{1+\alpha}}^{\frac{\beta}{1+\beta}}\int\limits_{u=0}^{\frac{e}{1-v}} f(u-uv,\ uv)\,u\,du\,dv$. **23.** $\int\limits_0^c\int\limits_0^b f(x, y)\,dy\,dx$.

Ans. $\int\limits_0^{\frac{b}{b+c}}\int\limits_0^{\frac{c}{1-v}} f(u-uv,\ uv)\,u\,du\,dv+\int\limits_{\frac{b}{b+c}}^{1}\int\limits_0^{\frac{b}{v}} f(u-uv,\ uv)\,u\,du\,dv$.

Calculating Surfaces by Means of Double Integrals

24. Compute the area of a figure bounded by the parabola $y^2=2x$ and the straight line $y=x$. *Ans.* $\frac{2}{3}$.

25. Compute the area of a figure bounded by the lines $y^2=4ax$, $x+y=3a$, $y=0$. *Ans.* $\frac{10}{3}a^2$.

26. Compute the area of a figure bounded by the lines $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$, $x+y=a$. *Ans.* $\frac{a^2}{3}$.

27. Compute the area of a figure bounded by the lines $y=\sin x$, $y=\cos x$, $x=0$. *Ans.* $\sqrt{2}-1$.

28. Compute the area of a loop of the curve $\varrho=a\sin 2\theta$. *Ans.* $\frac{\pi a^2}{8}$.

29. Compute the entire area bounded by the lemniscate $\varrho^2=a^2\cos 2\varphi$. *Ans.* a^2.

30. Compute the area of a loop of the curve $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{2xy}{c^2}$.

Hint. Change to new variables $x=\varrho a\cos\theta$ and $y=\varrho b\sin\theta$. *Ans.* $\frac{a^2b^2}{c^2}$.

Calculating Volumes

31. Compute the volumes of solids bounded by the following surfaces: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, $x=0$, $y=0$, $z=0$. *Ans.* $\frac{abc}{6}$. 32. $z=0$, $x^2+y^2=1$, $x+y+z=3$. *Ans.* 3π. 33. $(x-1)^2+(y-1)^2=1$, $xy=z$, $z=0$. *Ans.* π. 34. $x^2+y^2-2ax=0$, $z=0$, $x^2+y^2=z^2$. *Ans.* $\frac{32}{9}a^3$. 35. $y=x^2$, $x=y^2$, $z=0$, $z=12+y-x^2$. *Ans.* $\frac{549}{140}$.

36. Compute the volumes of solids bounded by the coordinate planes, the plane $2x+3y-12=0$ and the cylinder $z=\frac{1}{2}y^2$. *Ans.* 16.

37. Compute the volumes of solids bounded by a circular cylinder of radius a, whose axis coincides with the z-axis, the coordinate planes and the plane $\frac{x}{a}+\frac{z}{a}=1$. *Ans.* $a^3\left(\frac{\pi}{4}-\frac{1}{3}\right)$.

38. Compute the volumes of solids bounded by the cylinders $x^2+y^2=a^2$, $x^2+z^2=a^2$. *Ans.* $\frac{16}{3}a^3$. 39. $y^2+z^2=x$, $x=y$, $z=0$. *Ans.* $\frac{\pi}{64}$. 40. $x^2+y^2+z^2=a^2$, $x^2+y^2=R^2$, $a>R$. *Ans.* $\frac{4}{3}\pi[a^3-(\sqrt{a^2-R^2})^3]$. 41. $az=x^2+y^2$, $z=0$, $x^2+y^2=2ax$. *Ans.* $\frac{3}{2}\pi a^3$. 42. $\rho^2=a^2\cos 2\theta$, $x^2+y^2+z^2=a^2$, $z=0$. (Compute the volume that is interior with respect to the cylinder.) *Ans.* $\frac{1}{9}a^3(3\pi+20-16\sqrt{2})$.

Calculating Surface Areas

43. Compute the area of that part of the surface of the cone $x^2+y^2=z^2$ which is cut out by the cylinder $x^2+y^2=2ax$. *Ans.* $2\pi a^2\sqrt{2}$.

44. Compute the area of that part of the plane $x+y+z=2a$, which lies in the first octant and is bounded by the cylinder $x^2+y^2=a^2$. *Ans.* $\frac{\pi a^2}{4}\sqrt{3}$.

45. Compute the surface area of a spherical segment (minor) if the radius of the sphere is a, while the radius of the base of the segment is b. *Ans.* $2\pi(a^2-a\sqrt{a^2-b^2})$.

46. Find the area of that part of the surface of the sphere $x^2+y^2+z^2=a^2$ which is cut out by the surface of the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$. *Ans.* $4\pi a^2-8a^2-\arcsin\frac{\sqrt{a^2-b^2}}{a}$.

47. Find the surface area of a solid that is the common part of two cylinders $x^2+y^2=a^2$, $y^2+z^2=a^2$. *Ans.* $16a^2$.

48. Compute the area of that part of the surface of the cylinder $x^2+y^2=2ax$, which lies between the plane $z=0$ and the cone $x^2+y^2=z^2$. *Ans.* $8a^2$.

49. Compute the area of that part of the surface of the cylinder $x^2+y^2=a^2$ which lies between the plane $z=mx$ and the plane $z=0$. *Ans.* $2ma^2$.

50. Compute the area of that part of the surface of the paraboloid $y^2+z^2=2ax$, which lies between the parabolic cylinder $y^2=ax$ and the plane $x=a$. *Ans.* $\frac{1}{3}\pi a^2(3\sqrt{3}-1)$.

Computing the Mass, the Coordinates of the Centre of Gravity, and the Moment of Inertia of Plane Solids

(In Problems 51-64 we consider the surface density constant and equal to unity.)

51. Determine the mass of a slab in the shape of a circle of radius a, if the density at any point P is inversely proportional to the distance of P from the axis of the cylinder (the proportionality factor is K). *Ans.* πaK.

52. Compute the coordinates of the centre of gravity of an equilateral triangle if we take its altitude for the x-axis and the vertex of the triangle for the coordinate origin. *Ans.* $x=\frac{a\sqrt{3}}{3}$; $y=0$.

53. Find the coordinates of the centre of gravity of a circular sector of radius a, taking the bisector of its angle as the x-axis. The angle of spread of the sector is 2α. *Ans.* $x_c=\frac{2a\sin\alpha}{3\alpha}$, $y_c=0$.

54. Find the coordinates of the centre of gravity of the upper half of the circle $x^2+y^2=a^2$. *Ans.* $x_c=0$; $y_c=\frac{4a}{3\pi}$.

55. Find the coordinates of the centre of gravity of the area of one arc of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$. *Ans.* $x_c=a\pi$, $y_c=\frac{5a}{6}$.

56. Find the coordinates of the centre of gravity of the area bounded by a loop of the curve $\varrho^2=a^2\cos 2\theta$. *Ans.* $x_c=\frac{\pi a\sqrt{2}}{8}$, $y_c=0$.

57. Find the coordinates of the centre of gravity of the area of the cardioid $\varrho=a(1+\cos\theta)$. *Ans.* $x_c=\frac{5a}{6}$, $y_c=0$.

58. Compute the moment of inertia of the area of a rectangle bounded by the straight lines $x=0$, $x=a$, $y=0$, $y=b$ relative to the origin. *Ans.* $\frac{ab(a^2+b^2)}{3}$.

59. Compute the moment of inertia of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$: a) relative to the y-axis; b) relative to the origin. *Ans.* a) $\frac{\pi a^3b}{4}$; b) $\frac{\pi ab}{4}(a^2+b^2)$.

60. Compute the moment of inertia of the area of the circle $\varrho=2a\cos\theta$ relative to the pole. *Ans.* $\frac{3}{2}\pi a^4$.

61. Compute the moment of inertia of the area of the cardioid $\varrho=a(1-\cos\theta)$ relative to the pole. *Ans.* $\frac{35\pi a^4}{16}$.

62. Compute the moment of inertia of the area of the circle $(x-a)^2+(y-b)^2=2a^2$ relative to the y-axis. *Ans.* $3\pi a^4$.

63. The density at any point of a square slab with side a is a proportional to the distance of this point from one of the vertices of the square. Compute the moment of inertia of the slab relative to the side passing through this vertex. *Ans.* $\frac{1}{40} ka^5 [7\sqrt{2}+3\ln(\sqrt{2}+1)]$ where k is the proportionality factor.

64. Compute the moment of inertia of the area of a figure, bounded by the parabola $y^2=ax$ and the straight line $x=a$, relative to the straight line $y=-a$. *Ans.* $\frac{8}{5}a^4$.

Triple Integrals

65. Compute $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ if the region of integration is bounded by the coordinate planes and the plane $x+y+z=1$. *Ans.* $\frac{\ln 2}{2}-\frac{5}{16}$.

66. Evaluate $\int\limits_0^a \left[\int\limits_0^x \left(\int\limits_0^y xyz\,dz\right) dy\right] dx$. *Ans.* $\frac{a^6}{48}$.

67. Compute the volume of a solid bounded by the sphere $x^2+y^2+z^2=4$ and the surface of the paraboloid $x^2+y^2=3z$. *Ans.* $\frac{19}{6}\pi$.

68.*) Compute the coordinates of the centre of gravity and the moments of inertia of a pyramid bounded by the planes $x=0$, $y=0$, $z=0$; $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. *Ans.* $x_c=\frac{a}{4}$, $y_c=\frac{b}{4}$, $z_c=\frac{c}{4}$; $I_x=\frac{a^3bc}{60}$, $I_y=\frac{b^3ac}{60}$, $I_z=\frac{c^3ab}{60}$, $I_0=\frac{abc}{60}(a^2+b^2+c^2)$.

69. Compute the moment of inertia of a circular right cone relative to its axis. *Ans.* $\frac{1}{10}\pi hr^4$ where h is the altitude and r is the radius of the base of the cone.

70. Compute the volume of a solid bounded by a surfac with equation $(x^2+y^2+z^2)^2=a^3x$. *Ans.* $\frac{1}{3}\pi a^3$.

71. Compute the moment of inertia of a circular cone relative to the diameter of the base. *Ans.* $\frac{\pi hr^2}{60}(2h^2+3r^2)$.

72. Compute the coordinates of the centre of gravity of a solid lying between a sphere of radius a and a conic surface with angle at the vertex 2α, if the vertex of the cone coincides with the centre of the sphere. *Ans.* $x_c=0$, $y_c=0$, $z_c=\frac{3}{8}a(1+\cos\alpha)$ (the z-axis is the axis of the cone, and the vertex lies at the origin).

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*) In Problems 68, 69 and 71 to 73 we consider the density constant and equal to unity.

73. Compute the coordinates of the centre of gravity of a solid bounded by a sphere of radius a and by two planes passing through the centre of the sphere and forming an angle of 60°. *Ans.* $\varrho=\frac{9}{16}a$, $\theta=0$, $\varphi=\frac{\pi}{2}$ (the line of intersection of the planes is taken for the z-axis, the centre of the sphere for the origin; ϱ, θ, φ are spherical coordinates).

74. Using the equation $\frac{1}{\sqrt{x}}=\frac{2}{\sqrt{\pi}}\int\limits_0^\infty e^{-\alpha^2 x}\,d\alpha$ $(\alpha>0)$ compute the integrals $\int\limits_0^\infty \frac{\cos x\,dx}{\sqrt{x}}$ and $\int\limits_0^\infty \frac{\sin x\,dx}{\sqrt{x}}$. *Ans.* $\sqrt{\frac{\pi}{2}}$; $\sqrt{\frac{\pi}{2}}$.

CHAPTER XV

LINE INTEGRALS AND SURFACE INTEGRALS

SEC. 1. LINE INTEGRALS

Let the point $P(x, y)$ be in motion along some plane line L from the point M to the point N. To P is applied a force $\boldsymbol{F}$ which varies in magnitude and direction with the motion of P; it is thus some function of the coordinates of P:

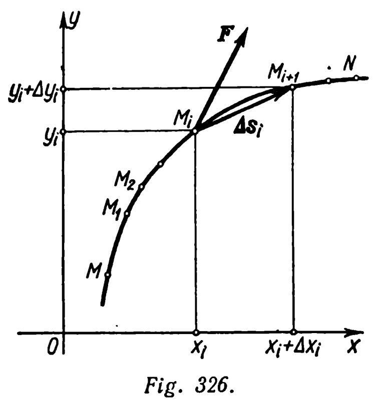


Fig. 326.

$$\boldsymbol{F} = \boldsymbol{F}(P).$$

Let us compute the work A of the force $\boldsymbol{F}$ as the point P is translated from M to N (Fig. 326). To do this, we divide the curve MN into n arbitrary parts by the points $M_0 = M$, M_1, M_2, ..., $M_n = N$ in the direction from M to N and we denote by $\Delta \boldsymbol{s}_i$ the vector $\overline{M_i M_{i+1}}$. We denote by $\boldsymbol{F}_i$ the magnitude of the force $\boldsymbol{F}$ at the point M_i. Then the scalar product $\boldsymbol{F}_i \Delta \boldsymbol{s}_i$ may be regarded as an approximate expression of the work of the force $\boldsymbol{F}$ along the arc $\overset{\frown}{M_i M_{i+1}}$:

$$A_i \approx \boldsymbol{F}_i \Delta \boldsymbol{s}_i.$$

Let

$$\boldsymbol{F} = X(x, y)\,\boldsymbol{i} + Y(x, y)\,\boldsymbol{j}$$

where $X(x, y)$ and $Y(x, y)$ are the projections of the vector $\boldsymbol{F}$ on the x- and y-axes. Denoting by Δx_i and Δy_i the increments of the coordinates x_i and y_i when changing from the point M_i to the point M_{i+1}, we get

$$\Delta \boldsymbol{s}_i = \Delta x_i\,\boldsymbol{i} + \Delta y_i\,\boldsymbol{j}.$$

Hence,

$$\boldsymbol{F}_i \Delta \boldsymbol{s}_i = X(x_i, y_i)\,\Delta x_i + Y(x_i, y_i)\,\Delta y_i.$$

The approximate value of the work A of the force $\boldsymbol{F}$ over the entire curve MN will be

$$A \approx \sum_{i=1}^{n} \boldsymbol{F}_i \Delta \boldsymbol{s}_i = \sum_{i=1}^{n} [X(x_i, y_i)\,\Delta x_i + Y(x_i, y_i)\,\Delta y_i]. \qquad (1)$$

Without making any precise statements, we shall say that if there exists a limit of the expression on the right as $\Delta s_i \longrightarrow 0$ (here, obviously, $\Delta x_i \longrightarrow 0$ and $\Delta y_i \longrightarrow 0$), then this limit expresses the work of the force $\boldsymbol{F}$ over the curve L from the point M to the point N:

$$A = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_i \to 0}} \sum_{i=1}^{n} [X(x_i, y_i)\Delta x_i + Y(x_i, y_i)\Delta y_i]. \tag{2}$$

The limit *) on the right is called the *line integral* of $X(x, y)$ and $Y(x, y)$ over the curve L and is denoted by

$$A = \int_L X(x, y)\,dx + Y(x, y)\,dy \tag{3}$$

or

$$A = \int_{(M)}^{(N)} X(x, y)\,dx + Y(x, y)\,dy. \tag{3'}$$

Limits of sums of type (2) frequently occur in mathematics and mechanics; here, $X(x, y)$ and $Y(x, y)$ are regarded as functions of two variables in some region D.

The letters M and N, which take the place of the limits of integration, are in brackets to signify that they are not numbers but symbols of the end points of the line over which the line integral is taken. The direction of the curve L from M to N is called the sense of integration.

If the curve L is a space curve, then the line integral of three functions $X(x, y, z)$, $Y(x, y, z)$, $Z(x, y, z)$ is defined similarly:

$$\int_L X(x, y, z)\,dx + Y(x, y, z)\,dy + Z(x, y, z)\,dz =$$

$$= \lim_{\substack{\Delta x_k \to 0 \\ \Delta y_k \to 0 \\ \Delta z_k \to 0}} \sum_{k=1}^{n} X(x_k, y_k, z_k)\Delta x_k + Y(x_k, y_k, z_k)\Delta y_k + Z(x_k, y_k, z_k)\Delta z_k.$$

The letter L under the integral sign indicates that the integration is performed along the curve L.

We note two properties of a line integral.

Property 1. A line integral is determined by the element of integration, the form of the curve of integration, and the sense of integration.

*) Here, the limit of the integral sum is to be understood in the same sense as in the case of the definite integral, see Sec. 2, Ch. XI.

A line integral changes sign when the sense of integration is reversed, since in that case the vector $\Delta \boldsymbol{s}$, and hence its projections Δx and Δy, changes sign.

Property 2. Divide the curve L by the point K into pieces L_1 and L_2 so that $\overset{\frown}{MN} = \overset{\frown}{MK} + \overset{\frown}{KN}$ (Fig. 327). Then, from formula (1) it follows directly that

$$\int_{(M)}^{(N)} X\,dx + Y\,dy = \int_{(M)}^{(K)} X\,dx + Y\,dy + \int_{(K)}^{(N)} X\,dx + Y\,dy.$$

This relationship holds for any number of terms.

It will further be noted that the definition of a line integral holds true also for the case when the curve L is closed.

In this case, the initial and terminal points of the curve coincide. Therefore, in the case of a closed curve we cannot write $\int_{(M)}^{(N)} X\,dx + Y\,dy$, but only $\int_L X\,dx + Y\,dy$; and we have to indicate the **direction of circulation** (sense of description) over the closed curve L. The line integral over a **closed contour** L is frequently denoted also by the symbol $\oint_L X\,dx + Y\,dy$.

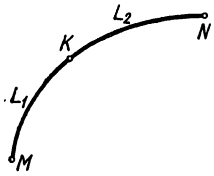


Fig. 327.

Note. We arrived at the concept of a line integral while considering the problem of the work of a force $\boldsymbol{F}$ on a curved path L.

Here, at all points of the curve L the force $\boldsymbol{F}$ was given as a vector function $\boldsymbol{F}$ of the coordinates of the point of application (x, y); the projections of the variable vector $\boldsymbol{F}$ on the coordinate axes are equal to the scalar (numerical, that is) functions $X(x, y)$ and $Y(x, y)$. For this reason, line integral of the form $\int_L X\,dx + Y\,dy$ may be regarded as an integral of the **vector** function F given by the projections X and Y.

The integral of a vector function $\boldsymbol{F}$ over the curve L is denoted by the symbol

$$\int_L \boldsymbol{F}\,ds.$$

If the vector $\boldsymbol{F}$ is defined by its projections X, Y, Z then this integral is equal to the line integral

$$\int_L X\,dx + Y\,dy + Z\,dz.$$

As a particular instance, if the vector $\boldsymbol{F}$ lies in the xy-plane, then the integral of this vector is equal to

$$\int_L X\,dx + Y\,dy.$$

When the line integral of a vector function $\boldsymbol{F}$ is taken along a closed curve L, this line integral is also called a *circulation* of the vector $\boldsymbol{F}$ over the closed contour L.

SEC. 2 EVALUATING A LINE INTEGRAL

In this section we shall make more precise the concept of the limit of the sum (1) of Sec. 1 and in this connection we shall make more precise the concept of the line integral and indicate a method for calculating it.

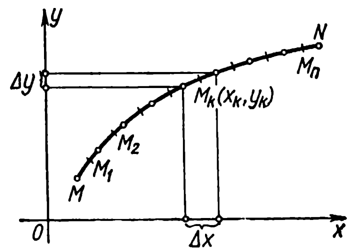


Fig. 328.

Let a curve L be represented by equations in parametric form:

$$x = \varphi(t), \quad y = \psi(t).$$

Consider the arc of the curve MN (Fig. 328). Let the points M and N correspond to the values of the parameter α and β. Divide the arc MN into subarcs Δs_i by the points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, ..., $M_n(x_n, y_n)$, and put $x_i = \varphi(t_i)$, $y_i = \psi(t_i)$.

Consider the line integral

$$\int_L X(x, y)\,dx + Y(x, y)\,dy \tag{1}$$

defined in the preceding section. We give without proof the **existence theorem of a line integral.** *If the functions $\varphi(t)$ and $\psi(t)$ are continuous and have continuous derivatives $\varphi'(t)$ and $\psi'(t)$, and also continuous are the functions $X[\varphi(t), \psi(t)]$ and $Y[\varphi(t), \psi(t)]$ as functions of t on the interval $[\alpha, \beta]$, then the following limits exist:*

$$\left.\begin{aligned} \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} X(\bar{x}_i, \bar{y}_i)\,\Delta x_i &= \int X(x, y)\,dx, \\ \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} Y(\bar{x}_i, \bar{y}_i)\,\Delta y_i &= \int Y(x, y)\,dy, \end{aligned}\right\} \tag{2}$$

where $\bar{x}_i$ and $\bar{y}_i$ are the coordinates of some point lying on the arc Δs_i. These limits do not depend on way the arc L is divided

into subarcs Δs_i, *provided that* $\Delta s_i \to 0$ *and do not depend on the choice of the point* $\overline{M}_i(\overline{x}_i, \overline{y}_i)$ *on the subarc* Δs_i; they are called *line integrals* and are denoted as

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} X(\overline{x}_i, \overline{y}_i)\,\Delta x_i = \int_L X(x, y)\,dx,$$

$$\lim_{\Delta y_i \to 0} \sum_{i=1}^{n} Y(\overline{x}_i, \overline{y}_i)\,\Delta y_i = \int_L Y(x, y)\,dy.$$

Note. From this theorem it follows that the sums defined in the preceding section, where the points $\overline{M}_i(\overline{x}_i, \overline{y}_i)$ are the extremities of the subarc Δs_i and the manner of partition of the arc L into subarcs Δs_i is arbitrary, approach the same limit—the line integral.

This theorem makes it possible to develop a method for computing a line integral.

Thus, by definition, we have

$$\int_{(M)}^{(N)} X(x, y)\,dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} X(\overline{x}_i, \overline{y}_i)\,\Delta x_i, \tag{3}$$

where

$$\Delta x_i = x_i - x_{i-1} = \varphi(t_i) - \varphi(t_{i-1}).$$

Transform this latter difference by the Lagrange formula

$$\Delta x_i = \varphi(t_i) - \varphi(t_{i-1}) = \varphi'(\tau_i)(t_i - t_{i-1}) = \varphi'(\tau_i)\,\Delta t_i,$$

where τ_i is some value of t that lies between the values $t_i - 1$ and t_i. Since the point $\overline{x}_i$, $\overline{y}_i$ on the subarc Δs_i may be chosen at pleasure, we shall choose it so that its coordinates correspond to the value of the parameter τ_i:

$$\overline{x}_i = \varphi(\tau_i), \quad \overline{y}_i = \psi(\tau_i).$$

Substituting into (3) the values of $\overline{x}_i$, $\overline{y}_i$ and Δx_i that we have found, we get

$$\int_{(M)}^{(N)} X(x, y)\,dx = \lim_{\Delta t_i \to 0} \sum_{i=1}^{n} X[\varphi(\tau_i)\,\psi(\tau_i)]\,\varphi'(\tau_i)\,\Delta t_i.$$

On the right is the limit of the integral sum for the continuous function of a single variable $X[\varphi(t), \psi(t)]\,\varphi'(t)$ on the interval $[\alpha, \beta]$.

Hence, this limit is equal to the definite integral of this function:

$$\int_{(M)}^{(N)} X(x, y)\,dx = \int_{\alpha}^{\beta} X[\varphi(t), \ \psi(t)]\,\varphi'(t)\,dt.$$

In analogous fashion we get the formula

$$\int_{(M)}^{(N)} Y(x, y)\,dy = \int_{\alpha}^{\beta} Y[\varphi(t), \ \psi(t)]\,\psi'(t)\,dt.$$

Adding these equations term by term, we obtain

$$\int_{(M)}^{(N)} X(x, y)\,dx + Y(x, y)\,dy = \int_{\alpha}^{\beta} \{X[\varphi(t)\,\psi(t)] + \\ + Y[\varphi(t), \ \psi(t)]\,\psi'(t)\}\,dt. \tag{4}$$

This is the desired formula for computing a line integral.

In similar manner we compute the line integral

$$\int X\,dx + Y\,dy + Z\,dz$$

over the space curve defined by the equations $x=\varphi(t)$, $y=\psi(t)$, $z=\chi(t)$.

Example 1. Compute the line integral of three functions: x^3, $3zy^2$, $-x^2y$ (or, which is the same thing, of the vector function $x^3\boldsymbol{i}+3zy^2\boldsymbol{j}-x^2y\boldsymbol{k}$) along a segment of a straight line issuing from the point $M(3, 2, 1)$ to the point $N(0, 0, 0)$ (Fig. 329).

Solution. To find the parametric equations of the line MN, along which the integration is to be performed, we write the equation of the straight line that passes through the given two points:

$$\frac{x}{3}=\frac{y}{2}=\frac{z}{1};$$

and denote all these relations by a single letter t; then we get the equations of the straight line in parametric form:

$$x=3t, \quad y=2t, \quad z=t.$$

Here, obviously, to the origin of the segment MN corresponds the value of the parameter $t=1$, and to the terminus of the segment, the value $t=0$. The derivatives of x, y, z with respect to the parameter t (which will be needed for evaluating the line integral) are easily found:

$$x'_t=3, \quad y'_t=2, \quad z'_t=1.$$

Now the desired line integral may be computed by formula (4):

$$\int_{(M)}^{(N)} x^3\,dx+3zy^2\,dy-x^2y\,dz=\int_1^0[(3t)^3\cdot 3+3t\,(2t)^2\cdot 2-(3t)^2\cdot 2t\cdot 1]\,dt=$$

$$=\int_1^0 87t^3\,dt=-\frac{87}{4}.$$

Example 2. Evaluate the line integral of a pair of functions: $6x^2y$, $10xy^2$ along a plane curve $y=x^3$ from the point $M(1, 1)$ to the point $N(2, 8)$ (Fig. 330).

Solution. To compute the required integral

$$\int_{(M)}^{(N)} 6x^2y\,dx+10xy^2\,dy$$

we must have the parametric equations of the given curve. However, the explicitly defined equation of the curve $y=x^3$ is a special case of the parametric equation:

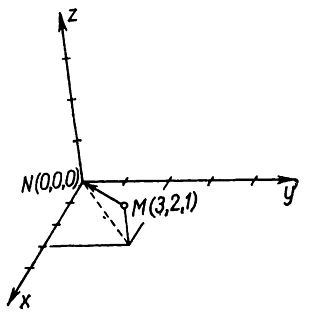


Fig. 329.

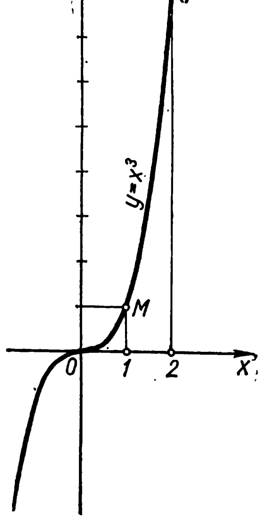


Fig. 330.

here, the abscissa x of the point of the curve serves as the parameter, and the parametric equations of the curve are

$$x=x,\quad y=x^3.$$

The parameter x varies from $x_1=1$ to $x_2=2$. The derivatives with respect to the parameter are readily evaluated:

$$x'_x=1,\quad y'_x=3x^2.$$

Hence,

$$\int_{(M)}^{(N)} 6x^2y\,dx+10xy^2\,dy=\int_1^2[6x^2x^3\cdot 1+10xx^6\cdot 3x^2]\,dx=$$

$$=\int_1^2(6x^5+30x^9)\,dx=[x^6+3x^{10}]_1^2=1084.$$

We now indicate certain applications of a line integral.

1. **The expression of the area of a region bounded by a curve in terms of a line integral.** In an xy-plane let there be given a region D (bounded by the contour L) such that any straight line parallel to one of the coordinate axes and passing through an interior point of the region cuts the boundary L of the region in no more than two points (which means that the region D is regular) (Fig. 331).

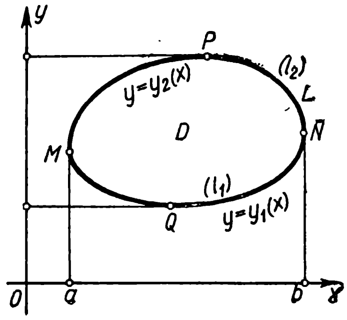


Fig. 331.

Suppose that the region D is projected on the x-axis in the interval $[a, b]$, and it is bounded below by the curve (l_1):

$$y=y_1(x),$$

and above by the curve (l_2):

$$y=y_2(x),$$

$$[y_1(x)\leqslant y_2(x)].$$

Then the area of the region D is

$$S=\int_a^b y_2(x)\,dx-\int_a^b y_1(x)\,dx.$$

But the first integral is a line integral over the curve $l_2(\widehat{MPN})$, since $y=y_2(x)$ is the equation of this curve; hence,

$$\int_a^b y_2(x)\,dx=\int_{MPN} y\,dx.$$

The second integral is a line integral over the curve $l_1(\widehat{MQN})$, that is,

$$\int_a^b y_1(x)\,dx=\int_{MQN} y\,dx.$$

By Property 1 of the line integral we have

$$\int_{MPN} y\,dx=-\int_{NPM} y\,dx.$$

Hence,

$$S=-\int_{NPM} y\,dx-\int_{MQN} y\,dx=-\int_L y\,dx. \quad (5)$$

Here, the curve L is traced in a **counterclockwise** direction.

If part of the boundary L is the segment M_1M, parallel to the y-axis, then $\int\limits_{(M_1)}^{(M)} y\,dx = 0$, and equation (5) holds true in this case as well (Fig. 332).

Similarly, it may be shown that

$$S = \int\limits_L x\,dy. \tag{6}$$

Adding (5) and (6) term by term and dividing by 2, we get another formula for computing the area S:

$$S = \frac{1}{2}\int\limits_L x\,dy - y\,dx. \tag{7}$$

Example 3. Compute the area of the ellipse

$$x = a\cos t, \quad y = b\sin t.$$

Solution. By formula (7) we find

$$S = \frac{1}{2}\int\limits_0^{2\pi} [a\cos t\, b\cos t - b\sin t\,(-a\sin t)]\,dt = \pi ab.$$

We note that formula (7) and formulas (5) and (6) as well hold true also for areas whose boundaries are cut by coordinate lines in more than two points (Fig. 333).

To prove this, we divide the given region (Fig. 333) into two regular regions by the line l^*.

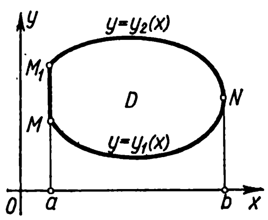


Fig. 332.

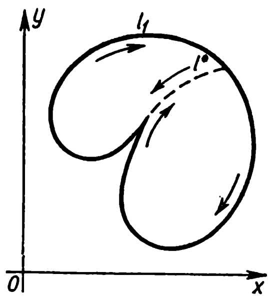


Fig. 333.

Formula (7) holds for each of these regions. Adding the left and right sides, we get (on the left) the area of the given region, on the right, a line integral (with coefficient $^1/_2$) taken over the entire boundary, since the line integral over the division line l^* is taken twice: in the direct and reverse senses; hence, it is equal to zero.

2. **Computing the work of a variable force F on some curved path L.** As was shown at the beginning of Sec. 1, the work done by a force $\boldsymbol{F}=X(x, y, z)\boldsymbol{i}+Y(x, y, z)\boldsymbol{j}+Z(x, y, z)\boldsymbol{k}$ along a line L = MN is equal to the line integral

$$A=\int_{(M)}^{(N)} X(x, y, z)\,dx+Y(x, y, z)\,dy+Z(x, y, z)\,dz.$$

Let us consider an instance that shows how to calculate the work of the force in concrete cases.

Example 4. Determine the work A of the force of gravity **F** when the mass m is translated from the point $M_1(a_1, b_1, c_1)$ to the point $M_2(a_2, b_2, c_2)$ along an arbitrary path L (Fig. 334).

Solution. The projections of the force of gravity **F** on the coordinate axes are

$$X=0,\quad Y=0,\quad Z=-mg.$$

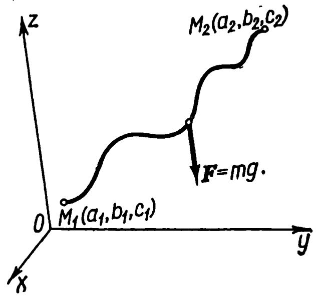


Fig. 334.

Hence, the desired work is

$$A=\int_{(M_1)}^{(M_2)} X\,dx+Y\,dy+Z\,dz=\int_{c_1}^{c_2}(-mg)\,dz=mg(c_1-c_2).$$

Consequently, in this case the line integral is independent of the path of integration and dependent only on the initial and terminal points. More precisely, the work of the force of gravity is dependent only on the difference between the heights of the terminal and initial points of the path.

SEC. 3. GREEN'S FORMULA

Let us establish a connection between a double integral over some plane region D and the line integral around the boundary L of this region.

In an xy-plane, let there be given a region D, which is regular both in the direction of the x-axis and the y-axis, bounded by a closed contour L. Let this region be bounded below by the curve $y=y_1(x)$, and above by the curve $y=y_2(x)$, $y_1(x)\leqslant y_2(x)$ $(a\leqslant x\leqslant b)$ (Fig. 331).

Together, both these curves represent the closed contour L. Let there be given, in the region D, continuous functions $X(x, y)$ and $Y(x, y)$ that have continuous partial derivatives. We consider the integral

$$\iint_D \frac{\partial X(x, y)}{\partial y}\,dx\,dy.$$

Representing it in the form of an iterated integral, we find

$$\iint_D \frac{dX}{dy}\,dx\,dy = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \frac{dX}{dy}\,dy\right] dx = \int_a^b X(x, y)\Big|_{y_1(x)}^{y_2(x)} dx =$$

$$= \int_a^b [X(x, y_2(x)) - X(x, y_1\cdot(x))]\,dx. \qquad (1)$$

We note that the integral

$$\int_a^b X(x, y_2(x))\,dx$$

is numerically equal to the line integral

$$\int_{(MPN)} X(x, y)\,dx$$

taken along the curve MPN, whose equations, in parametric form, are

$$x = x, \quad y = y_2(x),$$

where x is a parameter.
Thus

$$\int_a^b X(x, y_2(x))\,dx = \int_{MPN} X(x, y)\,dx. \qquad (2)$$

Similarly, the integral

$$\int_a^b X(x, y_1(x))\,dx$$

is numerically equal to the line integral along the arc MQN:

$$\int_a^b X(x, y_1(x))\,dx = \int_{(MQN)} X(x, y)\,dx. \qquad (3)$$

Substituting expressions (2) and (3) into formula (1), we obtain

$$\iint_D \frac{\partial X}{\partial y}\,dx\,dy = \int_{MPN} X(x, y)\,dx - \int_{MQN} X(x, y)\,dx. \qquad (4)$$

But

$$\int_{MQN} X(x, y)\,dx = -\int_{NQM} X(x, y)\,dx$$

(see Sec. 1, Property 1). And so formula (4) may be written thus:

$$\iint_D \frac{\partial X}{\partial y}\,dx\,dy = \int_{MPN} X(x, y)\,dx + \int_{MQN} X(x, y)\,dx.$$

But the sum of the line integrals on the right is equal to the line integral taken along the entire closed curve L in the clockwise direction. Hence, the last equation can be reduced to the form

$$\iint_D \frac{\partial X}{\partial y}\,dx\,dy = \int_{L\text{ (in the clockwise sense)}} X(x, y)\,dx. \qquad (5)$$

If part of the boundary is the segment l_3 parallel to the y-axis, then $\int_{l_3} X(x, y)\,dx = 0$, and equation (5) holds true in this case as well.

Analogously, we find

$$\iint_D \frac{\partial Y}{\partial x}\,dx\,dy = -\int_{L\text{ (in the clockwise sense)}} Y(x, y)\,dy. \qquad (6)$$

Subtracting (6) from (5), we obtain

$$\iint_D \left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right)dx\,dy = \int_{L\text{ (in the clockwise sense)}} X\,dx + Y\,dy.$$

If the contour is traversed in the counterclockwise sense, then *)

$$\iint_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)dx\,dy = \int_L X\,dx + Y\,dy.$$

This is *Green's formula*, named after the English physicist and mathematician D. Green (1793-1841) **).

We assumed that the region D is regular. But, as in the area problem (see Sec. 2), it may be shown that this formula holds true for any region that may be divided into regular regions.

SEC. 4. CONDITIONS FOR A LINE INTEGRAL BEING INDEPENDENT OF THE PATH OF INTEGRATION

Consider the line integral

$$\int_{(M)}^{(N)} X\,dx + Y\,dy,$$

*) If in a line integral along a closed contour the direction of circulation is not indicated, it is assumed that it is in the counterclockwise sense. If the direction of circulation is clockwise, this must be specified.

**) This formula is a special case of a more general formula discovered by the Russian mathematician M. V. Ostrogradsky.

taken around some plane curve L connecting the points M and N. We assume that the functions $X(x, y)$ and $Y(x, y)$ have continuous partial derivatives in the region D under consideration. Let us find out under what conditions the line integral above is independent of the shape of the curve L and is dependent only on the position of the initial and terminal points M and N.

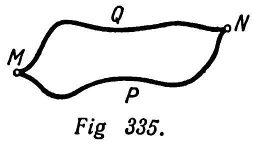


Fig 335.

Consider two arbitrary curves MPN and MQN lying in the given region D and connecting the points M and N (Fig. 335). Let

$$\int_{MPN} X\,dx + Y\,dy = \int_{MQN} X\,dx + Y\,dy, \qquad (1)$$

that is,

$$\int_{MPN} X\,dx + Y\,dy - \int_{MQN} X\,dx + Y\,dy = 0.$$

Then, on the basis of Properties 1 and 2 of line integrals (Sec. 1), we have

$$\int_{MPN} X\,dx + Y\,dy + \int_{NQM} X\,dx + Y\,dy = 0,$$

which is a line integral around the closed contour L:

$$\int_{L} X\,dx + Y\,dy = 0.$$

In this formula, the line integral is taken around the closed contour L, which is made up of the curves MPN and NQM. This contour L may obviously be considered arbitrary.

Thus, from the condition that for any two points M and N the line integral is independent of the shape of the curve connecting them and is dependent only on the position of these points, it follows that the *line integral along any closed contour is equal to zero*.

The converse conclusion is also true: if a line integral around any closed contour is equal to zero, then this line integral is independent of the shape of the curve connecting the two points, and *depends only upon the position of these points*. Indeed, equation (1) follows from equation (2).

In Example 4 of Sec. 2, the line integral is independent of the path of integration; in Example 3 the line integral depends on the path of integration because here the integral around the closed contour is not equal to zero, but yields an area bounded by the

contour in question; in Examples 1 and 2 the line integrals are likewise dependent on the path of integration.

The natural question arises: what conditions must the functions $X(x, y)$ and $Y(x, y)$ satisfy in order that the line integral $\int X\,dx+Y\,dy$ along any closed contour be equal to zero. The answer is given by the following theorem.

Theorem. *At all points of some region D, let the functions* $X(x, y)$, $Y(x, y)$, *together with their partial derivatives* $\frac{\partial X(x, y)}{\partial y}$ *and* $\frac{\partial Y(x, y)}{\partial x}$ *be continuous. Then, for the line integral along any closed contour L lying in this region to be zero, that is, for*

$$\int_L X(x, y)\,dx+Y(x, y)\,dy=0, \tag{2}$$

it is necessary and sufficient to fulfil the equation

$$\frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x} \tag{3}$$

at all points of the region D.

Proof. Consider an arbitrary closed contour L in a region D and write Green's formula for it:

$$\iint_D \left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right) dx\,dy=\int_L X\,dx+Y\,dy.$$

If condition (3) is fulfilled, then the double integral on the left is identically zero and, hence,

$$\int_L X\,dx+Y\,dy=0.$$

This proves the **sufficiency** of condition (3).

Now we prove the **necessity** of this condition; that is, we prove that if (2) is fulfilled for any closed curve L in the region D, then condition (3) is also fulfilled at each point of this region.

Let us assume, on the contrary, that equation (2) is fulfilled, that is,

$$\int_L X\,dx+Y\,dy=0,$$

and that condition (3) is not fulfilled;

$$\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\neq 0$$

at least in one point. For example, at some point $P(x_0, y_0)$ let

there be the inequality

$$\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}>0.$$

Since there is a continuous function on the left, it will be positive and greater than some number $\delta>0$ at all points of some sufficiently small region D' containing the point $P(x_0, y_0)$. Take the double integral of the difference $\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}$ over this region. It will have a positive value. Indeed,

$$\iint_{D'}\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)dx\,dy>\iint_{D'}\delta\,dx\,dy=\delta\iint_{D'}dx\,dy=\delta D'>0.$$

But by Green's formula the left side of the last inequality is equal to a line integral along the boundary L' of the region D', which, by assumption, is zero. Hence, the last inequality contradicts condition (2) and therefore the assumption that $\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}$ is different from zero in at least one point is not correct. Whence it follows that

$$\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}=0$$

at all points of the given region D.

The theorem is thus proved completely.

In Sec. 9, Ch. XIII, it was proved that fulfillment of the condition

$$\frac{\partial Y(x, y)}{\partial x}=\frac{\partial X(x, y)}{\partial y}$$

is tantamount to the fact that the expression $X\,dx+Y\,dy$ is an **exact differential of some function** $u(x, y)$, or

$$X\,dx+Y\,dy=du(x, y)$$

and

$$X(x, y)=\frac{\partial u}{\partial x}, \quad Y(x, y)=\frac{\partial u}{\partial y}.$$

But in this case the vector

$$\boldsymbol{F}=X\boldsymbol{i}+Y\boldsymbol{j}=\frac{\partial u}{\partial x}\boldsymbol{i}+\frac{\partial u}{\partial y}\boldsymbol{j}$$

is the gradient of the function $u(x, y)$; the function $u(x, y)$, the gradient of which is equal to the vector $X\boldsymbol{i}+Y\boldsymbol{j}$, is called the *potential* of this vector.

We shall prove that *in this case the line integral* $I=\int\limits_{(M)}^{(N)} X\,dx+Y\,dy$ *along any curve L connecting the points M and N is equal to the difference between the values of the function u at these points*:

$$\int\limits_{(M)}^{(N)} X\,dx+Y\,dy=\int\limits_{(M)}^{(N)} du(x,\ y)=u(N)-u(M).$$

Proof. If $X\,dx+Y\,dy$ is the exact differential of the function $u(x,\ y)$, then $X=\frac{\partial u}{\partial x}$; $Y=\frac{\partial u}{\partial y}$ and the line integral takes on the form

$$I=\int\limits_{(M)}^{(N)} \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy.$$

To evaluate this integral we write the parametric equations of the curve L connecting the points M and N:

$$x=\varphi(t),\qquad y=\psi(t).$$

We shall consider that to the value of the parameter $t=t_0$ there corresponds the point M, and to $t=T$, the point N. Then the line integral reduces to the following definite integral:

$$I=\int\limits_{t_0}^{T}\left[\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right]dt.$$

The expression in the brackets is a function of t, and this function is the total derivative of the function $u[\varphi(t),\ \psi(t)]$ with respect to t. Therefore

$$I=\int\limits_{t_0}^{T}\frac{\partial u}{\partial t}dt=u[\varphi(t),\ \psi(t)]\,\Big|_{t_0}^{T}=u[\varphi(t),\ \psi(t)]$$
$$-u[\varphi(t_0),\ \psi(t_0)]=u(N)-u(M).$$

As we see, *the line integral of an exact differential is independent of the shape of the curve along which the integration is performed.*

We have a similar assertion for a line integral over a space curve (see below, Sec. 7).

Note. It is sometimes necessary to consider line integrals of some function $X(x, y)$ along the length of an arc L:

$$\int_L X(x, y)\,ds = \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} X(x_i, y_i)\,\Delta s_i, \tag{4}$$

where ds is the differential of the arc. Such integrals are evaluated in similar fashion to the line integrals considered above. Let the curve L be represented by the parametric equations

$$x = \varphi(t), \quad y = \psi(t),$$

where $\varphi(t)$, $\psi(t)$, $\varphi'(t)$, $\psi'(t)$ are continuous functions of t.

Let α and β be values of the parameter t corresponding to the origin and terminus of the arc L.

Since

$$ds = \sqrt{\varphi'(t)^2 + \psi'(t)^2}\,dt,$$

we get a formula for evaluating integral (4):

$$\int_L X(x, y)\,ds = \int_\alpha^\beta X[\varphi(t), \psi(t)]\sqrt{\varphi'(t)^2 + \psi'(t)^2}\,dt.$$

We can consider the line integral along the arc of the space curve $x = \varphi(t)$, $y = \psi(t)$, $z = \chi(t)$:

$$\int_L X(x, y, z)\,ds = \int_\alpha^\beta X[\varphi(t), \psi(t), \chi(t)]\sqrt{\varphi'(t)^2 + \psi'(t)^2 + \chi'(t)^2}\,dt.$$

By the use of line integrals along an arc we can determine, for example, the coordinates of the centre of gravity of lines.

Reasoning as in Sec. 8, Ch. XII, we obtain a formula for evaluating the coordinates of the centre of gravity of a space curve.

$$x_c = \frac{\int_L x\,ds}{\int_L ds}, \quad y_c = \frac{\int_L y\,ds}{\int_L ds}, \quad z_c = \frac{\int_L z\,ds}{\int_L ds}. \tag{5}$$

Example. Find the coordinates of the centre of gravity of one turn of the helix

$$x = a\cos t, \quad y = a\sin t, \quad z = bt \quad (0 \leqslant t < 2\pi),$$

if its linear density is constant.

Solution. Applying formula (5), we find

$$x_c=\frac{\int\limits_0^{2\pi} a\cos t\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}\,dt}{\int\limits_0^{2\pi}\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}\,dt}=$$

$$=\frac{\int\limits_0^{2\pi} a\cos t\sqrt{a^2+b^2}\,dt}{\int\limits_0^{2\pi}\sqrt{a^2+b^2}\,dt}=\frac{a\sqrt{a^2+b^2}\,0}{2\pi\sqrt{a^2+b^2}}=0.$$

Similarly, $y_c=0$,

$$z_c=\frac{\int\limits_0^{2\pi} bt\sqrt{a^2\sin^2 t+a^2\cos^2 t+b^2}\,dt}{2\pi\sqrt{a^2+b^2}}=\frac{b\cdot 4\pi^2\sqrt{a^2+b^2}}{2\pi\sqrt{a^2+b^2}}=\pi b.$$

Thus, the coordinates of the centre of gravity of one turn of the helix are

$$x_c=0,\quad y_c=0,\quad z_c=\pi b.$$

SEC. 5. SURFACE INTEGRALS

Let a region V be given in an xyz-coordinate system. Let a surface σ bounded by a certain space line λ be given in V.

With respect to the surface σ we shall assume that at each point P of it the positive direction of the normal is determined by the unit vector $\boldsymbol{n}(P)$, the direction cosines of which are continuous functions of the coordinates of the surface points.

At each point of the surface let there be defined a vector,

$$\boldsymbol{F}=X(x,\ y,\ z)\,\boldsymbol{i}+Y(x,\ y,\ z)\,\boldsymbol{j}+Z(x,\ y,\ z)\,\boldsymbol{k},$$

where X, Y, Z are continuous functions of the coordinates.

Divide the surface in some way into subregions $\Delta\sigma_i$. In each subregion take an arbitrary point P_i and consider the sum

$$\sum_i (\boldsymbol{F}(P_i)\,\boldsymbol{n}(P_i))\,\Delta\sigma_i, \tag{1}$$

where $\boldsymbol{F}(P_i)$ is the value of the vector $\boldsymbol{F}$ at the point P_i of the subregion $\Delta\sigma_i$; $\boldsymbol{n}(P_i)$ is the unit normal vector at this point and $\boldsymbol{Fn}$ is the scalar product of these vectors.

The limit of the sum (1) extended over all subregions $\Delta\sigma_i$ as the diameters of all such subregions approach zero is called the

surface integral and is denoted by the symbol

$$\iint_{\sigma} \boldsymbol{Fn}\, d\sigma.$$

Thus, by definition*)

$$\lim_{\text{diam } \Delta\sigma_i \to 0} \sum \boldsymbol{F}_i \boldsymbol{n}_i \, \Delta\sigma_i = \iint_{\sigma} \boldsymbol{Fn}\, d\sigma. \tag{2}$$

Each term of the sum (1)

$$\boldsymbol{F}_i \boldsymbol{n}_i \Delta\sigma_i = F_i \Delta\sigma_i \cos(\boldsymbol{n}_i, \boldsymbol{F}_i) \tag{3}$$

may be interpreted mechanically as follows: this product is equal to the volume of a cylinder with base $\Delta\sigma_i$ and altitude $F_i \cos(\boldsymbol{n}_i, \boldsymbol{F}_i)$. If the vector $\boldsymbol{F}$ is the rate of flow of a liquid through the surface σ, then the product (3) is equal to the quantity of liquid flowing through the subregion $\Delta\sigma_i$ in unit time in the direction of the vector $\boldsymbol{n}_i$ (Fig. 336).

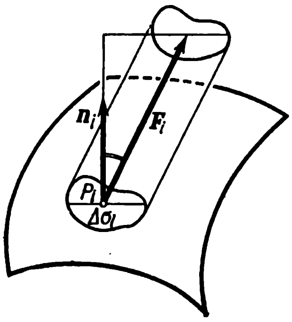


Fig. 336.

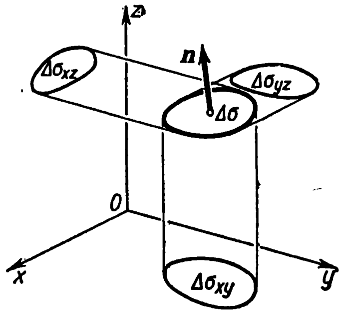


Fig. 337.

The expression $\iint_{\sigma} \boldsymbol{Fn}\, d\sigma$ yields the total quantity of liquid flowing in unit time through the surface σ in the positive direction if by the vector $\boldsymbol{F}$ we assume the flow-rate vector of the liquid at the given point. Therefore, the surface integral (2) is called the *flux of the vector field* $\boldsymbol{F}$ *through the surface* σ.

From the definition of a surface integral it follows that if the surface σ is divided into the parts $\sigma_1, \sigma_2, \ldots, \sigma_k$, then

$$\iint_{\sigma} \boldsymbol{Fn}\, d\sigma = \iint_{\sigma_1} \boldsymbol{Fn}\, d\sigma + \iint_{\sigma_2} \boldsymbol{Fn}\, d\sigma + \ldots + \iint_{\sigma_k} \boldsymbol{Fn}\, d\sigma.$$

*)If the surface σ is such that at each point of it there exists a tangent plane that constantly varies as the point P is translated over the surface, and if the vector function $\boldsymbol{F}$ is continuous on this surface, then this limit exists (we accept this existence theorem of a surface integral without proof).

Let us express the unit vector $\boldsymbol{n}$ in terms of its projections on the coordinate axes:

$$\boldsymbol{n}=\cos(n,\ x)\,\boldsymbol{i}+\cos(n,\ y)\,\boldsymbol{j}+\cos(n,\ z)\,\boldsymbol{k}.$$

Substituting into the integral (2) the expressions of the vectors $\boldsymbol{F}$ and $\boldsymbol{n}$ in terms of their projections, we get

$$\iint_{\sigma}\boldsymbol{Fn}\,d\sigma=\iint_{\sigma}[X\cos(n,\ x)+Y\cos(n,\ y)+Z\cos(n,\ z)]\,d\sigma. \quad (2')$$

The product $\Delta\sigma\cos(n,\ z)$ is the projection of subregion $\Delta\sigma$ on the xy-plane (Fig. 337); an analogous assertion holds true for the following products as well:

$$\Delta\sigma\cos(n,\ x)=\Delta\sigma_{yz},\quad \Delta\sigma\cos(n,\ y)=\Delta\sigma_{xz},\quad \Delta\sigma\cos(n,\ z)=\Delta\sigma_{xy}, \quad (4)$$

where $\Delta\sigma_{yz}$, $\Delta\sigma_{xz}$, $\Delta\sigma_{xy}$ are the projections of the subregion $\Delta\sigma$ on the appropriate coordinate planes.

On this basis, integral (2′) can also be written in the form

$$\iint_{\sigma}\boldsymbol{Fn}\,d\sigma=\iint_{\sigma}[X\cos(n,\ x)+Y\cos(n,\ y)+Z\cos(n,\ z)]\,d\sigma=$$
$$=\iint_{\sigma}X\,dy\,dz+Y\,dz\,dx+Z\,dx\,dy. \quad (2'')$$

SEC. 6. EVALUATING SURFACE INTEGRALS

Computing the integral over a curved surface reduces to evaluating a double integral over a plane region.

To illustrate, the following is a method of computing the integral

$$\iint_{\sigma}Z\cos(n,\ z)\,d\sigma.$$

Let the surface σ be such that any straight line parallel to the z-axis cuts it in one point. Then the equation of the surface may be written in the form

$$z=f(x,\ y).$$

Denoting by D the projection of the surface σ on the xy-plane, we get (by the definition of a surface integral)

$$\iint_{\sigma}Z(x,\ y,\ z)\cos(n,\ z)\,d\sigma=\lim_{\operatorname{diam}\Delta\sigma_i\to 0}\sum_{i=1}^{n}Z(x_i,\ y_i,\ z_i)\cos(n_i,\ z)\,\Delta\sigma_i.$$

Noting, further, the last of formulas (4), Sec. 5, we obtain

$$\iint_{\sigma} Z\cos(n,\ z)\,d\sigma = \lim_{\operatorname{diam}\Delta\sigma_{xy}\to 0}\sum_{i=1}^{n} Z(x_i,\ y_i,\ f(x_i,\ y_i))(\Delta\sigma_{xy})_i =$$

$$= \pm \lim_{\operatorname{diam}\Delta\sigma\to 0}\sum_{i=1}^{n} Z(x_i,\ y_i,\ f(x_i,\ y_i))\,|\Delta\sigma_{x,\,y}|_i;$$

the last expression is the integral sum for a double integral of the function $Z(x,\ y,\ f(x,\ y))$ over the region D. Therefore,

$$\iint_{\sigma} Z\cos(n,\ z)\,d\sigma = \pm\iint_{D} Z(x,\ y,\ f(x,\ y))\,dx\,dy.$$

The plus sign in front of the double integral is taken if $\cos(n,\ z)\geqslant 0$, the minus sign, if $\cos(n,\ z)\leqslant 0$.

If the surface σ does not satisfy the condition indicated at the beginning of this section, then it is divided into parts that satisfy this condition, and the integral is computed over each part separately.

The following integrals are computed in similar fashion:

$$\iint X\cos(n,\ x)\,d\sigma; \qquad \iint Y\cos(n,\ y)\,d\sigma.$$

The foregoing proof justifies the notation of a surface integral in the form of (2″), Sec. 5.

Here, the right side of (2″) may be regarded as the sum of double integrals over the appropriate projections of the region σ and the signs of these double integrals (or, otherwise stated, the signs of the products $dy\,dz$, $dx\,dz$, $dx\,dy$) are taken in accord with the foregoing rule.

Example 1. Let a closed surface σ be such that any straight line parallel to the z-axis cuts it in no more than two points.

Consider the integral

$$\iint_{\sigma} z\cos(n,\ z)\,d\sigma.$$

We shall call the outer normal the positive direction of the normal.

In this case, the surface may be divided into two parts: lower and upper; their equations are, respectively,

$$z=f_1(x,\ y) \quad\text{and}\quad z=f_2(x,\ y).$$

Denote by D the projection σ on the xy-plane (Fig. 338); then

$$\iint_{\sigma} z\cos(n,\ z)\,d\sigma = \iint_{D} f_2(x,\ y)\,dx\,dy - \iint_{D} f_1(x,\ y)\,dx\,dy.$$

The minus sign in the second integral is taken because in a surface integral the sign of $dx\,dy$ on a surface $z=f_1(x, y)$ must be taken negative, since for it $\cos(n, z)$ is negative.

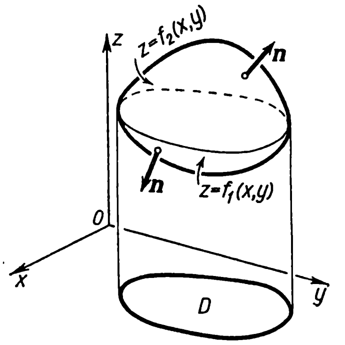


Fig. 338.

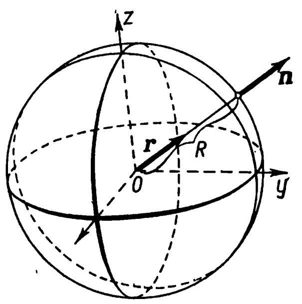


Fig. 339.

But the difference between the integrals on the right in the last formula yields a volume bounded by the surface σ. This means that the volume of the solid bounded by the closed surface σ is equal to the following integral over the surface

$$v=\iint_{\sigma} z\cos(n, z)\,d\sigma.$$

Example 2. A positive electric charge e placed at the coordinate origin creates a vector field such that at each point of space the vector $\boldsymbol{F}$ is defined by the Coulomb law as

$$\boldsymbol{F}=k\frac{e}{r^2}\boldsymbol{r},$$

where r is the distance of the given point from the origin, $\boldsymbol{r}$ is the unit vector directed along the radius vector of the given point (Fig. 339); and k is a constant coefficient.

Determine the vector-field flux through a sphere of radius R with centre at the origin of coordinates.

Solution. Taking into account that $r=R=\text{const}$, we will have

$$\iint_{\sigma} k\frac{e}{r^2}\boldsymbol{rn}\,d\sigma=\frac{ke}{R^2}\iint_{\sigma}\boldsymbol{rn}\,d\sigma.$$

But the last integral is equal to the area of the surface σ. Indeed, by the definition of an integral (noting that $\boldsymbol{rn}=1$), we obtain

$$\iint_{\sigma}\boldsymbol{rn}\,d\sigma=\lim_{\Delta\sigma_k\to 0}\sum \boldsymbol{r}_k\boldsymbol{n}_k\Delta\sigma_k=\lim_{\Delta\sigma_k\to 0}\sum\Delta\sigma_k=\sigma.$$

Hence, the flux is $\frac{ke}{R^2}\sigma=\frac{ke}{R^2}\cdot 4\pi R^2=4\pi ke$.

SEC. 7. STOKES' FORMULA

Let there be a surface σ such that any straight line parallel to the z-axis cuts it in one point. Denote by λ the boundary of the surface σ. Take the positive direction of the normal $\boldsymbol{n}$ so that it forms an acute angle with the positive z-axis (Fig. 340).

Let the equation of the surface be $z=f(x, y)$. The direction cosines of the normal are expressed by the formulas (see Sec. 6, Ch. IX):

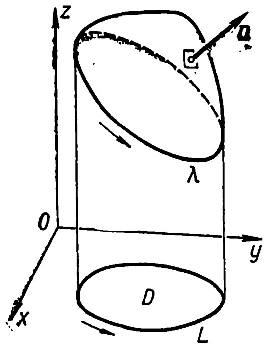


Fig. 340.

$$\left.\begin{aligned} \cos(n, x) &= \frac{-\frac{\partial f}{\partial x}}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}};\\ \cos(n, y) &= \frac{-\frac{\partial f}{\partial y}}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}};\\ \cos(n, z) &= \frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}. \end{aligned}\right\} \quad (1)$$

We shall assume that the surface σ lies entirely in some region V. Let there be a function $X(x, y, z)$ given in V that is continuous together with first-order partial derivatives. Consider the line integral along the curve λ:

$$\int_{\lambda} X(x, y, z)\,dx.$$

On the line λ, $z=f(x, y)$, where x, y are the coordinates of the points of the line L, which is a projection of the line λ on the xy-plane (Fig. 340). Thus, we can write the equation

$$\int_{\lambda} X(x, y, z)\,dx=\int_{L} X(x, y, f(x, y))\,dx. \quad (2)$$

The last integral is a line integral along L. Transform this integral by Green's formula, putting

$$X(x, y, f(x, y))=\overline{X}(x, y), \quad 0=\overline{Y}(x, y).$$

Substituting into Green's formula the expressions of $\overline{X}$ and $\overline{Y}$, we obtain

$$-\iint_{D} \frac{\partial X(x, y, f(x, y))}{\partial y}\,dx\,dy=\int_{L} X(x, y, f(x, y))\,dx, \quad (3)$$

where the region D is bounded by the line L. On the basis of the derivative of the composite function $X(x, y, f(x, y))$, where y enters both directly and in terms of the function $z=f(x, y)$, we find

$$\frac{\partial X(x, y, f(x, y))}{\partial y}=\frac{\partial X(x, y, z)}{\partial y}+\frac{\partial X(x, y, z)}{\partial z}\frac{\partial f(x, y)}{\partial y}. \quad (4)$$

Substituting expression (4) into the left side of (3), we obtain

$$-\iint_D \left[\frac{\partial X(x, y, z)}{\partial y}+\frac{\partial X(x, y, z)}{\partial z}\cdot\frac{\partial f(x, y)}{\partial y}\right] dx\,dy=$$

$$=\int_L X(x, y, f(x, y))\,dx.$$

Taking into account (2), the last equation may be rewritten as

$$\int_\lambda X(x, y, z)\,dx=-\iint_D \frac{\partial X}{\partial y}\,dx\,dy-\iint_D \frac{\partial X}{\partial z}\frac{\partial f}{\partial y}\,dx\,dy. \quad (5)$$

The last two integrals can be transformed into surface integrals. Indeed, from formula (2″), Sec. 5, it follows that if we have some function $A(x, y, z)$, the following equation is true:

$$\iint_\sigma A(x, y, z)\cos(n, z)\,d\sigma=\iint_D A\,dx\,dy.$$

On the basis of this equation, the integrals on the right side of (5) are transformed as follows:

$$\left.\begin{aligned}\iint_D \frac{\partial X}{\partial y}\,dx\,dy&=\iint_\sigma \frac{\partial X}{\partial y}\cos(n, z)\,d\sigma,\\ \iint_D \frac{\partial X}{\partial z}\frac{\partial f}{\partial y}\,dx\,dy&=\iint_\sigma \frac{\partial X}{\partial z}\frac{\partial f}{\partial y}\cos(n, z)\,d\sigma.\end{aligned}\right\} \quad (6)$$

Transform the last integral using formulas (1) of this section: dividing the second of these equations by the third termwise, we find

$$\frac{\cos(n, y)}{\cos(n, z)}=-\frac{\partial f}{\partial y}$$

or

$$\frac{\partial f}{\partial y}\cos(n, z)=-\cos(n, y).$$

Hence,

$$\iint_D \frac{\partial X}{\partial z}\frac{\partial f}{\partial y}\,dx\,dy=-\iint_\sigma \frac{\partial X}{\partial z}\cos(n, y)\,d\sigma. \quad (7)$$

Substituting expressions (6) and (7) into equation (5), we get

$$\int_{\lambda} X(x, y, z)\,dx = -\iint_{\sigma} \frac{\partial X}{\partial y}\cos(n, z)\,d\sigma + \iint_{\sigma} \frac{\partial X}{\partial z}\cos(n, y)\,d\sigma. \quad (8)$$

The direction of circulation of the contour λ must agree with the chosen direction of the positive normal n. Namely, if an observer looks from the end of the normal, he sees the circulation along the curve λ as being counterclockwise.

Formula (8) holds true for any surface if this surface can be divided into parts whose equations have the form $z=f(x, y)$.

Similarly, we can write the formulas

$$\int_{\lambda} Y(x, y, z)\,dy = \iint_{\sigma} \left[-\frac{\partial Y}{\partial z}\cos(n, x) + \frac{\partial Y}{\partial x}\cos(n, z)\right] d\sigma, \quad (8')$$

$$\int_{\lambda} Z(x, y, z)\,dz = \iint_{\sigma} \left[-\frac{\partial Z}{\partial x}\cos(n, y) + \frac{\partial Z}{\partial y}\cos(n, x)\right] d\sigma. \quad (8'')$$

Adding the left and right sides of (8), (8′), and (8″), we get the formula

$$\int_{\lambda} X\,dx + Y\,dy + Z\,dz = \iint_{\sigma} \left[\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\cos(n, z) + \right.$$
$$\left. + \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)\cos(n, x) + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right)\cos(n, y)\right] d\sigma. \quad (9)$$

This formula is called *Stokes' formula* after the English physicist and mathematician D. Stokes (1819-1903). It establishes a relationship between the integral over the surface σ and the line integral along the boundary λ of this surface, the circulation about the curve λ being performed according to the same rule as that given earlier.

The vector $\boldsymbol{B}$, defined by the projections

$$B_x = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}; \quad B_y = \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}; \quad B_z = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y},$$

is called the *curl* or *rotation* of the vector function $\boldsymbol{F} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ and is denoted by the symbol rot $\boldsymbol{F}$.

Thus, in vector notation, formula (9) will have the form

$$\int_{\lambda} \boldsymbol{F}\,d\boldsymbol{s} = \iint_{\sigma} \boldsymbol{n}\,\mathrm{rot}\,\boldsymbol{F}\,d\sigma, \quad (9')$$

and Stokes' theorem is formulated thus:

The circulation of a vector around the contour of some surface is equal to the flux of the curl through this surface.

Note. If the surface σ is a piece of plane parallel to the xy-plane, then $\Delta z = 0$, and we get Green's formula as a special case of Stokes' formula.

From formula (9) it follows that if

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = 0, \quad \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} = 0, \quad \frac{\partial X}{\partial z} - \frac{dZ}{dx} = 0, \tag{10}$$

then the line integral along any closed space curve λ is zero:

$$\int\limits_{\lambda} X\,dx + Y\,dy + Z\,dz = 0. \tag{11}$$

Whence it follows that the line integral is independent of the shape of the curve of integration.

As in the case of a plane curve, it may be shown that the indicated conditions are not only sufficient but also necessary.

In the fulfillment of these conditions, the expression under the integral sign is an exact differential of some function $u(x, y, z)$:

$$X\,dx + Y\,dy + Z\,dz = du(x, y, z)$$

and, consequently,

$$\int\limits_{(M)}^{(N)} X\,dx + Y\,dy + Z\,dz = \int\limits_{(M)}^{(N)} du = u(N) - u(M).$$

This is proved exactly like the corresponding formula for a function of two variables (see Sec. 4).

Example 1. Write the basic equations of the dynamics of a material point:

$$m\frac{dv_x}{dt} = X; \quad m\frac{dv_y}{dt} = Y; \quad m\frac{dv_z}{dt} = Z.$$

Here, m is the mass of the point, X, Y, Z are the projections of a force, acting on the point, onto the coordinate axes; $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$, $v_z = \frac{dz}{dt}$ are the projections of velocity v on the axes.

Multiply the left and right sides of these equations by the expressions

$$v_x\,dt = dx, \quad v_y\,dt = dy, \quad v_z\,dt = dz.$$

Adding the given equations term by term, we obtain

$$m(v_x\,dv_x + v_y\,dv_y + v_z\,dv_z) = X\,dx + Y\,dy + Z\,dz;$$

$$m\frac{1}{2}d(v_x^2 + v_y^2 + v_z^2) = X\,dx + Y\,dy + Z\,dz,$$

Since $v_x^2+v_y^2+v_z^2=v^2$, we can write

$$d\left(\frac{1}{2}mv^2\right)=X\,dx+Y\,dy+Z\,dz.$$

Take the integral along the trajectory connecting the points M_1 and M_2:

$$\frac{1}{2}mv_2^2-\frac{1}{2}mv_1^2=\int_{(M_1)}^{(M_2)} X\,dx+Y\,dy+Z\,dz,$$

where v_1 and v_2 are the velocities at the points M_1 and M_2.

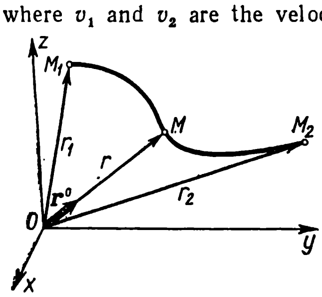


Fig. 341.

This last equation expresses the theorem of live forces: the increase in kinetic energy when passing from one point to another is equal to the work of the force acting on the mass m.

Example 2. Determine the work of the force of Newtonian attraction to a fixed centre of mass m in the translation of unit mass from $M_1(a_1, b_1, c_1)$ to $M_2(a_2, b_2, c_2)$.

Solution. Let the origin be in the fixed centre of attraction. Denote by $\boldsymbol{r}$ the radius vector of the point M (Fig. 341) corresponding to an arbitrary position of unit mass, and by $\boldsymbol{r}^0$ the unit vector directed along the vector $\boldsymbol{r}$. Then $\boldsymbol{F}=-\frac{km}{r^2}\boldsymbol{r}^0$, where k is the constant of gravitation. The projections of the force $\boldsymbol{F}$ on the coordinate axes will be

$$X=-km\frac{1}{r^2}\frac{x}{r};\quad Y=-km\frac{1}{r^2}\frac{y}{r},$$

$$Z=-km\frac{1}{r^2}\frac{z}{r}.$$

Then the work of the force $\boldsymbol{F}$ over the path M_1M_2 is

$$A=-km\int_{(M_1)}^{(M_2)}\frac{x\,dx+y\,dy+z\,dz}{r^3}=$$

$$=-km\int_{(M_1)}^{(M_2)}\frac{r\,dr}{r^3}=km\int_{(M_1)}^{(M_2)}d\left(\frac{1}{r}\right)$$

(since $r^2=x^2+y^2+z^2$, $r\,dr=x\,dx+y\,dy+z\,dz$). If we denote by r_1 and r^2 the lengths of the radius vectors of the points M_1 and M_2, then

$$A=km\left(\frac{1}{r^2}-\frac{1}{r_1}\right).$$

Thus, here again the line integral does not depend on the shape of the curve of integration, but only on the position of the initial and terminal points. The function $u=\frac{km}{r}$ is called the *potential* of the gravitational field

generated by the mass m. In the given case,

$$X=\frac{\partial u}{\partial x},\quad Y=\frac{\partial u}{\partial y},\quad Z=\frac{\partial u}{\partial z},$$

$$A=u(M_2)-u(M_1).$$

That is, the work done in moving unit mass is equal to the difference between the values of the potential at the terminal and initial points.

SEC. 8. OSTROGRADSKY'S FORMULA

Let there be given, in space, a regular three-dimensional region V bounded by a closed surface σ and projected on an xy-plane into a regular two-dimensional region D. We shall assume that the surface σ may be divided into three parts σ_1, σ_2, and σ_3 such that the equations of the first two have the form

$$z=f_1(x, y) \text{ and } z=f_2(x, y),$$

where $f_1(x, y)$ and $f_2(x, y)$ are functions continuous is the region D and the third part σ_3 is a cylindrical surface with generator parallel to the z-axis.

Consider the integral

$$I=\iiint\limits_V \frac{\partial Z(x, y, z)}{\partial z}\,dx\,dy\,dz.$$

First perform the integration with respect to z:

$$I=\iint\limits_D\left(\int\limits_{f_1(x, y)}^{f_2(x, y)}\frac{\partial Z}{\partial z}\,dz\right)dx\,dy=$$

$$=\iint\limits_D Z(x, y, f_2(x, y))\,dx\,dy-\iint\limits_D Z(x, y, f_1(x, y))\,dx\,dy. \quad (1)$$

On the normal to the surface, choose a definite direction, namely that which coincides with the direction of the outer normal to the surface σ. Then $\cos(n, z)$ will be positive on the surface σ_2 and negative on the surface σ_1; on the surface σ_3 it will be zero.

The double integrals on the right of (1) are equal to the corresponding surface integrals:

$$\iint\limits_D Z(x, y, f_2(x, y))\,dx\,dy=\iint\limits_{\sigma_2} Z(x, y, z)\cos(n, z)\,d\sigma, \quad (2')$$

$$\iint\limits_D Z(x, y, f_1(x, y))\,dx\,dy=\iint\limits_{\sigma_1} Z(x, y, z)(-\cos(n, z))\,d\sigma.$$

In the last integral we wrote $[-\cos(n, z)]$ because the elements of surface σ_1 and σ_2 and the element of area Δs of the region D are connected by the relation $\Delta s = \Delta\sigma\,[-\cos(n, z)]$, since the angle (n, z) is obtuse.

Thus,

$$\iint_D Z(x, y, f_1(x, y))\,dx\,dy = -\iint_{\sigma_1} Z(x, y, f_1(x, y))\cos(n, z)\,d\sigma. \quad (2'')$$

Substituting (2′) and (2″) into (1), we obtain

$$\iiint_V \frac{\partial Z(x, y, z)}{\partial z}\,dx\,dy\,dz =$$

$$= \iint_{\sigma_2} Z(x, y, z)\cos(n, z)\,d\sigma + \iint_{\sigma_1} Z(x, y, z)\cos(n, z)\,d\sigma.$$

For the sake of convenience in subsequent formulas, we shall rewrite the last equation as follows $\left[\text{adding} \iint_{\sigma_3} Z(x, y, z)\cos(n, z)\,d\sigma = 0,\right.$ since the equation $\cos(n, z) = 0$ is fulfilled on the surface $\left.\sigma_3\right]$:

$$\iiint_V \frac{\partial Z(x, y, z)}{\partial z}\,dx\,dy\,dz =$$

$$= \iint_{\sigma_2} Z\cos(n, z)\,d\sigma + \iint_{\sigma_1} Z\cos(n, z)\,d\sigma + \iint_{\sigma_3} Z\cos(n, z)\,d\sigma.$$

But the sum of integrals on the right of this equation is an integral over the entire closed surface σ; therefore,

$$\iiint_V \frac{\partial Z}{\partial z}\,dx\,dy\,dz = \iint_\sigma Z(x, y, z)\cos(n, z)\,d\sigma.$$

Analogously, we can obtain the relations

$$\iiint_V \frac{\partial Y}{\partial y}\,dx\,dy\,dz = \iint_\sigma Y(x, y, z)\cos(n, y)\,d\sigma,$$

$$\iiint_V \frac{\partial X}{\partial x}\,dx\,dy\,dz = \iint_\sigma X(x, y, z)\cos(n, x)\,d\sigma.$$

Adding together the last three equations term by term, we get *Ostrogradsky's formula* *):

$$\iiint\limits_V \left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)dx\,dy\,dz=$$

$$=\iint\limits_\sigma (X\cos(n,\ x)+Y\cos(n,\ y)+Z\cos(n,\ z))\,d\sigma. \qquad (2)$$

The expression $\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}$ is called the *divergence* of the vector (or the divergence of the vector function):

$$\boldsymbol{F}=X\boldsymbol{i}+Y\boldsymbol{j}+Z\boldsymbol{k}$$

and is denoted by the symbol div $\boldsymbol{F}$:

$$\operatorname{div}\boldsymbol{F}=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}.$$

We note that this formula holds good for any region which may be divided into subregions that satisfy the conditions indicated at the beginning of this section.

Let us examine a hydromechanical interpretation of this formula.

Let the vector $\boldsymbol{F}=X\boldsymbol{i}+Y\boldsymbol{j}+Z\boldsymbol{k}$ be the velocity vector of a liquid flowing through the region V. Then the surface integral in formula (2) is an integral of the projection of the vector $\boldsymbol{F}$ on the outer normal $\boldsymbol{n}$; it yields the quantity of liquid flowing out of the region V through the surface σ in unit time (or flowing into V if this integral is negative). This quantity is expressed in terms of the triple integral of div $\boldsymbol{F}$.

If div $\boldsymbol{F}\equiv 0$, then the double integral over any closed surface is equal to zero, that is, the quantity of liquid flowing out of (or into) something through any closed surface σ will be zero (no sources). More precisely, the quantity of liquid flowing into a region is equal to the quantity of liquid flowing out of this region.

In vector notation, Ostrogradsky's formula has the form

$$\iiint\limits_V \operatorname{div}\boldsymbol{F}\,dv=\iint\limits_\sigma \boldsymbol{F}\boldsymbol{n}\,ds \qquad (1')$$

*) This formula (sometimes called the Ostrogradsky-Gauss formula) was discovered by the noted Russian mathematician M. V. Ostrogradsky (1801-1861) and published in 1828 in an article entitled "A Note on the Theory of Heat".

and is read: *the integral of the divergence of a vector field* $\boldsymbol{F}$ *extended over some volume is equal to the vector flux through the surface bounding the given volume.*

SEC. 9. THE HAMILTONIAN OPERATOR AND CERTAIN APPLICATIONS OF IT

Suppose we have a function $u=u(x, y, z)$. At each point of the region in which the function $u(x, y, z)$ is defined and differentiable, the following gradient is determined:

$$\operatorname{grad} u=\boldsymbol{i}\frac{\partial u}{\partial x}+\boldsymbol{j}\frac{\partial u}{\partial y}+\boldsymbol{k}\frac{\partial u}{\partial z}. \tag{1}$$

The gradient of the function $u(x, y, z)$ is sometimes denoted as follows:

$$\nabla u=\boldsymbol{i}\frac{\partial u}{\partial x}+\boldsymbol{j}\frac{\partial u}{\partial y}+\boldsymbol{k}\frac{\partial u}{\partial z}. \tag{2}$$

The symbol ∇ is read "del".

1) It is convenient to write equation (2) symbolically as

$$\nabla u=\left(\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z}\right)u \tag{2'}$$

and to consider the symbol

$$\nabla=\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z} \tag{3}$$

as a "symbolic vector". This symbolic vector is called the *Hamiltonian operator* or del operator (∇-operator). From formulas (2) and (2′) it follows that "multiplication" of the symbolic vector ∇ by the scalar function u gives the gradient of this function:

$$\nabla u=\operatorname{grad} u. \tag{4}$$

2) We can form the scalar product of the symbolic vector ∇ by the vector $\boldsymbol{F}=\boldsymbol{i}X+\boldsymbol{j}Y+\boldsymbol{k}Z$:

$$\nabla\boldsymbol{F}=\left(\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z}\right)(\boldsymbol{i}X+\boldsymbol{j}Y+\boldsymbol{k}Z)=$$
$$=\frac{\partial}{\partial x}X+\frac{\partial}{\partial y}Y+\frac{\partial}{\partial z}Z=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=\operatorname{div}\boldsymbol{F}$$

(see Sec. 8). Thus,

$$\nabla\boldsymbol{F}=\operatorname{div}\boldsymbol{F}. \tag{5}$$

3) Form the vector product of the symbolic vector ∇ by the vector $\boldsymbol{F}=\boldsymbol{i}X+\boldsymbol{j}Y+\boldsymbol{k}Z$:

$$\nabla\times\boldsymbol{F}=\left(\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z}\right)\times(\boldsymbol{i}X+\boldsymbol{j}Y+\boldsymbol{k}Z)=$$

$$=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ X & Y & Z\end{vmatrix}=\boldsymbol{i}\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ Y & Z\end{vmatrix}-\boldsymbol{j}\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ X & Z\end{vmatrix}+\boldsymbol{k}\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ X & Y\end{vmatrix}=$$

$$=\boldsymbol{i}\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)-\boldsymbol{j}\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)+\boldsymbol{k}\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)=$$

$$=\boldsymbol{i}\left(\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z}\right)+\boldsymbol{j}\left(\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}\right)+\boldsymbol{k}\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)=\operatorname{rot}\boldsymbol{F}$$

(see Sec. 7). Thus,

$$\nabla\times\boldsymbol{F}=\operatorname{rot}\boldsymbol{F}. \tag{6}$$

From the foregoing it follows that vector operations may be greatly condensed by the use of the symbolic vector ∇. Let us consider several more formulas.

4) The vector field $\boldsymbol{F}(x, y, z)=\boldsymbol{i}X+\boldsymbol{j}Y+\boldsymbol{k}Z$ is called a *potential vector field* if the vector $\boldsymbol{F}$ is the gradient of some scalar function $u(x, y, z)$:

$$\boldsymbol{F}=\operatorname{grad} u$$

or

$$\boldsymbol{F}=\boldsymbol{i}\frac{\partial u}{\partial x}+\boldsymbol{j}\frac{\partial u}{\partial y}+\boldsymbol{k}\frac{\partial u}{\partial z}.$$

In this case the projections of the vector $\boldsymbol{F}$ will be

$$X=\frac{\partial u}{\partial x},\quad Y=\frac{\partial u}{\partial y},\quad Z=\frac{\partial u}{\partial z}.$$

From these equations it follows (see Ch. VIII, Sec. 12) that

$$\frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x},\quad \frac{\partial Y}{\partial z}=\frac{\partial Z}{\partial y},\quad \frac{\partial X}{\partial z}=\frac{\partial Z}{\partial x}$$

or

$$\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}=0,\quad \frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}=0,\quad \frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x}=0.$$

Hence, for the vector $\boldsymbol{F}$ under consideration,

$$\operatorname{rot}\boldsymbol{F}=0.$$

Thus, we get

$$\operatorname{rot}(\operatorname{grad} u)=0. \tag{7}$$

Applying the del operator ∇, we can write (7) as follows [on the basis of (4) and (5)]:

$$(\nabla \times \nabla u) = 0. \qquad (7')$$

Taking advantage of the property that for multiplication of a vector product by a scalar it is sufficient to multiply this scalar by one of the factors, we write

$$(\nabla \times \nabla)\, u = 0. \qquad (7'')$$

Here, the del operator again has the properties of an ordinary vector; the vector product of a vector into itself is zero.

The vector field $\boldsymbol{F}(x, y, z)$, for which $\operatorname{rot} \boldsymbol{F} = 0$, is called *irrotational*. From (7) it follows that every potential field is irrotational.

The converse also holds: if some vector field $\boldsymbol{F}$ is irrotational, then it is potential. The truth of this statement follows from reasoning given at the end of Sec. 7.

5) A vector field $\boldsymbol{F}(x, y, z)$ for which

$$\operatorname{div} \boldsymbol{F} = 0,$$

that is, a vector field in which there are no sources (see Sec. 8) is called *solenoidal*. We shall prove that

$$\operatorname{div}(\operatorname{rot} \boldsymbol{F}) = 0 \qquad (8)$$

or that the rotational field is free of sources.

Indeed, if $\boldsymbol{F} = \boldsymbol{i}X + \boldsymbol{j}Y + \boldsymbol{k}Z$, then

$$\operatorname{rot} \boldsymbol{F} = \boldsymbol{i}\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)$$

and therefore

$$\operatorname{div}(\operatorname{rot} \boldsymbol{F}) = \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) = 0.$$

Using the del operator, we can write equation (8) as

$$\nabla(\nabla \times \boldsymbol{F}) = 0. \qquad (8')$$

The left side of this equation may be regarded as a vector-scalar (mixed) product of three vectors: ∇, ∇, $\boldsymbol{F}$, of which two are the same. This product is obviously equal to zero.

6) Let there be a scalar field $u = u(x, y, z)$. Determine the gradient field:

$$\operatorname{grad} u = \boldsymbol{i}\frac{\partial u}{\partial x} + \boldsymbol{j}\frac{\partial u}{\partial y} + \boldsymbol{k}\frac{\partial u}{\partial z}.$$

Then find

$$\operatorname{div}(\operatorname{grad} u)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)$$

or

$$\operatorname{div}(\operatorname{grad} u)=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}. \qquad (9)$$

The right side of this expression is called the *Laplacian operator* of the function u and is denoted by

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}. \qquad (10)$$

Hence, (9) may be written as

$$\operatorname{div}(\operatorname{grad} u)=\Delta u. \qquad (11)$$

Using the del operator ∇ we can write (11) as

$$(\nabla\nabla u)=\Delta u. \qquad (11')$$

We note that the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad (12)$$

or

$$\Delta u=0 \qquad (12')$$

is called *Laplace's equation*. The function that satisfies the Laplace equation is called a *harmonic function*.

Exercises on Chapter XV

Compute the following line integrals:

1. $\int y^2\,dx+2xy\,dy$ over the circumference $x=a\cos t$, $y=a\sin t$. *Ans.* 0.

2. $\int y\,dx-x\,dy$ over an arc of the ellipse $x=a\cos t$, $y=b\sin t$. *Ans.* $-2\pi ab$.

3. $\int\left(\frac{x}{x^2+y^2}\,dx-\frac{y}{x^2+y^2}\,dy\right)$ over a circle with centre at the origin. *Ans.* 0.

4. $\int\left(\frac{y\,dx+x\,dy}{x^2+y^2}\right)$ over a segment of the straight line $y=x$ from $x=1$ to $x=2$. *Ans.* $\ln 2$.

5. $\int yz\,dx+xz\,dy+xy\,dz$ over an arc of the helix $x=a\cos t$, $y=\sin t$, $z=kt$ as t varies from 0 to 2π. *Ans.* 0.

6. $\int x\,dy-y\,dx$ over an arc of the hypocycloid $x=a\cos^3 t$, $y=a\sin^3 t$. *Ans.* $\frac{3}{4}\pi a^2$ (the double area of the hypocycloid).

7. $\int x\,dy-y\,dx$ over the loop of the folium of Descartes $x=\frac{3at}{1+t^3}$, $y=\frac{3at^2}{1+t^3}$. *Ans.* $\frac{3}{2}a^2$ (the double area of the region bounded by the indicated loop).

8. $\int x\,dy-y\,dx$ over the curve $x=a(t-\sin t)$, $y=a(1-\cos t)(0\leqslant t\leqslant 2\pi)$. *Ans.* $-6\pi a^2$ (the double area of the region bounded by one arc of a cycloid and the x-axis).

Prove that:

9. $\operatorname{grad}(c\varphi)=c\operatorname{grad}\varphi$ where c is a constant.

10. $\operatorname{grad}(c_1\varphi+c_2\psi)=c_1\operatorname{grad}\varphi+c_2\operatorname{grad}\psi$ where c is a constant.

11. $\operatorname{grad}(\varphi\psi)=\varphi\operatorname{grad}\psi+\psi\operatorname{grad}\varphi$.

12. Find $\operatorname{grad} r$, $\operatorname{grad} r^2$, $\operatorname{grad}\frac{1}{r}$, $\operatorname{grad} f(r)$ where $r=\sqrt{x^2+y^2+z^2}$. *Ans.* $\frac{\boldsymbol{r}}{r}$, $2\boldsymbol{r}$, $-\frac{\boldsymbol{r}}{r^3}$, $f'(r)\frac{\boldsymbol{r}}{r}$.

13. Prove that $\operatorname{div}(\boldsymbol{A}+\boldsymbol{B})=\operatorname{div}\boldsymbol{A}+\operatorname{div}\boldsymbol{B}$.

14. Compute $\operatorname{div}\boldsymbol{r}$, where $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$.
Ans. 3.

15. Compute $\operatorname{div}(\boldsymbol{A}\varphi)$, where $\boldsymbol{A}$ is a vector function and φ is a scalar function. *Ans.* $\varphi\operatorname{div}\boldsymbol{A}+(\operatorname{grad}\varphi\,\boldsymbol{A})$.

16. Compute $\operatorname{div}(\boldsymbol{r}\cdot\boldsymbol{c})$, where $\boldsymbol{c}$ is a constant vector. *Ans.* $\frac{(\boldsymbol{c}\cdot\boldsymbol{r})}{r}$.

17. Compute $\operatorname{div}\boldsymbol{B}(\boldsymbol{r}\boldsymbol{A})$. *Ans.* $\boldsymbol{AB}$.

Prove that:

18. $\operatorname{rot}(c_1\boldsymbol{A}_1+c_2\boldsymbol{A}_2)=c_1\operatorname{rot}\boldsymbol{A}_1+c_2\operatorname{rot}\boldsymbol{A}_2$ where c_1 and c_2 are constants.

19. $\operatorname{rot}(A\boldsymbol{c})=\operatorname{grad}A\times\boldsymbol{c}$ where $\boldsymbol{c}$ is a constant vector.

20. $\operatorname{rot}\operatorname{rot}\boldsymbol{A}=\operatorname{grad}\operatorname{div}\boldsymbol{A}-\nabla\boldsymbol{A}$.

21. $\boldsymbol{A}\times\operatorname{rot}\varphi=\operatorname{rot}(\varphi\boldsymbol{A})$.

Surface Integrals

22. Prove that $\iint\cos(n,z)\,d\sigma=0$ if σ is a closed surface and n is a normal to it.

23. Find the moment of inertia of the surface of a segment of a sphere with equation $x^2+y^2+z^2=R^2$ cut off by the plane $z=H$ relative to the z-axis. *Ans.* $\frac{2\pi R}{3}(2R^3-3R^2H+H^3)$.

24. Find the moment of inertia of the surface of the paraboloid of revolution $x^2+y^2=2cz$ cut off by the plane $z=c$ relative to the z-axis. *Ans.* $\left[\frac{55+9\sqrt{3}}{65}c^2\right]$.

25. Compute the coordinates of the centre of gravity of a part of the surface of the cone $x^2+y^2=\frac{R^2}{H^2}z^2$ cut off by the plane $z=H$. *Ans.* 0, 0, $\frac{2}{3}H$.

26. Compute the coordinates of the centre of gravity of a segment of the surface of the sphere $x^2+y^2+z^2=R^2$ cut off by the plane $z=H$. *Ans.* $\left(0,\ 0,\ \frac{R+H}{2}\right)$.

27. Find $\iint\limits_{\sigma} [x\cos(nx)+y\cos(ny)+z\cos(nz)]\,d\sigma$, where σ is a closed surface. *Ans.* $3V$, where V is the volume of the solid bounded by the surface σ.

28. Find $\iint\limits_{S} z\,dx\,dy$ where S is the external side of a sphere $x^2+y^2+z^2=$ $=R^2$. *Ans.* $\frac{4}{3}\pi r^3$.

29. Find $\iint\limits_{S} x^2\,dy\,dz+y^2\,dz\,dx+z^2\,dx\,dy$ where S is the external side of the surface of a sphere $x^2+y^2+z^2=R^2$. *Ans.* πR^4.

30. Find $\iint\limits_{S} \sqrt{x^2+y^2}\,ds$ where S is the lateral surface of a cone $\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=0$, $0\leqslant z\leqslant b$. *Ans.* $\frac{2\pi a^2\sqrt{a^2+b^2}}{3}$.

31. Using the Stokes formula, transform the integral $\int\limits_{L} y\,dx+z\,dy+x\,dz$. *Ans.* $-\iint\limits_{S} (\cos\alpha+\cos\beta+\cos\gamma)\,ds$.

Find the line integrals, applying the Stokes formula and directly: **32.** $\int\limits_{L} (y+z)\,dx+(z+x)\,dy+(x+y)\,dz$ where L is the circle $x^2+y^2+z^2=$ $=a^2$, $x+y+z=0$. *Ans.* 0. **33.** $\int\limits_{L} x^2y^3dx+dy+z\,dz$ where L is the circle $x^2+y^2=R^2$, $z=0$. *Ans.* $-\frac{\pi R^6}{8}$.

Applying the Ostrogradsky formula, transform the surface integrals into volume integrals: **34.** $\iint\limits_{S} (x\cos\alpha+y\cos\beta+z\cos\gamma)\,ds$. *Ans.* $\iiint\limits_{V} 3\,dx\,dy\,dz$.

35. $\iint\limits_{S} (x^2+y^2+z^2)(dy\,dz+dx\,dz+dx\,dy)$. *Ans.* $\iiint\limits_{V} (x+y+z)\,dx\,dy\,dz$.

36. $\iint\limits_{S} xy\,dx\,dy+yz\,dy\,dz+zx\,dz\,dx$. *Ans.* 0. **37.** $\iint\limits_{S} \frac{\partial u}{\partial x}\,dy\,dz+\frac{\partial u}{\partial y}\,dx\,dz+$ $+\frac{\partial u}{\partial z}\,dx\,dy$. *Ans.* $\iiint\limits_{V}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)dx\,dy\,dz$.

Using the Ostrogradsky formula compute the following integrals: **38.** $\iint\limits_{S} (x\cos\alpha+y\cos\beta+z\cos\gamma)\,ds$ where S is the surface of the ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. *Ans.* $4\pi abc$. **39.** $\iint\limits_S (x^3\cos\alpha+y^3\cos\beta+z^3\cos\gamma)\,ds$ where S is the surface of the sphere $x^2+y^2+z^2=R^2$. *Ans.* $\frac{12}{5}\pi R^5$. **40.** $\iint\limits_S x^2\,dy\,dz+y^2\,dz\,dx+z^2\,dx\,dy$ where S is the surface of the cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ $(0\leqslant z\leqslant b)$. *Ans.* $\frac{\pi a^2b^2}{2}$. **41.** $\iint\limits_S x\,dy\,dz+y\,dx\,dz+z\,dx\,dy$ where S is the surface of the cylinder $x^2+y^2=a^2$, $-H\leqslant z\leqslant H$. *Ans.* $3\pi a^2H$.

42. Prove the identity $\iint\limits_D\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)dx\,dy=\int\limits_C\frac{\partial u}{\partial n}\,ds$, where C is a contour bounding the region D, and $\frac{\partial u}{\partial n}$ is the directional derivative of the outer normal.

Solution.

$$\iint\limits_D\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)dx\;dy=\int\limits_C -Y\,dx+X\,dy=\int\limits_C\,[-Y\cos(s,\ x)+X\sin(s,\ x)]\,ds,$$

where $(s,\ x)$ is the angle between the tangent line to the contour C and the x-axis. If we denote by $(n,\ x)$ the angle between the normal and the x-axis, then $\sin(s,\ x)=\cos(n,\ x)$, $\cos(s,\ x)=-\sin(n,\ x)$. Hence,

$$\iint\limits_D\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)dx\;dy=\int\limits_C[X\cos(n,\ x)+Y\sin(n,\ x)]\,ds.$$

Setting $X=\frac{\partial u}{\partial x}$, $\quad Y=\frac{\partial u}{\partial y}$, we get

$$\iint\limits_D\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)dx\;dy=\int\limits_C\left[\frac{\partial u}{\partial x}\cos(n,\ x)+\frac{\partial u}{\partial y}\sin(n,\ x)\right]ds$$

or

$$\iint\limits_D\left[\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right]dx\;dy=\int\limits_C\frac{\partial u}{\partial n}\,ds.$$

The expression $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}$ is called the Laplacian operator.

43. Prove the identity (called Green's formula)

$$\iiint\limits_V(v\,\Delta u-u\,\Delta v)\,dx\,dy\,dz=\iint\limits_\sigma\left(v\,\frac{\partial u}{\partial n}-u\,\frac{\partial v}{\partial n}\right)d\sigma,$$

where u and v are continuous functions with continuous derivatives to the second order in the region D.

The symbols Δu and Δv denote

$$\Delta u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2},\quad \Delta v=\frac{\partial^2v}{\partial x^2}+\frac{\partial^2v}{\partial y^2}+\frac{\partial^2v}{\partial z^2}.$$

These expressions are called Laplacian operators in space.

Solution. In the formula

$$\iiint\limits_V \left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right) dx\,dy\,dz=\iint\limits_\sigma [X\cos(n,\ x)+Y\cos(n,\ y)+Z\cos(n,\ z)]\,d\sigma$$

we put

$$X=vu_x'-uv_x',$$
$$Y=vu_y'-uv_y',$$
$$Z=vu_z'-uv_z'.$$

Then

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=v\,(u_{xx}''+u_{yy}''+u_{zz}'')-u(v_{xx}''+v_{yy}''+v_{zz}'')=v\Delta u-u\Delta v,$$

$$X\cos(n,\ x)+Y\cos(n,\ y)+Z\cos(n,\ z)=$$
$$=v\,(u_x'\cos nx+u_y'\cos ny+u_z'\cos nz)-u\,(v_x'\cos nx+v_y'\cos ny+v_z'\cos nz)=$$
$$=v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}.$$

Hence,

$$\iiint\limits_V (v\Delta u-u\Delta v)\,dx\,dy\,dz=\iint\limits_\sigma\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)d\sigma.$$

44. Prove the identity

$$\iiint\limits_V \Delta u\,dx\,dy\,dz=\iint\limits_\sigma \frac{\partial u}{\partial n}\,d\sigma,$$

where $\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$ (Laplacian).

Solution. In Green's formula, which was derived in the preceding section, put $v=1$. Then $\Delta v=0$, and we get the desired identity.

45. If $u(x, y, z)$ is a harmonic function in some region, that is, a function which at every point of this region satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0,$$

then

$$\iint\limits_\sigma \frac{\partial u}{\partial n}d\sigma=0$$

where σ is a closed surface.

Solution. This follows directly from the formula of Problem 44.

46. Let $u(x,\ y,\ z)$ be a harmonic function in some region V and let there be, in V, a sphere $\bar{\sigma}$ with centre at the point $M(x_1,\ y_1,\ z_1)$ and with radius R. Prove that

$$u(x_1,\ y_1,\ z_1)=\frac{1}{4\pi R^2}\iint\limits_{\bar{\sigma}} u\,d\sigma.$$

Solution. Consider the region Ω bunded by two spheres σ, $\overline{\sigma}$ of radius R and ϱ ($\varrho < R$) with centres at the point $M(x_1, y_1, z_1)$. Apply Green's formula (found in Problem 43) to this region, taking for u the above-indicated function, and for the function v,

$$v=\frac{1}{r}=\frac{1}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}.$$

By direct differentiation and substitution we are convinced that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$. Consequently,

$$\iint\limits_{\underline{\sigma}+\overline{\sigma}}\left(\frac{1}{r}\frac{\partial u}{\partial n}-u\frac{\partial \frac{1}{r}}{\partial n}\right)d\sigma=0$$

or

$$\iint\limits_{\underline{\sigma}}\left(\frac{1}{r}\frac{\partial u}{\partial n}-u\frac{\partial\left(\frac{1}{r}\right)}{\partial n}\right)d\sigma+\iint\limits_{\overline{\sigma}}\left(\frac{1}{r}\frac{\partial u}{\partial n}-u\frac{\partial\left(\frac{1}{r}\right)}{\partial n}\right)d\sigma=0.$$

On the surfaces $\overline{\sigma}$ and $\underline{\sigma}$ the quantity $\frac{1}{r}$ is constant $\left(\frac{1}{R}\text{ and }\frac{1}{\varrho}\right)$ and so can be taken outside the integral sign. By virtue of the result obtained in Problem 45, we have

$$\frac{1}{R}\iint\limits_{\underline{\sigma}}\frac{\partial u}{\partial n}d\sigma=0;$$

$$\frac{1}{\varrho}\iint\limits_{\overline{\sigma}}\frac{\partial u}{\partial n}d\sigma=0.$$

Hence,

$$-\iint\limits_{\underline{\sigma}}u\frac{\partial\left(\frac{1}{r}\right)}{\partial n}+\iint\limits_{\overline{\sigma}}u\frac{\partial\left(\frac{1}{r}\right)}{\partial n}d\sigma=0,$$

but

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial n}=\frac{d\left(\frac{1}{r}\right)}{dr}=-\frac{1}{r^2}.$$

Therefore,

$$+\iint\limits_{\sigma}u\frac{1}{r^2}d\sigma-\iint\limits_{\overline{\sigma}}u\frac{1}{r^2}d\sigma=0$$

or

$$\frac{1}{\varrho^2}\iint\limits_{\underline{\sigma}}u\,d\sigma=\frac{1}{R^2}\iint\limits_{\overline{\sigma}}u\,d\sigma. \tag{1}$$

Apply the theorem of the mean to the integral on the right:

$$\frac{1}{\varrho^2}\iint\limits_{\sigma} u\,d\sigma=\frac{u(\xi,\ \eta,\ \zeta)}{\varrho^2}\iint\limits_{\sigma} d\sigma, \qquad (2)$$

where $u(\xi, \eta, \zeta)$ is a point on the surface of a sphere of radius ϱ with centre at the point $M(x_1, y_1, z_1)$.

We make ϱ approach zero; then $u(\xi, \eta, \zeta) \to u(x_1, y_1, z_1)$;

$$\frac{1}{\varrho^2}\iint\limits_{\sigma} d\sigma=\frac{4\pi\varrho^2}{\varrho^2}=4\pi.$$

Hence, as $\varrho \to 0$ we get

$$\frac{1}{\varrho^2}\iint\limits_{\sigma} u\,d\sigma \to u(x_1,\ y_1,\ z_1)\,4\pi.$$

Further, since the left side of (1) is independent of ϱ, it follows that a $\varrho \to 0$ we finally get

$$\frac{1}{R^2}\iint\limits_{\sigma} u\,d\sigma=4\pi u(x_1,\ y_1,\ z_1)$$

or

$$u(x_1,\ y_1,\ z_1)=\frac{1}{4\pi R^2}\iint\limits_{\sigma} u\,d\sigma.$$

CHAPTER XVI

SERIES

SEC. 1. SERIES. SUM OF A SERIES

Definition 1. Let there be given an infinite sequence of numbers *)

$$u_1, u_2, u_3, \ldots, u_n, \ldots$$

The expression

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots \qquad (1)$$

is called a *numerical series*. Here, the numbers $u_1, u_2, \ldots, u_n, \ldots$ are called the *terms of the series*.

Definition 2. The sum of a finite number of terms (the first n terms) of a series is called the *nth partial sum of the series:*

$$s_n = u_1 + u_2 + \ldots + u_n.$$

Consider the partial sums

$$s_1 = u_1,$$
$$s_2 = u_1 + u_2,$$
$$s_3 = u_1 + u_2 + u_3,$$
$$\cdots\cdots\cdots\cdots$$
$$s_n = u_1 + u_2 + u_3 + \ldots + u_n.$$

If there exists a finite limit

$$s = \lim_{n \to \infty} s_n,$$

it is called the *sum of the series* (1) and we say that the *series converges*.

If $\lim\limits_{n \to \infty} s_n$ does not exist (for example, $s_n \longrightarrow \infty$ as $n \longrightarrow \infty$), then we say that the *series* (1) *diverges and has no sum.*

Example. Consider the series

$$a + aq + aq^2 + \ldots + aq^{n-1} + \ldots \qquad (2)$$

This is a **geometric progression** with first term a and ratio q $(a \neq 0)$.

---

*) A sequence is considered specified if we know the law by which it is possible to determine any term u_n for a given n.

The sum of the first n terms of the geometric progression is (when $q \neq 1$)

$$s_n = \frac{a - aq^n}{1-q}$$

or

$$s_n = \frac{a}{1-q} - \frac{aq^n}{1-q}.$$

1) If $|q| < 1$, then $q^n \to 0$ as $n \to \infty$ and, consequently,

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(\frac{a}{1-q} - \frac{aq^n}{1-q} \right) = \frac{a}{1-q}.$$

Hence, in the case of $|q| < 1$, the series (2) converges and its sum is

$$s = \frac{a}{1-q}.$$

2) If $|q| > 1$, then $|q^n| \to \infty$ as $n \to \infty$ and then $\frac{a - aq^n}{1-q} \to \pm\infty$ as $n \to \infty$, that is, $\lim\limits_{n \to \infty} s_n$ does not exist. Thus, when $|q| > 1$, the series (2) diverges.

3) If $q = 1$, then the series (2) has the form

$$a + a + a + \ldots$$

In this case

$$s_n = na, \quad \lim_{n \to \infty} s_n = \infty,$$

and the series diverges.

4) If $q = -1$, then the series (2) has the form

$$a - a + a - a + \ldots$$

In this case

$$s_n = \begin{cases} 0 \text{ when } n \text{ is even,} \\ a \text{ when } n \text{ is odd.} \end{cases}$$

Thus, s_n has no limit and the series diverges.

Thus, a geometric progression (with first term different from zero) converges only when the ratio of the progression is less than unity in absolute value.

Theorem 1. *If a series obtained from a given series* (1) *by suppression of some of its terms converges, then the given series itself converges.*

Conversely, if a given series converges, then a series obtained from the given series by suppression of several terms also converges. In other words, *the convergence of a series is not affected by the suppression of a finite number of its terms.*

Proof. Let s_n be the sum of the first n terms of the series (1), c_k, the sum of k suppressed terms (we note that for a sufficiently large n, all suppressed terms are contained in the sum s_n), and σ_{n-k} is the sum of the terms of the series that enter into the

sum s_n but do not enter into c_k. Then we have

$$s_n = c_k + \sigma_{n-k},$$

where c_k is a constant that is independent of n.

From the last relationship it follows that if $\lim\limits_{n\to\infty} \sigma_{n-k}$ exists, then $\lim\limits_{n\to\infty} s_n$ exists as well; if $\lim\limits_{n\to\infty} s_n$ exists, then $\lim\limits_{n\to\infty} \sigma_{n-k}$ also exists; which proves the theorem.

We conclude this section with two simple properties of series.

Theorem 2. *If a series*

$$a_1 + a_2 + \dots \qquad (3)$$

converges and its sum is s, then the series

$$ca_1 + ca_2 + \dots \qquad (4)$$

where c is some fixed number, also converges, and its sum is cs.

Proof. Denote the nth partial sum of the series (3) by s_n, and that of the series (4), by σ_n. Then

$$\sigma_n = ca_1 + \dots + ca_n = c(a_1 + \dots + a_n) = cs_n.$$

Whence it is clear that the limit of the nth partial sum of the series (4) exists, since

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} (cs_n) = c \lim_{n\to\infty} s_n = cs.$$

Thus, the series (4) converges and its sum is equal to cs.

Theorem 3. *If the series*

$$a_1 + a_2 + \dots \qquad (5)$$

and

$$b_1 + b_2 + \dots \qquad (6)$$

converge and their sums, respectively, are $\bar{s}$ and $\bar{\bar{s}}$, then the series

$$(a_1 + b_1) + (a_2 + b_2) + \dots \qquad (7)$$

and

$$(a_1 - b_1) + (a_2 - b_2) + \dots \qquad (8)$$

also converge and their sums are $\bar{s} + \bar{\bar{s}}$ and $\bar{s} - \bar{\bar{s}}$, respectively.

Proof. We prove the convergence of the series (7). Denoting its nth partial sum by σ_n and the nth partial sums of the series (5) and (6) by $\bar{s}_n$ and $\bar{\bar{s}}_n$, respectively, we get

$$\sigma_n = (a_1 + b_1) + \dots + (a_n + b_n) =$$
$$= (a_1 + \dots + a_n) + (b_1 + \dots + b_n) = \bar{s}_n + \bar{\bar{s}}_n.$$

Passing to the limit in this equation as $n \to \infty$, we get

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} (\bar{s}_n + \bar{\bar{s}}_n) = \lim_{n\to\infty} \bar{s}_n + \lim_{n\to\infty} \bar{\bar{s}}_n = \bar{s} + \bar{\bar{s}}.$$

Thus, the series (7) converges and its sum is $\bar{s} + \bar{\bar{s}}$.

It is analogously proved that the series (8) also converges and its sum is equal to $\bar{s} - \bar{\bar{s}}$.

Of the series (7) and (8) it is said that they were obtained by means of termwise addition or, respectively, termwise subtraction of the series (5) and (6).

SEC. 2. NECESSARY CONDITION FOR CONVERGENCE OF A SERIES

One of the basic questions, when investigating series, is that of whether the given series converges or diverges. We shall establish sufficient conditions for one to decide this question. We shall also examine the necessary condition for convergence of a series; in other words, we shall establish a condition for which the series will diverge if it is not fulfilled.

Theorem. *If a series converges, its nth term approaches zero as n becomes infinite.*

Proof. Let the series

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

converge; that is, let us have the equality

$$\lim_{n\to\infty} s_n = s,$$

where s is the sum of the series (a finite fixed number). But then we also have the equation

$$\lim_{n\to\infty} s_{n-1} = s,$$

since $(n-1)$ also tends to infinity as $n \to \infty$. Subtracting the second equation from the first termwise, we obtain

$$\lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = 0$$

or

$$\lim_{n\to\infty} (s_n - s_{n-1}) = 0.$$

But

$$s_n - s_{n-1} = u_n.$$

Hence,

$$\lim_{n \to 0} u_n = 0,$$

which is what was to be proved.

Corollary. *If the nth term of a series does not tend to zero as $n \longrightarrow \infty$, then the series diverges.*

Example. The series

$$\frac{1}{3}+\frac{2}{5}+\frac{3}{7}+\ldots+\frac{n}{2n+1}+\ldots$$

diverges, since

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \left(\frac{n}{2n+1}\right) = \frac{1}{2} \neq 0.$$

We stress the fact that this condition is only a necessary condition, but not a sufficient condition; in other words, *from the fact that the nth term approaches zero, it does not follow that the series converges*, for the series may diverge.

For example, the so-called *harmonic series*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$$

diverges, although

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0.$$

To prove this, write the harmonic series in more detail:

$$1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}+$$

$$+\underbrace{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}+\frac{1}{17}+\ldots \tag{1}$$

We also write the auxiliary series

$$1+\frac{1}{2}+\overbrace{\frac{1}{4}+\frac{1}{4}}+\overbrace{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}+$$

$$+\overbrace{\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}}+\overbrace{\frac{1}{32}+\ldots+\frac{1}{32}}^{16 \text{ terms}}+\ldots \tag{2}$$

The series (2) is constructed as follows: its first term is equal to unity, its second is $^1/_2$, its third and fourth are $^1/_4$, the fifth to the eighth terms are equal to $^1/_8$, the terms 9 to 16 are equal to $^1/_{16}$, the terms 17 to 32 are equal to $^1/_{32}$, etc.

Denote by $s_n^{(1)}$ the sum of the first n terms of the harmonic series (1) and by $s_n^{(2)}$ the sum of the first n terms of the series (2).

Since each term of the series (1) is greater than the corresponding term of the series (2) or equal to it, then for $n > 2$

$$s_n^{(1)} > s_n^{(2)}. \tag{3}$$

We compute the partial sums of the series (2) for values of n equal to 2^1, 2^2, 2^3, 2^4, 2^5:

$$s_2 = 1 + \frac{1}{2} = \frac{3}{2},$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2 \cdot \frac{1}{2},$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3 \cdot \frac{1}{2},$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \underbrace{\left(\frac{1}{8} + \dots + \frac{1}{8}\right)}_{4 \text{ terms}} + \underbrace{\left(\frac{1}{16} + \dots + \frac{1}{16}\right)}_{8 \text{ terms}} = 1 + 4 \cdot \frac{1}{2},$$

$$s_{32} = 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \underbrace{\left(\frac{1}{8} + \dots + \frac{1}{8}\right)}_{4 \text{ terms}} + \underbrace{\left(\frac{1}{16} + \dots + \frac{1}{16}\right)}_{8 \text{ terms}} + \underbrace{\left(\frac{1}{32} + \dots + \frac{1}{32}\right)}_{16 \text{ terms}} = 1 + 5 \cdot \frac{1}{2};$$

in the same way we find that $s_{2^6} = 1 + 6 \cdot \frac{1}{2}$, $s_{2^7} = 1 + 7 \cdot \frac{1}{2}$ and, generally, $s_{2^k} = 1 + k \cdot \frac{1}{2}$.

Thus, for sufficiently large k, the partial sums of the series (2) can be made greater than any positive number; that is,

$$\lim_{n \to \infty} s_n^{(2)} = \infty,$$

but then from the relation (3) it also follows that

$$\lim_{n \to \infty} s_n^{(1)} = \infty$$

which means that the harmonic series (1) diverges.

SEC. 3. COMPARING SERIES WITH POSITIVE TERMS

Suppose we have two series with **positive** terms:

$$u_1+u_2+u_3+\ldots+u_n+\ldots, \tag{1}$$

$$v_1+v_2+v_3+\ldots+v_n+\ldots \tag{2}$$

For them the following assertions hold true.

Theorem 1. *If the terms of the series* (1) *are not greater than the corresponding terms of the series* (2); *that is,*

$$u_n \leqslant v_n \quad (n=1,\ 2,\ \ldots) \tag{3}$$

and the series (2) *converges, then the series* (1) *also converges.*

Proof. Denote by s_n and σ_n, respectively, the partial sums of the first and second series:

$$S_n=\sum_{i=1}^{n} u_i, \qquad \sigma_n=\sum_{i=1}^{n} v_i.$$

From the condition (3) it follows that

$$S_n \leqslant \sigma_n. \tag{4}$$

Since the series (2) converges, its partial sum has a limit σ:

$$\lim_{n\to\infty} \sigma_n=\sigma.$$

From the fact that the terms of the series (1) and (2) are positive, it follows that $\sigma_n<\sigma$, and then by virtue of (4)

$$s_n<\sigma.$$

We have thus proved that the partial sums s_n are bounded. We note that as n increases, the partial sum s_n increases, and from the fact that the sequence of partial sums is bounded and increases, it follows that it has a limit *)

$$\lim_{n\to\infty} s_n=s,$$

and it is obvious that

$$s \leqslant \sigma.$$

Using Theorem 1, we can judge of the convergence of certain series.

---

*) To convince ourselves that the variable s_n has a limit, let us recall a condition for the existence of a limit of a sequence (see Ch. II): "if a variable is bounded and increases, it has a limit." Here, the sequence of sums s_n is bounded and increases. Hence it has a limit, i. e., the series converges.

Example 1. The series

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots+\frac{1}{n^n}+\ldots$$

converges because its terms are smaller than the corresponding terms of the series

$$1+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots+\frac{1}{2^n}+\ldots$$

But the last series converges because its terms, beginning with the second, form a geometric progression with common ratio $\frac{1}{2}$. The sum of this series is equal to $1\frac{1}{2}$. Hence, by virtue of Theorem 1, the given series also converges, and the sum does not exceed $1\frac{1}{2}$.

Theorem 2. *If the terms of the series* (1) *are not smaller than the corresponding terms of the series* (2); *that is,*

$$u_n \geqslant v_n \tag{5}$$

and the series (2) *diverges, then the series* (1) *also diverges.*

Proof. From condition (5) it follows that

$$s_n \geqslant \sigma_n. \tag{6}$$

Since the terms of the series (2) are positive, its partial sum σ_n increases with increasing n, and since it diverges, it follows that

$$\lim_{n\to\infty} \sigma_n = \infty.$$

But then, by virtue of (6),

$$\lim_{n\to\infty} s_n = \infty,$$

the series (1) diverges.

Example 2. The series

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$$

diverges because its terms (from the second on) are greater than the corresponding terms of the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots,$$

which, as we know, diverges.

Note. Both the conditions that we have proved (Theorems 1 and 2) hold only for series with positive terms. They also hold

true when some of the terms of the first or second series are zero. But these conditions do not hold if some of the terms of the series are negative numbers.

SEC. 4. D'ALEMBERT'S TEST

Theorem (d'Alembert's Test). *If in a series with positive terms*

$$u_1+u_2+u_3+\ldots+u_n+\ldots \tag{1}$$

the ratio of the $(n+1)$*st term to the nth term, as* $n\to\infty$, *has a (finite) limit* l, *that is,*

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=l, \tag{2}$$

then:

1) *the series converges for* $l<1$,
2) *the series diverges for* $l>1$.

(For $l=1$, the theorem cannot determine the convergence or divergence of the series.)

Proof. 1) Let $l<1$. Consider a number q that satisfies the relationship $l<q<1$ (Fig. 342).

From the definition of a limit and relation (2) it follows that for all values of n after a certain integer N, that is, for $n\geqslant N$, we will have the inequality

$$\frac{u_{n+1}}{u_n}<q. \tag{2'}$$

Indeed, since the quantity $\frac{u_{n+1}}{u_n}$ tends to the limit l, the difference between the quantity $\frac{u_{n+1}}{u_n}$ and the number l may (after a certain N) be made less (in absolute value) than any positive number, in particular less than $q-l$; that is,

$$\left|\frac{u_{n+1}}{u_n}-l\right|<q-l.$$

Inequality (2′) follows from this last inequality. Writing this inequality for various values of n, from N onwards, we get

$$\left.\begin{array}{l} u_{N+1}<qu_N, \\ u_{N+2}<qu_{N+1}<q^2u_N, \\ u_{N+3}<qu_{N+2}<q^2u_N, \\ \ldots\ldots\ldots\ldots \end{array}\right\} \tag{3}$$

Now consider the two series

$$u_1+u_2+u_3+\ldots+u_N+u_{N+1}+u_{N+2}+\ldots, \qquad (1)$$

$$u_N+qu_N+q^2u_N+\ldots \qquad (1')$$

The series (1′) is a geometric progression with positive common ratio $q<1$. Hence, this series converges. The terms of the

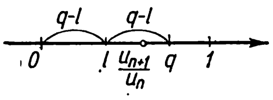


Fig. 342.

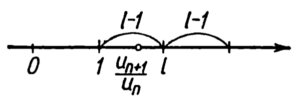


Fig. 343.

series (1), after u_{N+1}, are less than the terms of the series (1′). By Theorem 1, Sec. 3, and Theorem 1, Sec. 1, it follows that the series (1) converges.

2) Let $l>1$.

Then from the equation $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=l$ (where $l>1$) it follows that, after a certain N, that is for $n\geqslant N$, we will have the inequality

$$\frac{u_{n+1}}{u_n}>1$$

(Fig. 343), or $u_{n+1}>u_n$ for all $n\geqslant N$. But this means that the terms of the series increase after the term $N+1$, and for this reason the general term of the series does not tend to zero. Hence, the series diverges.

Note 1. The series will also diverge when $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\infty$. This follows from the fact that if $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\infty$, then after a certain $n=N$ we will have the inequality $\frac{u_{n+1}}{u_n}>1$, or $u_{n+1}>u_n$.

Example 1. Test the following series for convergence:

$$1+\frac{1}{1\cdot2}+\frac{1}{1\cdot2\cdot3}+\ldots+\frac{1}{1\cdot2\cdot3\cdot\ldots\cdot n}+\ldots$$

Solution. Here,

$$u_n=\frac{1}{1\cdot2\ldots n}=\frac{1}{n!}, \qquad u_{n+1}=\frac{1}{1\cdot2\ldots n\,(n+1)}=\frac{1}{(n+1)!};$$

$$\frac{u_{n+1}}{u_n}=\frac{n!}{(n+1)!}=\frac{1}{n+1}.$$

Hence,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{1}{n+1}=0<1.$$

The series converges.

Example 2. Test for convergence the series

$$\frac{2}{1}+\frac{2^2}{2}+\frac{2^3}{3}+\ldots+\frac{2^n}{n}+\ldots$$

Solution. Here,

$$u_n=\frac{2^n}{n};\quad u_{n+1}=\frac{2^{n+1}}{n+1};\quad \frac{u_{n+1}}{u_n}=2\frac{n}{n+1};\quad \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}2\frac{n}{n+1}=2>1.$$

The series diverges and its general term u_n approaches infinity.

Note 2. D'Alembert's test tells us whether a given positive series converges; but it does so only when $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}$ exists and is different from 1. But if this limit does not exist or if it does exist and $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=1$, then d'Alembert's test does not enable us to tell whether the series converges or diverges, because in this case the series may prove to be both convergent and divergent. Some other test is needed to determine the convergence of such series.

It will be noted, however, that if $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=1$, but the ratio $\frac{u_{n+1}}{u_n}$ for all n (after a certain one) is greater than unity, the series diverges. This follows from the fact that if $\frac{u_{n+1}}{u_n}>1$, then $u_{n+1}>u_n$ and the general term does not approach zero as $n\to\infty$.

To illustrate, let us examine some examples.

Example 3. Test for convergence the series

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots+\frac{n}{n+1}+\ldots$$

Solution. Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\frac{n+1}{n+2}}{\frac{n}{n+1}}=\lim_{n\to\infty}\frac{n^2+2n+1}{n^2+2n}=1.$$

In this case the series diverges because $\frac{u_{n+1}}{u_n}>1$ for all n:

$$\frac{u_{n+1}}{u_n}=\frac{n^2+2n+1}{n^2+2n}>1.$$

Example 4. Using the d'Alembert test, examine the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

We note that $u_n=\frac{1}{n}$, $u_{n+1}=\frac{1}{n+1}$ and, consequently,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{n}{n+1}=1.$$

Thus, d'Alembert's test does not allow us to determine the convergence or divergence of the given series. But we earlier found out by a different expedient that a harmonic series diverges.

Example 5. Test for convergence the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots$$

Solution. Here,

$$u_n=\frac{1}{n(n+1)},\quad u_{n+1}=\frac{1}{(n+1)(n+2)},$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{n(n+1)}{(n+1)(n+2)}=\lim_{n\to\infty}\frac{n}{n+2}=1.$$

D'Alembert's test does not permit us to infer that the series converges; but by other reasoning we can establish the fact that this series converges. Noting that

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1},$$

we can write the given series in the form

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\ldots$$

The partial sum of the first n terms, after removing brackets and cancelling, is

$$s_n=1-\frac{1}{n+1}.$$

Hence,

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1.$$

That is, the series converges and its sum is 1.

SEC. 5. CAUCHY'S TEST

Theorem (Cauchy's Test). *If for a series with positive terms*

$$u_1+u_2+u_3+\ldots+u_n+\ldots \tag{1}$$

the quantity $\sqrt[n]{u_n}$ *has a finite limit* l *as* $n\to\infty$, *that is,*

$$\lim_{n\to\infty}\sqrt[n]{u_n}=l,$$

then: 1) *for* $l<1$, *the series converges;*
2) *for* $l>1$, *the series diverges.*

Proof. 1) Let $l<1$. Consider the number q that satisfies the relation $l<q<1$.

After some $n=N$ we will have the relation

$$|\sqrt[n]{u_n}-l|<q-l;$$

whence it follows that

$$\sqrt[n]{u_n}<q$$

or

$$u_n<q^n$$

for all $n \geqslant N$.

Now consider two series:

$$u_1+u_2+u_3+\ldots+u_N+u_{N+1}+u_{N+2}+\ldots, \quad (1)$$

$$q^N+q^{N+1}+q^{N+2}+\ldots \quad (1')$$

The series (1′) converges since its terms form a decreasing geometric progression. The terms of the series (1), after u_N, are less than the terms of the series (1′). Consequently, the series (1) converges.

2) Let $l>1$. Then, after some $n=N$, we will have

$$\sqrt[n]{u_n}>1$$

or

$$u_n>1.$$

But if all the terms of this series, after u_N, exceed 1, then the series diverges, since its general term does not tend to zero.

Example. Test for convergence the series

$$\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\ldots+\left(\frac{n}{2n+1}\right)^n+\ldots$$

Solution. Apply the Cauchy test:

$$\lim_{n\to\infty}\sqrt[n]{u_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n+1}\right)^n}=\lim_{n\to\infty}\frac{n}{2n+1}=\frac{1}{2}<1.$$

The series converges.

Note. As in the d'Alembert test, the case

$$\lim_{n\to\infty}\sqrt[n]{u_n}=l=1$$

requires further investigation. Among the series that satisfy this condition are convergent and divergent series. Thus, for the har-

monic series (which is known to be divergent)

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = 1.$$

To be sure, we shall prove that $\lim \ln \sqrt[n]{\frac{1}{n}} = 0$. Indeed,

$$\lim_{n\to\infty} \ln \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\ln n}{n}.$$

Here, the numerator and denominator of the fraction approach infinity. Applying l'Hospital's rule, we find

$$\lim_{n\to\infty} \ln \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\ln n}{n} = \lim_{n\to\infty} \frac{-\frac{1}{n}}{1} = 0.$$

Thus, $\ln \sqrt[n]{\frac{1}{n}} \to 0$, but then $\sqrt[n]{\frac{1}{n}} \to 1$, i. e.,

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = 1.$$

For the series

$$\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots$$

we also have the equality

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} \sqrt[n]{\frac{1}{n}} = 1,$$

but this series converges, since if we suppress the first term, the terms of the remaining series will be less than the corresponding terms of the converging series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} + \ldots$$

(see Example 5, Sec. 4).

SEC. 6. THE INTEGRAL TEST FOR CONVERGENCE OF A SERIES

Theorem. *Let the terms of the series*

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots \tag{1}$$

be positive and not increasing, that is,

$$u_1 \geqslant u_2 \geqslant u_3 \geqslant \ldots,$$

and let $f(x)$ be a continuous nonincreasing function such that

$$f(1)=u_1; \quad f(2)=u_2; \quad \ldots; \quad f(n)=u_n. \qquad (2)$$

Then the following assertions hold true.

1) *if the improper integral*

$$\int_1^{\infty} f(x)\,dx$$

converges (see Sec. 7, Ch. XI), *then the series* (1) *converges too*;

2) *if the given integral diverges, then the series* (1) *diverges as well.*

Proof. Depict the terms of the series geometrically by plotting on the x-axis the numbers $1, 2, 3, \ldots, n, n+1, \ldots$ of the terms of the series, and on the y-axis, the corresponding values of the terms of the series $u_1, u_2, \ldots, u_n, \ldots$ (Fig. 344).

In the same coordinate system plot the graph of the continuous nonincreasing function

$$y=f(x)$$

which satisfies condition (2).

An examination of Fig. 344 shows that the first of the constructed rectangles has base equal to 1 and altitude $f(1)=u_1$. The area of this rectangle is thus u_1. The area of the second one is u_2, and so on; finally, the area of the last (nth) of the constructed rectangles is u_n. The sum of the areas of the constructed rectangles is equal to the sum s_n of the first n terms of the series. On the other hand, the step-like figure formed by these rectangles embraces a region bounded by the curve $y=f(x)$ and the straight lines $x=1$, $x=n+1$, $y=0$; the area of this region is equal to $\int_1^{n+1} f(x)\,dx$. Hence,

$$s_n > \int_1^{n+1} f(x)\,dx. \qquad (3)$$

Let us now consider Fig. 345. Here the first of the constructed rectangles on the left has altitude u_1; and so its area is u_1. The area of the second rectangle is u_2, and so forth. The area of the last of the constructed rectangles is u_{n+1}. Hence, the sum of the areas of all constructed rectangles is equal to the sum of all terms of the series beginning from the second to the $(n+1)$st

or $s_{n+1}-u_1$. On the other hand, it is readily seen that the step-like figure formed by these rectangles is contained within the

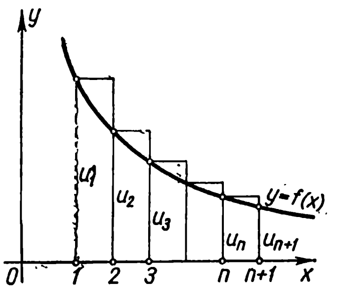


Fig. 344.

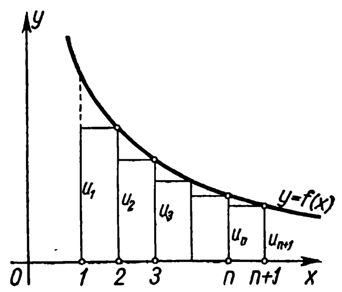


Fig. 345.

curvilinear figure bounded by the curve $y=f(x)$ and the straight lines $x=1$, $x=n+1$, $y=0$. The area of this curvilinear figure is equal to $\int\limits_1^{n+1} f(x)\,dx$. Hence,

$$s_{n+1}-u_1<\int\limits_1^{n+1} f(x)\,dx,$$

whence

$$s_{n+1}<\int\limits_1^{n+1} f(x)\,dx+u_1. \tag{4}$$

Let us now consider both cases.

1. We assume that the integral $\int\limits_1^{\infty} f(x)\,dx$ converges, that is, has a finite value.

Since

$$\int\limits_1^{n+1} f(x)\,dx<\int\limits_1^{\infty} f(x)\,dx,$$

it follows, by virtue of inequality (4), that

$$s_n<s_{n+1}<\int\limits_1^{\infty} f(x)\,dx+u_1.$$

Thus, the partial sum s_n remains bounded for all values of n. But it increases with insreasing n, since all the terms u_n are

positive. Consequently, s_n (as $n \to \infty$) has the finite limit

$$\lim_{n \to \infty} s_n = s$$

and the series converges.

2. Assume, further, that $\int\limits_1^{\infty} f(x)\,dx = \infty$. This means that $\int\limits_1^{n+1} f(x)\,dx$ increases without bound as n increases. But then, by virtue of inequality (3), s_n likewise increases indefinitely with n; the series diverges.

The theorem is thus proved completely.

Example. Test for convergence the series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

Solution. Apply the integral test, putting

$$f(x) = \frac{1}{x^p}.$$

This function satisfies all the conditions of the theorem. Consider the integral

$$\int\limits_1^N \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} x^{1-p} \Big|_1^N = \frac{1}{1-p}(N^{1-p} - 1) & \text{when } p \neq 1, \\ \ln x \Big|_1^N = \ln N & \text{when } p = 1. \end{cases}$$

Allow N to approach infinity and determine whether the improper integral converges in various cases.

It will then be possible to judge about the convergence or divergence of the series for various values of p.

For $p > 1$, $\int\limits_1^{\infty} \frac{dx}{x^p} = \frac{1}{p-1}$, the integral is finite and, hence, the series converges;

for $p < 1$, $\int\limits_1^{\infty} \frac{dx}{x^p} = \infty$, the integral is infinite, and the series diverges;

for $p = 1$, $\int\limits_1^{\infty} \frac{dx}{x} = \infty$, the integral is infinite, and the series diverges.

We note that neither the d'Alembert test nor the Cauchy test, which were considered earlier, decide whether the series is convergent or not, since

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p=1,$$

$$\lim_{n\to\infty}\sqrt[n]{u_n}=\lim_{n\to\infty}\sqrt[n]{\frac{1}{n^p}}=\lim_{n\to\infty}\left(\sqrt[n]{\frac{1}{n}}\right)^p=1^p=1.$$

SEC. 7. ALTERNATING SERIES. LEIBNIZ' THEOREM

So far we have been considering series whose terms are all positive. In this section we consider series whose terms have **alternating signs**, that is, series of the form

$$u_1-u_2+u_3-u_4+\ldots, \tag{1}$$

where $u_1, u_2, \ldots, u_n, \ldots$ are positive.

Leibniz' Theorem. *If in the alternating series*

$$u_1-u_2+u_3-u_4+\ldots\quad(u_n>0) \tag{1}$$

the terms are such that

$$u_1>u_2>u_3>\ldots \tag{2}$$

and

$$\lim_{n\to\infty}u_n=0, \tag{3}$$

then the series (1) *converges, its sum is positive and does not exceed the first term.*

Proof. Consider the sum of the first $n=2m$ terms of the series (1):

$$s_{2m}=(u_1-u_2)+(u_3-u_4)+\ldots+(u_{2m-1}-u_{2m}).$$

From condition (2) it follows that the expression in each of the brackets is positive. Hence, the sum s_{2m} is positive,

$$s_{2m}>0,$$

and increases with increasing m.

Now write this sum as follows:

$$s_{2m}=u_1-(u_2-u_3)-(u_4-u_5)-\ldots-(u_{2m-2}-u_{2m-1})-u_{2m}.$$

By virtue of condition (2), each of the parentheses is positive. Therefore, subtracting these parentheses from u_1 we get a number less than u_1, or

$$s_{2m}<u_1.$$

We have thus established that s_{2m} increases with increasing m and is bounded above. Whence it follows that s_{2m} has the limit s:

$$\lim_{m\to\infty} s_{2m} = s,$$

and

$$0 < s < u_1.$$

However, we have not yet proved the convergence of the series; we have only proved that a sequence of "even" partial sums has as its limit the number s. We now prove that "odd" partial sums also approach the limit s.

Consider the sum of the first $n = 2m+1$ terms of the series (1):

$$s_{2m+1} = s_{2m} + u_{2m+1}.$$

Since, by condition (3), $\lim\limits_{m\to\infty} u_{2m+1} = 0$, it follows that

$$\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} s_{2m} + \lim_{m\to\infty} u_{2m+1} = \lim_{m\to\infty} s_{2m} = s.$$

We have thus proved that $\lim\limits_{n\to\infty} s_n = s$ both for even n and for odd n. Hence, the series (1) converges.

Note 1. The Leibniz theorem may be illustrated geometrically as follows. Plot the following partial sums on a number line (Fig. 346):

$s_1 = u_1$, $s_2 = u_1 - u_2 = s_1 - u_2$, $s_3 = s_2 + u_3$, $s_4 = s_3 - u_4$, $s_5 = s_4 + u_5$, etc.

The points corresponding to partial sums will approach a certain point s, which depicts the sum of the series. Here, the points corresponding to the even partial sums lie on the left of s, and those corresponding to odd sums, on the right of s.

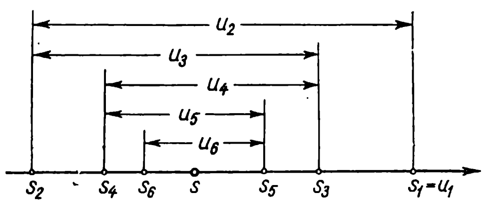


Fig. 346.

Note 2. If an alternating series satisfies the statement of the Leibniz theorem, then it is easy to evaluate the error that results if we replace its sum, s, by the partial sum s_n. In this substitution we suppress all terms after u_{n+1}. But these numbers form by themselves an alternating series, whose sum (in absolute value) is less than the first term of this series (that is, less than u_{n+1}). Thus, the error obtained when replacing s by s_n does not exceed (in absolute value) the first of the suppressed terms.

Example 1. The series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

converges, since

1) $1>\frac{1}{2}>\frac{1}{3}>\ldots;$

2) $\lim\limits_{n\to\infty} u_n=\lim\limits_{n\to\infty}\frac{1}{n}=0.$

The sum of the first n terms of this series

$$s_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+(-1)^n\frac{1}{n}$$

differs from the sum s of the series by a quantity less than $\frac{1}{n+1}$.

Example 2. The series

$$1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\ldots$$

converges by virtue of the Leibniz theorem.

SEC. 8. PLUS-AND-MINUS SERIES. ABSOLUTE AND CONDITIONAL CONVERGENCE

We give the name *plus-and-minus* series to a series that has both positive and negative terms.

Obviously, the **alternating** series considered in Sec. 7 is a **special case** of plus-and-minus series*).

We shall consider some properties of alternating series.

In contrast to the agreement made in the preceding section we will now assume that the numbers $u_1, u_2, \ldots, u_n \ldots$ can be both positive and negative.

First, let us give an important sufficient condition for the convergence of an alternating series.

Theorem 1. *If the alternating series*

$$u_1+u_2+\ldots+u_n+\ldots \tag{1}$$

is such that a series made up of the absolute values of its terms,

$$|u_1|+|u_2|+\ldots+|u_n|+\ldots, \tag{2}$$

converges, then the given alternating series also converges.

Proof. Let s_n and σ_n be the sums of the first n terms of the series (1) and (2).

*) In this English edition we shall use the term alternating series for both types.— *Tr.*

Also, let s'_n be the sum of all the positive terms, and s''_n, the sum of the absolute values of all the negative terms of the first n terms of the given series; then

$$s_n = s'_n - s''_n;\quad \sigma_n = s'_n + s''_n.$$

By hypothesis, σ_n has the limit σ; s'_n and s''_n are positive increasing quantities less than σ. Consequently, they have the limits s' and s''. From the relationship $s_n = s'_n - s''_n$ it follows that s_n also has a limit and that this limit is equal to $s' - s''$, which means that the alternating series (1) converges.

The above-proved theorem enables one to judge about the convergence of some alternating series. In this case, the test for convergence of the alternating series reduces to investigating a series with positive terms.

Consider two examples.

Example 1. Test for convergence the series

$$\frac{\sin\alpha}{1^2} + \frac{\sin 2\alpha}{2^2} + \frac{\sin 3\alpha}{3^2} + \ldots + \frac{\sin n\alpha}{n^2} + \ldots, \tag{3}$$

where α is any number.

Solution. Also consider the series

$$\left|\frac{\sin\alpha}{1^2}\right| + \left|\frac{\sin 2\alpha}{2^2}\right| + \left|\frac{\sin 3\alpha}{3^2}\right| + \ldots + \left|\frac{\sin n\alpha}{n^2}\right| + \ldots \tag{4}$$

and

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots \tag{5}$$

The series (5) converges (see Sec. 6). The terms of the series (4) are not greater than the corresponding terms of the series (5); hence, the series (4) also converges. But then, in virtue of the theorem just proved, the given series (3) likewise converges.

Example 2. Test for convergence the series

$$\frac{\cos\frac{\pi}{4}}{3} + \frac{\cos 3\frac{\pi}{4}}{3^2} + \frac{\cos 5\frac{\pi}{4}}{3^3} + \ldots + \frac{\cos(2n-1)\frac{\pi}{4}}{3^n} + \ldots \tag{6}$$

Solution. In addition to this series, consider the series

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} + \ldots \tag{7}$$

This series converges because it is a decreasing geometric progression with ratio $\frac{1}{3}$. But then the given series (6) converges, since the absolute values of its terms are less than those of the corresponding terms of the series (7).

We note that the convergence condition that was proved earlier is only a **sufficient** condition for convergence of an alternating series, but not a necessary condition: there are alternating series which converge, but series formed from the absolute values of their terms diverge. In this connection, it is useful to introduce the concepts of absolute and conditional convergence of an alternating series and, on the basis of these concepts, to classify alternating series.

Definition. The alternating series

$$u_1+u_2+u_3+\ldots+u_n+\ldots \qquad (1)$$

is called *absolutely convergent* if a series made up of the absolute values of its terms converges:

$$|u_1|+|u_2|+|u_3|+\ldots+|u_n|+\ldots \qquad (2)$$

If the alternating series (1) converges, while the series (2) composed of the absolute values of its terms diverges, then the given alternating series (1) is called *a conditionally convergent series.*

Example 3. The alternating series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

is **conditionally** convergent, since a series composed of the absolute values of its terms is a harmonic series,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots,$$

which diverges. The series itself converges (this can be readily verified by Leibniz' test).

Example 4. The alternating series

$$1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\ldots$$

is **absolutely** convergent, since a series made up of the absolute values of its terms,

$$1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots,$$

converges, as established in Sec. 4.

Theorem 1 is frequently stated (with the help of the concept of absolute convergence) as follows: *every absolutely convergent series is a convergent series.*

In conclusion, we note (without proof) the following properties of absolutely convergent and conditionally convergent series.

Theorem 2. *If a series converges absolutely, it remains absolutely convergent for any rearrangement of its terms. The sum of the series is independent of the order of its terms.*

This property does not hold for conditionally convergent series.

Theorem 3. *If a series converges conditionally, then no matter what number A is given, the terms of this series can be rearranged in such manner that its sum is exactly equal to A. What is more, it is possible so to rearrange the terms of a conditionally convergent series that the series resulting after the rearrangement is divergent.*

The proofs of these theorems are beyond the scope of this course.

To illustrate the fact that the sum of a conditionally convergent series can change upon rearrangement of its terms, consider the following example.

Example 5. The alternating series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots \tag{8}$$

converges conditionally. Denote its sum by s. It is obvious that $s>0$. Rearrange the terms of the series (8) so that two negative terms follow one positive term:

$$\underbrace{1-\frac{1}{2}-\frac{1}{4}}+\underbrace{\frac{1}{3}-\frac{1}{6}-\frac{1}{8}}+\dots+\underbrace{\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}}+\dots \tag{9}$$

We shall prove that the resultant series converges, but that its sum s' is half the sum of the series (8): $\frac{1}{2}s$. Denote by s_n and s_n' the partial sums of the series (8) and (9). Consider the sum of $3k$ terms of the series (9):

$$\begin{aligned} s_{3k}' &= \left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\dots+\left(\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}\right)= \\ &=\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\dots+\left(\frac{1}{4k-2}-\frac{1}{4k}\right)= \\ &=\frac{1}{2}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right)\right]= \\ &=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{2k-1}-\frac{1}{2k}\right)=\frac{1}{2}s_{2k}. \end{aligned}$$

Consequently,

$$\lim_{k\to\infty} s_{3k}'=\lim_{k\to\infty}\frac{1}{2}s_{2k}=\frac{1}{2}s.$$

Further,

$$\lim_{k\to\infty} s_{3k+1}'=\lim_{k\to\infty}\left(s_{3k}'+\frac{1}{2k+1}\right)=\frac{1}{2}s,$$

$$\lim_{k\to\infty} s_{3k+2}'=\lim\left(s_{3k}'+\frac{1}{2k+1}-\frac{1}{4k+2}\right)=\frac{1}{2}s.$$

And we obtain

$$\lim_{n \to \infty} s'_n = s' = \frac{1}{2} s.$$

Thus, in this case the sum of the series changed after its terms were rearranged (it diminished by a factor of 2).

SEC. 9. FUNCTIONAL SERIES

The series $u_1 + u_2 + \ldots + u_n + \ldots$ is called a *functional series* if its terms are functions of x.

Consider the functional series

$$u_1(x) + u_2(x) + u_3(x) + \ldots + u_n(x) + \ldots \qquad (1)$$

Assigning to x definite numerical values, we get different numerical series, which may prove to be convergent or divergent.

The set of all those values of x for which the functional series converges is called the *domain of convergence* of the series.

Obviously, in the domain of convergence of a series its sum is some function of x. Therefore, the sum of a functional series is denoted by $s(x)$.

Example. Consider the functional series

$$1 + x + x^2 + \ldots + x^n + \ldots$$

This series converges for all values of x in the interval $(-1, 1)$, that is, for all x that satisfy the condition $|x| < 1$. For each value of x in the interval $(-1, 1)$, the sum of the series is equal to $\frac{1}{1-x}$ (the sum of a decreasing geometric progression with ratio x). Thus, in the interval $(-1, 1)$ the given series defines the function

$$s(x) = \frac{1}{1-x}$$

which is the sum of the series; that is,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

Denote by $s_n(x)$ the sum of the first n terms of the series (1). If this series converges and its sum is equal to $s(x)$, then

$$s(x) = s_n(x) + r_n(x),$$

where $r_n(x)$ is the sum of the series $u_{n+1}(x) + u_{n+2}(x) + \ldots$, i.e.,

$$r_n(x) = u_{n+1}(x) + u_{n+2}(x) + \ldots$$

Here, the quantity $r_n(x)$ is called the *remainder of the series* (1). For all values of x in the domain of convergence of the series we have the relation $\lim_{n\to\infty} s_n(x) = s(x)$; therefore,

$$\lim_{n\to\infty} r_n(x) = \lim_{n\to\infty} [s(x) - s_n(x)] = 0,$$

which means that the remainder $r_n(x)$ of a convergent series approaches zero as $n \to \infty$.

SEC. 10. DOMINATED SERIES

Definition. The functional series

$$u_1(x) + u_2(x) + u_3(x) + \ldots + u_n(x) + \ldots \qquad (1)$$

is called *dominated* in some range of x if there exists a convergent numerical series

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n + \ldots \qquad (2)$$

with positive terms such that for all values of x from this range the following relations are fulfilled:

$$|u_1(x)| \leqslant \alpha_1, \quad |u_2(x)| \leqslant \alpha_2, \quad \ldots, \quad |u_n(x)| \leqslant \alpha_n, \quad \ldots \qquad (3)$$

In other words, a series is called *dominated* if each of its terms does not exceed, in absolute value, the corresponding term of some convergent numerical series with positive terms.

For example, the series

$$\frac{\cos x}{1} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots + \frac{\cos nx}{n^2} + \ldots$$

is a series majorised on the entire x-axis. Indeed, for all values of x, the relation

$$\left|\frac{\cos nx}{n^2}\right| \leqslant \frac{1}{n^2} \quad (n = 1, 2, \ldots),$$

is fulfilled and the series

$$\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots,$$

as we know, converges.

From the definition it follows straightway that a series dominated in some range converges absolutely at all points of this range (see Sec. 8). Also, a dominated series has the following important property.

Theorem. *Let the functional series*

$$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

be dominated on the interval $[a, b]$. *Let* $s(x)$ *be the sum of this series and* $s_n(x)$ *the sum of the first n terms of this series. Then for each arbitrarily small number* $\varepsilon > 0$ *there will be a positive integer N such that for all* $n \geqslant N$ *the following inequality will be fulfilled,*

$$|s(x) - s_n(x)| < \varepsilon,$$

no matter what the x of the interval $[a, b]$.

Proof. Denote by σ the sum of the series (2):

$$\sigma = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n + \alpha_{n+1} + \ldots,$$

then

$$\sigma = \sigma_n + \varepsilon_n,$$

where σ_n is the sum of the first n terms of the series (2), and ε_n is the sum of the remaining terms of this series; that is,

$$\varepsilon_n = \alpha_{n+1} + \alpha_{n+2} + \ldots$$

Since this series converges, it follows that

$$\lim_{n \to \infty} \sigma_n = \sigma$$

and, consequently,

$$\lim_{n \to \infty} \varepsilon_n = 0.$$

Let us now represent the sum of the functional series (1) in the form

$$s(x) = s_n(x) + r_n(x),$$

where

$$s_n(x) = u_1(x) + \ldots + u_n(x),$$
$$r_n(x) = u_{n+1}(x) + u_{n+2}(x) + u_{n+3}(x) + \ldots$$

From condition (3) it follows that

$$|u_{n+1}(x)| \leqslant \alpha_{n+1}, \quad |u_{n+2}(x)| \leqslant \alpha_{n+2}, \quad \ldots,$$

and therefore

$$|r_n(x)| \leqslant \varepsilon_n$$

for all x of the range under consideration.

Thus,

$$|s(x) - s_n(x)| < \varepsilon_n$$

for all x of the interval $[a, b]$, and $\varepsilon_n \to 0$ as $n \to \infty$.

Note 1. This result may be represented geometrically as follows.

Consider the graph of the function $y = s(x)$. About this curve construct a band of width $2\varepsilon_n$; in other words, construct the

curves $y=s(x)+\varepsilon_n$ and $y=s(x)-\varepsilon_n$ (Fig. 347). Then for any ε_n the graph of the function $s_n(x)$ will lie completely in the band under consideration. The graphs of all successive partial sums will likewise lie within this band.

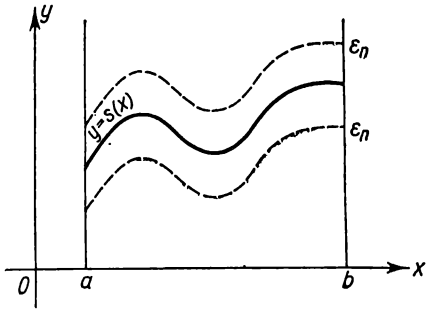


Fig. 347.

Note 2. Not every functional series convergent on the interval $[a, b]$ has the property indicated in the foregoing theorem. However, there are nondominated series such that possess this property. A series that possesses this property is called *a uniformly convergent series on the interval* $[a, b]$.

Thus, *the functional series* $u_1(x)+u_2(x)+\ldots+u_n(x)+$ $+\ldots$ *is called a uniformly convergent series on the interval* $[a, b]$ *if for any arbitrarily small* $\varepsilon>0$ *there is an integer* N *such that for all* $n \geqslant N$ *the inequality*

$$|s(x)-s_n(x)|<\varepsilon$$

will be fulfilled for any x *of the interval* $[a, b]$.

From the theorem that has been proved it follows that a dominated series is a series that uniformly converges.

SEC. 11. THE CONTINUITY OF THE SUM OF A SERIES

Let there be a series made up of continuous functions

$$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots,$$

convergent on some interval $[a, b]$.

In Chapter II we proved a theorem which stated that the sum of a finite number of continuous functions is a continuous function. This property does not hold for the sum of a series (consisting of an infinite number of terms). Some functional series with continuous terms have for the sum a continuous function, while in the case of other functional series with continuous terms, the sum is a discontinuous function.

Example. Consider the series

$$\left(x^{\frac{1}{3}}-x\right)+\left(x^{\frac{1}{5}}-x^{\frac{1}{3}}\right)+\left(x^{\frac{1}{7}}-x^{\frac{1}{5}}\right)+\ldots+\left(x^{\frac{1}{2n+1}}-x^{\frac{1}{2n-1}}\right)+\ldots$$

The terms of this series (each term is bracketed) are continuous functions for all values of x. We shall prove that this series converges and that its sum is a discontinuous function.

We find the sum of the first n terms of the series:

$$s_n = x^{\frac{1}{2n+1}} - x.$$

Find the sum of the series:
if $x > 0$, then

$$s = \lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} (x^{\frac{1}{2n+1}} - x) = 1 - x,$$

if $x < 0$, then

$$s = \lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} (-|x|^{\frac{1}{2n+1}} - x) = -1 - x,$$

if $x = 0$, then $s_n = 0$, and so $s = \lim\limits_{n\to\infty} s_n = 0$. Thus, we have

$$s(x) = 1 - x \quad \text{for } x > 0,$$
$$s(x) = -1 - x \quad \text{for } x < 0,$$
$$s(x) = 0 \quad \text{for } x = 0.$$

And so the sum of the given series is a discontinuous function. Its graph is shown in Fig. 348 along with the graphs of the partial sums $s_1(x)$, $s_2(x)$, and $s_3(x)$.

The following theorem holds true for dominated series.

Theorem. *The sum of a series of continuous functions dominated on some interval* $[a, b]$ *is a function continuous on this interval.*

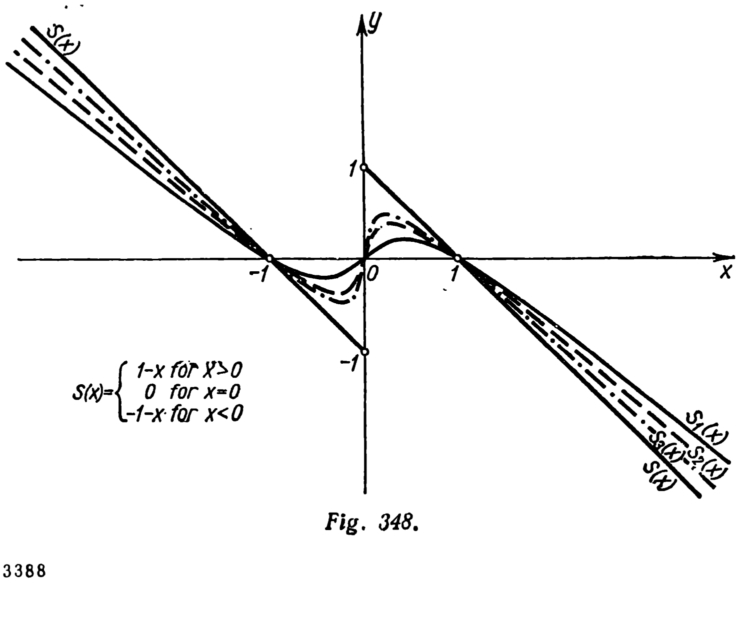


Fig. 348.

Proof. Let there be a series of continuous functions dominated on the interval $[a, b]$:

$$u_1(x)+u_2(x)+u_3(x)+\dots \tag{1}$$

Let us represent its sum in the form

$$s(x)=s_n(x)+r_n(x),$$

where

$$s_n(x)=u_1(x)+\dots+u_n(x)$$

and

$$r_n(x)=u_{n+1}(x)+u_{n+2}(x)+\dots$$

On the interval $[a, b]$ take an arbitrary value of the argument x and give it an increase Δx such that the point $x+\Delta x$ should also lie on the interval $[a, b]$.

We introduce the notations

$$\Delta s=s(x+\Delta x)-s(x);$$
$$\Delta s_n=s_n(x+\Delta x)-s_n(x);$$

then

$$\Delta s=\Delta s_n+r_n(x+\Delta x)-r_n(x),$$

from which we have

$$|\Delta s|\leqslant|\Delta s_n|+|r_n(x+\Delta x)|+|r_n(x)|. \tag{2}$$

This inequality is true for any integer n.

To prove the continuity of $s(x)$, we have to show that for any preassigned and arbitrarily small $\varepsilon>0$ there will be a number $\delta>0$ such that for all $|\Delta x|<\delta$ we will have $|\Delta s|<\varepsilon$.

Since the given series (1) is dominated, it follows that for any preassigned $\varepsilon>0$ there will be found an integer N such that for all $n\geqslant N$ (and as a particular case, $n=N$) the inequality

$$|r_N(x)|<\frac{\varepsilon}{3} \tag{3}$$

will be fulfilled for any x of the interval $[a, b]$. The value $x+\Delta x$ lies on the interval $[a, b]$ and therefore the following inequality is fulfilled:

$$|r_N(x+\Delta x)|<\frac{\varepsilon}{3}. \tag{3'}$$

Further, for the chosen N the partial sum $s_N(x)$ is a continuous function (the sum of a **finite** number of continuous functions) and, consequently, a positive number δ may be chosen such that for every Δx that satisfies the condition $|\Delta x|<\delta$ the following inequality is fulfilled:

$$|\Delta s_N|<\frac{\varepsilon}{3}. \tag{4}$$

By inequalities (2), (3), (3′), and (4), we have

$$|\Delta s| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

that is,

$$|\Delta s| < \varepsilon \quad \text{for} \quad |\Delta x| < \delta,$$

which means that $s(x)$ is a continuous function at the point x (and, consequently, at any point of the interval $[a, b]$).

Note. From this theorem it follows that if the sum of a series is discontinuous on some interval $[a, b]$, then the series is not dominated on this interval. In particular, the series given in the example is not dominated (on any interval containing the point $x=0$, that is to say, a point of discontinuity of the sum of the series).

We note, finally, that the converse statement is not true: there are series, not dominated on an interval, which, however, converge on this interval to a continuous function. For instance, every series uniformly convergent on the interval $[a, b]$ (even if it is not dominated) has a continuous function for its sum (if, of course, all terms of the series are continuous).

SEC. 12. INTEGRATION AND DIFFERENTIATION OF SERIES

Theorem 1. *Let there be a series of continuous functions*

$$u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots, \tag{1}$$

dominated on the interval $[a, b]$ and let $s(x)$ be the sum of this series. Then the integral of $s(x)$ between the limits from α to x, which limits belong to the interval $[a, b]$, is equal to the sum of such integrals of the terms of the given series; that is,

$$\int_\alpha^x s(x)\,dx = \int_\alpha^x u_1(x)\,dx + \int_\alpha^x u_2(x)\,dx + \ldots + \int_\alpha^x u_n(x)\,dx + \ldots$$

Proof. The function $s(x)$ may be represented in the form

$$s(x) = s_n(x) + r_n(x)$$

or

$$s(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + r_n(x).$$

Then

$$\int_\alpha^x s(x)\,dx = \int_\alpha^x u_1(x)\,dx + \int_\alpha^x u_2(x) + \ldots +$$

$$+ \int_\alpha^x u_n(x)\,dx + \int_\alpha^x r_n(x)\,dx \tag{2}$$

(the integral of the sum of a **finite** number of terms is equal to the sum of the integrals of these terms).

Since the original series (1) is dominated, it follows that for every x we have $|r_n(x)| < \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$. Therefore,

$$\left|\int_\alpha^x r_n(x)\,dx\right| \leqslant \int_\alpha^x |r_n(x)|\,dx < \int_\alpha^x \varepsilon_n\,dx = \varepsilon_n(x-\alpha) \leqslant \varepsilon_n(b-\alpha).$$

Since $\varepsilon_n \to 0$, it follows that

$$\lim_{n\to\infty}\int_\alpha^x r_n(x)\,dx = 0.$$

But from equation (2) we have

$$\int_\alpha^x r_n(x)\,dx = \int_\alpha^x s(x)\,dx - \left[\int_\alpha^x u_1(x)\,dx + \ldots + \int_\alpha^x u_n(x)\,dx\right].$$

Hence

$$\lim_{n\to\infty}\left\{\int_\alpha^x s(x)\,dx - \left[\int_\alpha^x u_1(x)\,dx + \ldots + \int_\alpha^x u_n(x)\,dx\right]\right\} = 0,$$

or

$$\lim_{n\to\infty}\left[\int_\alpha^x u_1(x)\,dx + \ldots + \int_\alpha^x u_n(x)\,dx\right] = \int_\alpha^x s(x)\,dx. \qquad (3)$$

The sum in the brackets is a partial sum of the series

$$\int_\alpha^x u_1(x)\,dx + \ldots + \int_\alpha^x u_n(x)\,dx + \ldots \qquad (4)$$

Since the partial sums of this series have a limit, this series converges and its sum, by virtue of equation (3), is equal to $\int_\alpha^x s(x)\,dx$, i. e.,

$$\int_\alpha^x s(x)\,dx = \int_\alpha^x u_1(x)\,dx + \int_\alpha^x u_2(x)\,dx + \ldots + \int_\alpha^x u_n(x)\,dx + \ldots,$$

this is the equation that had to be proved.

Note 1. If a series is not dominated, term-by-term integration of it is not always possible. This is to be understood in the sense that the integral $\int_\alpha^x s(x)\,dx$ of the sum of the series (1) is not always

equal to the sum of the integrals of its terms [that is, to the sum of the series (4)].

Theorem 2. *If a series,*

$$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots, \tag{5}$$

made up of functions having continuous derivatives on the interval $[a, b]$ *converges (on this interval) to the sum* $s(x)$ *and the series*

$$u_1'(x)+u_2'(x)+\ldots+u_n'(x)+\ldots \tag{6}$$

made up of the derivatives of its terms is dominated on the same interval, then the sum of the series of derivatives is equal to the derivative of the sum of the original series; that is,

$$s'(x)=u_1'(x)+u_2'(x)+u_3'(x)+\ldots+u_n'(x)+\ldots$$

Proof. Denote by $F(x)$ the sum of the series (6):

$$F(x)=u_1'(x)+u_2'(x)+\ldots+u_n'(x)+\ldots,$$

and prove that

$$F(x)=s'(x).$$

Since the series (6) is dominated, it follows, by the preceding theorem, that

$$\int_\alpha^x F(x)\,dx=\int_\alpha^x u_1'(x)\,dx+\int_\alpha^x u_2'(x)\,dx+\ldots+\int_\alpha^x u_n'(x)\,dx+\ldots$$

Performing the integration, we get

$$\int_\alpha^x F(x)\,dx=[u_1(x)-u_1(\alpha)]+$$
$$+[u_2(x)-u_2(\alpha)]+\ldots+[u_n(x)-u_n(\alpha)]+\ldots$$

But, by hypothesis,

$$s(x)=u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots,$$
$$s(\alpha)=u_1(\alpha)+u_2(\alpha)+\ldots+u_n(\alpha)+\ldots,$$

no matter what the numbers x and α on the interval $[a, b]$. Therefore,

$$\int_\alpha^x F(x)\,dx=s(x)-s(\alpha).$$

Differentiating both sides of this equation with respect to x, we obtain

$$F(x)=s'(x).$$

We have thus proved that when the conditions of the theorem are fulfilled, the derivative of the sum of the series is equal to the sum of the derivatives of the terms of the series.

Note 2. The requirement of dominance (majorisation) of a series of derivatives is extremely essential, and if not fulfilled it can make term-by-term differentiation of the series impossible. This is illustrated by a dominated series that does not admit term-by-term differentiation.

Consider the series

$$\frac{\sin 1^4 x}{1^2}+\frac{\sin 2^4 x}{2^2}+\frac{\sin 3^4 x}{3^2}+\ldots+\frac{\sin n^4 x}{n^2}+\ldots$$

This series converges to a continuous function because it is dominated. Indeed, for every x its terms are (in absolute value) less than the terms of the numerical convergent series with positive terms

$$\frac{1}{1}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$$

Write a series composed of the derivatives of the terms of the original series:

$$\cos x+2^2\cos 2^4 x+\ldots+n^2\cos n^4 x+\ldots$$

This series diverges. Thus, for instance, for $x=0$ it turns into the series

$$1+2^2+3^2+\ldots+n^2+\ldots$$

(It may be shown that it diverges not only for $x=0$.)

SEC. 13. POWER SERIES. INTERVAL OF CONVERGENCE

Definition 1. *A power series* is a functional series of the form

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots, \tag{1}$$

where a_0, a_1, a_2, $\ldots$, a_n, $\ldots$ are constants called coefficients of the series.

The domain of convergence of a power series is always some interval, which, in a particular case, can degenerate into a point. To convince ourselves of this, let us first prove the following theorem, which is very important for the whole theory of power series.

Theorem 1 (Abel's Theorem). 1) *If a power series converges for some nonzero value x_0, then it converges absolutely for any value of x, for which*

$$|x|<|x_0|;$$

2) *if a series diverges for some value x_0', then it diverges for every x for which*

$$|x|>|x_0'|.$$

Proof. 1) Since, by assumption, the numerical series

$$a_0+a_1x_0+a_2x_0^2+\ldots+a_nx_0^n+\ldots \qquad (1')$$

converges, it follows that its common term $a_nx_0^n \rightarrow 0$ as $n \rightarrow \infty$, and this means that there exists a positive number M such that all the terms of the series are less than M in absolute value.

Rewrite the series (1) in the form

$$a_0+a_1x_0\left(\frac{x}{x_0}\right)+a_2x_0^2\left(\frac{x}{x_0}\right)^2+\ldots+a_nx_0^n\left(\frac{x}{x_0}\right)^n+\ldots \qquad (1a)$$

and consider a series of the absolute values of its terms:

$$|a_0|+|a_1x_0|\left|\frac{x}{x_0}\right|+|a_2x_0^2|\left|\frac{x}{x_0}\right|^2+\ldots+|a_nx_0^n|\left|\frac{x}{x_0}\right|^n+\ldots \qquad (2)$$

The terms of this series are less than the corresponding terms of the series

$$M+M\left|\frac{x}{x_0}\right|+M\left|\frac{x}{x_0}\right|^2+\ldots+M\left|\frac{x}{x_0}\right|^n+\ldots \qquad (3)$$

For $|x|<|x_0|$ the latter series is a geometric progression with ratio $\left|\frac{x}{x_0}\right|<1$ and, consequently, converges. Since the terms of the series (2) are less than the corresponding terms of the series (3), the series (2) also converges, and this means that the series (1*a*) or (1) converges absolutely.

2) It is now easy to prove the second part of the theorem: let the series (1) diverge at some point x_0'. Then it will diverge at any point x that satisfies the condition $|x|>|x_0'|$. Indeed, if at some point x that satisfies this condition the series converged, then by virtue of the first part (just proved) of the theorem, it should converge at the point x_0' as well, since $|x_0'|<|x|$. But this contradicts the condition that at the point x_0' the series diverges. Hence the series diverges at the point x as well. The theorem is thus completely proved.

Abel's theorem makes it possible to judge the position of the points of convergence and divergence of a power series. Indeed, if x_0 is a point of convergence, then the entire interval $(-|x_0|, |x_0|)$ is filled with points of absolute convergence. If x_0' is a point of divergence, then the whole infinite half-line to the right of the point $|x_0'|$ and the whole half-line to the left of the point $-|x_0'|$ consist of points of divergence.

From this it may be concluded that there exists a number R such that for $|x|<R$ we have points of absolute convergence and for $|x|>R$, points of divergence.

We thus have the following theorem on the structure of the domain of convergence of a power series:

Theorem 2. *The domain of convergence of a power series is an interval with centre at the coordinate origin.*

Definition 2. The *interval of convergence* of a power series is an interval from $-R$ to $+R$ such that for any point x lying inside

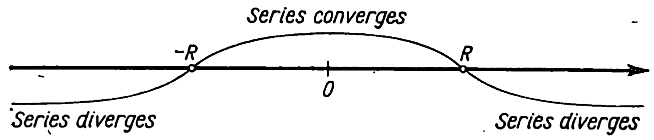


Fig. 349.

this interval, the series converges and converges absolutely, while for points x lying outside it, the series diverges (Fig. 349). The number R is called the *radius of convergence* of the power series.

At the end points of the interval (at $x=R$ and at $x=-R$) the question of the convergence or divergence of a given series is decided separately for each specific series.

We note that in some series the interval of convergence degenerates into a point ($R=0$), while in others it encompasses the entire x-axis ($R=\infty$).

We give a method for determining the radius of convergence of a power series.

Let there be a series

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots \tag{1}$$

Consider a series made up of the absolute values of its terms:

$$|a_0|+|a_1||x|+|a_2||x|^2+|a_3||x|^3+ \\ +|a_4||x|^4+\ldots+|a_n||x|^n+\ldots \tag{4}$$

To determine the convergence of this series (with positive terms!), apply the d'Alembert test.

Let us assume that there exists a limit:

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_nx^n}\right|=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x|=L|x|.$$

Then by the d'Alembert test the series (4) converges, if $L|x|<1$; that is, if $|x|<\frac{1}{L}$, and diverges if $L|x|>1$, that is, if $|x|>\frac{1}{L}$.

Consequently, series (1) converges absolutely when $|x|<\frac{1}{L}$. But if $|x|>\frac{1}{L}$, then $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=|x|L>1$ and series (4) diverges, and

its general term does not tend to zero.*) But then neither does the general term of the given power series (1) tend to zero, and this means that (on the basis of the necessary condition of convergence) this power series diverges $\left(\text{when } |x| > \frac{1}{L}\right)$.

From the foregoing it follows that the interval $\left(-\frac{1}{L}, \frac{1}{L}\right)$ is the interval of convergence of the power series (1):

$$R = \frac{1}{L} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

Similarly, to determine the interval of convergence we can make use of the Cauchy test, and then

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Example 1. To determine the interval of convergence of the series

$$1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

Solution. Applying d'Alembert's test directly, we get

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = |x|.$$

Thus, the series converges when $|x| < 1$ and diverges when $|x| > 1$. At the extremities of the interval $(-1, 1)$ it is impossible to investigate the series by means of d'Alembert's test. However, it is immediately apparent that when $x = -1$ and when $x = 1$ the series diverges.

Example 2. Determine the interval of convergence of the series

$$\frac{2x}{1} - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \ldots$$

Solution. We apply the d'Alembert test:

$$\lim_{n\to\infty} \left| \frac{\dfrac{(2x)^{n+1}}{n+1}}{\dfrac{(2x)^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| |2x| = |2x|.$$

The series converges if $|2x| < 1$, that is, if $|x| < \frac{1}{2}$; when $x = \frac{1}{2}$ the series converges; when $x = -\frac{1}{2}$ the series diverges.

*) It will be recalled that in proving d'Alembert's test (see Sec. 4) we found that if $\lim \frac{u_{n+1}}{u_n} > 1$, then the general term of the series increases and, consequently, does not tend to zero.

Example 3. Determine the interval of convergence of the series

$$x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$$

Solution. Applying the d'Alembert test we get

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}n!}{x^n(n+1)!}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=0<1.$$

Since the limit is independent of x and is less than unity, the series converges for all values of x.

Example 4. The series $1+x+(2x)^2+(3x)^3+\ldots+(nx)^n+\ldots$ diverges for all values of x except $x=0$ because $(nx)^n\to\infty$ as $n\to\infty$ no matter what the x, as long as it is different from zero.

Theorem 3. *The power series*

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots \qquad (1)$$

is dominated on any interval $[-\varrho, \varrho]$ *that lies completely inside the interval of convergence.*

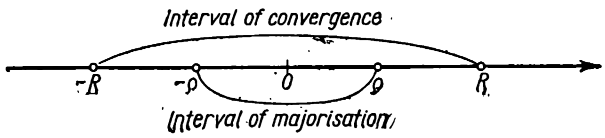


Fig. 350.

Proof. It is given that $\varrho<R$ (Fig. 350) and therefore the number series (with positive terms)

$$|a_0|+|a_1|\varrho+|a_2|\varrho^2+\ldots+|a_n|\varrho^n \qquad (5)$$

converges. But when $|x|<\varrho$, the terms of the series (1) do not exceed, in absolute value, the corresponding terms of series (5). Hence, series (1) is dominated on the interval $[-\varrho, \varrho]$.

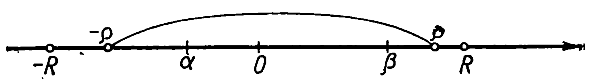


Fig. 351.

Corollary 1. *On every interval lying entirely within the interval of convergence, the sum of a power series is a continuous function.* Indeed, the series on this interval is majorised, and its terms are continuous functions of x. Consequently, on the basis of Theorem 1, Sec. 11, the sum of this series is a continuous function.

Corollary 2. *If the limits of integration* α, β *lie within the interval of convergence of a power series, then the integral of the sum of the series is equal to the sum of the integrals of the terms of the series*, because the region of integration may be taken in the interval $[-\varrho, \varrho]$, where the series is dominated (Fig. 351) (see Theorem 2, Sec. 12, on the possibility of term-by-term integration of a dominated series).

SEC. 14. DIFFERENTIATION OF POWER SERIES

Theorem 1. *If a power series*

$$s(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots + a_nx^n + \ldots \tag{1}$$

has an interval of convergence $(-R, R)$, *then the series*

$$\varphi(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots, \tag{2}$$

obtained by termwise differentiation of the series (1) has the same interval of convergence $(-R, R)$; *here,*

$$\varphi(x) = s'(x), \text{ if } |x| < R,$$

i.e., inside the interval of convergence the derivative of the sum of the power series (1) *is equal to the sum of the series obtained by termwise differentiation of the series* (1).

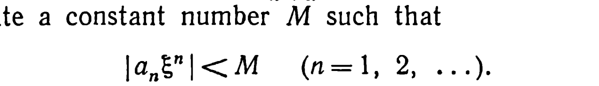


Fig. 352.

Proof. We shall prove that the series (2) is majorised on any interval $[-\varrho, \varrho]$ that lies completely within the interval of convergence.

Take a point ξ such that $\varrho < \xi < R$ (Fig. 352). The series (1) converges at this point, hence $\lim\limits_{n\to\infty} a_n\xi^n = 0$; it is therefore possible to indicate a constant number M such that

$$|a_n\xi^n| < M \quad (n = 1, 2, \ldots).$$

If $|x| \leqslant \varrho$, then

$$|na_nx^{n-1}| \leqslant |na_n\varrho^{n-1}| = n\,|a_n\xi^{n-1}|\left|\frac{\varrho}{\xi}\right|^{n-1} < n\frac{M}{\xi}q^{n-1},$$

where

$$q = \frac{\varrho}{\xi} < 1.$$

Thus, in absolute value, the terms of the series (2), when $|x|\leqslant\varrho$, are less than the terms of a positive number series with constant terms:

$$\frac{M}{\xi}(1+2q+3q^2+\ldots+nq^{n-1}+\ldots).$$

But this latter series converges, as will be evident if we apply the d'Alembert test:

$$\lim_{n\to\infty}\frac{nq^{n-1}}{(n-1)q^{n-2}}=q<1.$$

Hence, the series (2) is majorised on the interval $[-\varrho, \varrho]$, and by Theorem 2, Sec. 12, its sum is a derivative of the sum of the given series on the interval $[-\varrho, \varrho]$, i.e.,

$$\varphi(x)=s'(x).$$

Since every interior point of the interval $(-R, R)$ may be included in some interval $[-\varrho, \varrho]$, it follows that the series (2) converges at every interior point of the interval $(-R, R)$.

We shall prove that outside the interval $(-R, R)$ the series (2) diverges. Assume that the series (2) converges when $x_1>R$. Integrating it termwise in the interval $(0, x_2)$, where $R<x_2<x_1$, we would find that the series (1) converges at the point x_2, but this contradicts the hypotheses of the theorem. Thus, the interval $(-R, R)$ is the interval of convergence of series (2). And the theorem is proved completely.

Series (2) may again be differentiated term by term, and this may be continued as many times as one pleases. We thus have the conclusion:

Theorem 2. *If a power series converges in an interval* $(-R, R)$, *its sum is a function which has, inside the interval of convergence, derivatives of any order, each of which is the sum of a series resulting from term-by-term differentiation of the given series an appropriate number of times; here, the interval of convergence of each series obtained by differentiation is the same interval* $(-R, R)$.

SEC. 15. SERIES IN POWERS OF $x-a$

Also called a power series is a functional series of the form

$$a_0+a_1(x-a)+a_2(x-a)^2+\ldots+a_n(x-a)^n+\ldots, \tag{1}$$

where the constants $a_0, a_1, \ldots, a_n, \ldots$ are likewise termed coefficients of the series. This is a power series arranged in powers of the binomial $x-a$.

When $a=0$, we have a power series in powers of x, which, consequently, is a special case of series (1).

To determine the region of convergence of series (1), substitute the variable

$$x-a=X.$$

Series (1) then takes on the form

$$a_0+a_1X+a_2X^2+\ldots+a_nX^n+\ldots, \tag{2}$$

we thus have a power series in powers of X.

Let the interval $-R<X<R$ be the interval of convergence of the series (2) (Fig. 353, α). It thus follows that series (1) will converge for values of x that satisfy the inequality $-R<x-a<R$ or $a-R<x<a+R$. Since series (2) diverges for $|X|>R$ the series (1) will diverge for $|x-a|>R$, that is, it will diverge outside the interval $a-R<x<a+R$ (Fig. 353, β).

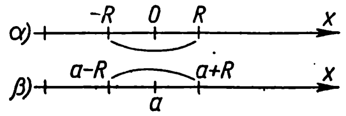


Fig. 353.

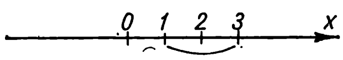


Fig. 354.

And so the interval $(a-R,\ a+R)$ with centre at the point a will be the interval of convergence of series (1). All the properties of a series in powers of x inside the interval of convergence $(-R,\ +R)$ are retained completely for a series in powers of $x-a$ inside the interval of convergence $(a-R,\ a+R)$. For example, after term-by-term integration of the power series (1), if the limits of integration lie within the interval of convergence $(a-R,\ a+R)$, we get a series whose sum is equal to the corresponding integral of the sum of the given series (1). In the case of termwise differentiation of the power series (1), for all x lying inside the interval of convergence $(a-R,\ a+R)$ we obtain a series whose sum is equal to the derivative of the sum of the given series (1).

Example. Find the region of convergence of the series

$$(x-2)+(x-2)^2+(x-2)^3+\ldots+(x-2)^n+\ldots$$

Solution. Putting $x-2=X$, we get the series

$$X+X^2+X^3+\ldots+X^n+\ldots$$

This series converges when $-1<X<+1$. Hence, the given series converges for all x that satisfy the inequality $-1<x-2<1$, that is, when $1<x<3$ (Fig. 354).

SEC. 16. TAYLOR'S SERIES AND MACLAURIN'S SERIES

In Sec. 6, Ch. IV, it was shown that for a function $f(x)$ that has all derivatives up to the $(n+1)$st order inclusive, Taylor's formula holds in the neighbourhood of the point $x=a$ (that is, in some interval containing the point $x=a$):

$$f(x)=f(a)+\frac{x-a}{1}f'(a)+$$
$$+\frac{(x-a)^2}{1\cdot 2}f''(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+R_n(x), \qquad (1)$$

where the so-called remainder term $R_n(x)$ is computed from the formula

$$R_n=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a+\theta(x-a)], \quad 0<\theta<1.$$

If the function $f(x)$ has derivatives of **all** orders in the neighbourhood of the point $x=a$, then in Taylor's formula the number n may be taken as large as we please. Suppose that in the neighbourhood under consideration the remainder term R_n tends to zero as $n\to\infty$:

$$\lim_{n\to\infty} R_n=0.$$

Then, passing to the limit in formula (1) as $n\to\infty$, we get an infinite series on the right which is called the *Taylor series*:

$$f(x)=f(a)+\frac{x-a}{1}f'(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\ldots \qquad (2)$$

This equation is valid only when $R_n(x)\to 0$ as $n\to\infty$. Then the series on the right converges and its sum is equal to the given function $f(x)$. Let us prove that this is indeed the case:

$$f(x)=P_n(x)+R_n(x),$$

where

$$P_n(x)=f(a)+\frac{x-a}{1!}f'(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a).$$

Since it is given that $\lim_{n\to\infty} R_n=0$, we have

$$f(x)=\lim_{n\to\infty} P_n(x).$$

But $P_n(x)$ is the nth partial sum of the series (2); its limit is equal to the sum of the series on the right side of (2). Hence, (2) is true:

$$f(x)=f(a)+\frac{x-a}{1}f'(a)+\frac{(x-a)^2}{2}f''(a)+\ldots+\frac{(x-a)^n}{n!}f^{(n)}(a)+\ldots$$

From the foregoing it follows that the *Taylor series is a given function* $f(x)$ *only when* $\lim R_n = 0$. If $\lim R_n \neq 0$, then the series is not the given function, although it may converge (to a different function).

If in the Taylor series we put $a=0$, we get a special case of this series known as *Maclaurin's series*:

$$f(x) = f(0) + \frac{x}{1} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots \quad (3)$$

If for some function we have a formally written Taylor's series, then in order to prove that this series is a given function it is either necessary to prove that the remainder term approaches zero, or to be convinced in some way that this series converges to the given function.

We note that for each of the elementary functions defined in Sec. 8, Ch. I, there exists an a and an R such that in the interval $(a-R, a+R)$ it may be expanded into a Taylor's series or (if $a=0$) into a Maclaurin's series.

SEC. 17. EXAMPLES OF EXPANSION OF FUNCTIONS IN SERIES

1. Expanding the function $f(x) = \sin x$ in a Maclaurin's series. In Sec. 7, Ch. IV, we obtained the formula

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} + R_{2n}.$$

Since it was proved that $\lim\limits_{n \to \infty} R_{2n} = 0$, it follows, by what has been said in the preceding section, that we get an expansion of $\sin x$ in a Maclaurin's series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} + \ldots \quad (1)$$

Since the remainder term approaches zero for any x, the given series converges and, for its sum, has the function $\sin x$ for any x.

Fig. 355 shows the graphs of the function $\sin x$ and of the first three partial sums of the series (1).

This series is used to compute the values of $\sin x$ for different values of x.

To illustrate, let us compute $\sin 10^\circ$ to the fifth decimal place. Since $10^\circ = \frac{\pi}{18} = 0.174533$, we have

$$\sin 10^\circ = \frac{\pi}{18} - \frac{1}{3!}\left(\frac{\pi}{18}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{18}\right)^5 - \frac{1}{7!}\left(\frac{\pi}{18}\right)^7 + \ldots$$

Confining ourselves to the first two terms, we get the following approximate equality:

$$\sin\frac{\pi}{18}\approx\frac{\pi}{18}-\frac{1}{6}\left(\frac{\pi}{18}\right)^3;$$

here, we are in error by δ, which in absolute value is less than the first of the suppressed terms; that is,

$$\delta<\frac{1}{5!}\left(\frac{\pi}{18}\right)^5<\frac{1}{120}(0.2)^5<4\cdot10^{-6}.$$

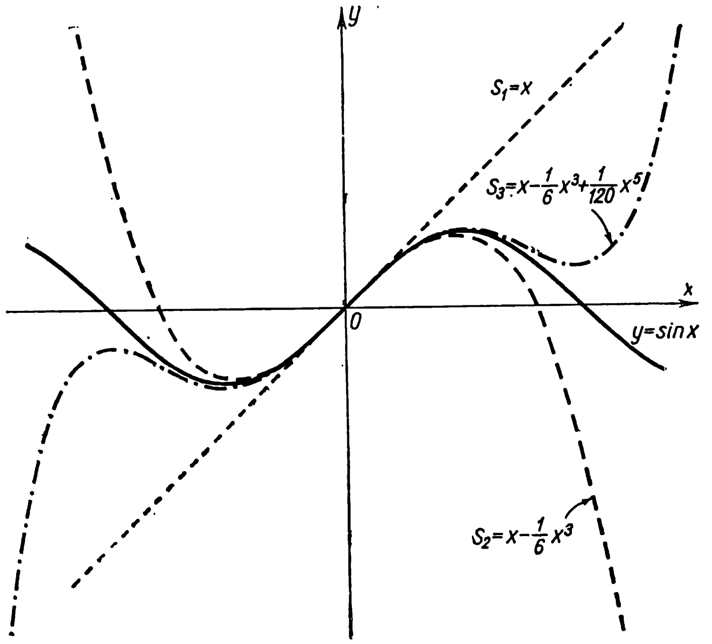


Fig. 355.

If each term in the expression for $\sin\frac{\pi}{18}$ is computed to six decimal places, we get

$$\sin\frac{\pi}{18}=0.173647.$$

We can be sure of the first four decimals.

2. Expanding the function $f(x)=e^x$ in a Maclaurin's series. On the basis of Sec. 7, Ch. IV, we have

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots, \tag{2}$$

since it was proved that $\lim_{n\to\infty} R_n(x)=0$ for any x. Hence, the series converges for all values of x and is the function e^x.

3. Expanding the function $f(x)=\cos x$ in a Maclaurin's series. From Sec. 7, Ch. IV, we have

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots; \qquad (3)$$

for all values of x the series converges and represents the function $\cos x$.

SEC. 18. EULER'S FORMULA

Up till now we have considered only series with real terms and have not dealt with series with complex terms. We shall not give the complete theory of series with complex terms, for this goes beyond the scope of this text. We shall consider only one important example in this field.

In Chapter VII we defined the function e^{x+iy} by the equation

$$e^{x+iy}=e^x(\cos y+i\sin y).$$

When $x=0$, we get Euler's formula:

$$e^{iy}=\cos y+i\sin y.$$

If we determine the exponential function e^{iy} with imaginary exponent by means of formula (2), Sec. 17, which represents the function e^x in the form of a power series, we will get the very same Euler equation. Indeed, determine e^{iy} by putting the expression iy in place of x in equation (2), Sec. 17:

$$e^{iy}=1+\frac{iy}{1!}+\frac{(iy)^2}{2!}+\frac{(iy)^3}{3!}+\ldots+\frac{(iy)^n}{n!}+\ldots \qquad (1)$$

Taking into account that $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i$, $i^6=-1$, and so forth, we transform formula (1) to the form

$$e^{iy}=1+\frac{iy}{1!}-\frac{y^2}{2!}-\frac{iy^3}{3!}+\frac{y^4}{4!}+\frac{iy^5}{5!}-\ldots$$

Separating in this series the reals from the imaginaries, we find

$$e^{iy}=\left(1-\frac{y^2}{2!}+\frac{y^4}{4!}-\ldots\right)+i\left(\frac{y}{1!}-\frac{y^3}{3!}+\frac{y^5}{5!}-\ldots\right).$$

The parentheses contain power series whose sums are equal to $\cos y$ and $\sin y$, respectively [see formulas (3) and (1) of the preceding section]. Consequently,

$$e^{iy}=\cos y+i\sin y.$$

Thus, we have again arrived at Euler's formula.

SEC. 19. THE BINOMIAL SERIES

1. Let us expand the following function in a Maclaurin's series:

$$f(x)=(1+x)^m,$$

where m is an arbitrary constant number.

Here the evaluation of the remainder term presents certain difficulties and so we shall approach the series expansion of this function somewhat differently.

Noting that the function $f(x)=(1+x)^m$ satisfies the differential equation

$$(1+x)f'(x)=mf(x) \tag{1}$$

and the condition

$$f(0)=1,$$

we find a power series whose sum $s(x)$ satisfies equation (1) and the condition $s(0)=1$:

$$s(x)=1+a_1x+a_2x^2+\ldots+a_nx^n+\ldots\,^{*)}. \tag{2}$$

Putting this series into equation (1), we get

$$(1+x)(a_1+2a_2x+3a_3x^2+\ldots+na_nx^{n-1}+\ldots)=$$
$$=m(1+a_1x+a_2x^2+\ldots+a_nx^n+\ldots).$$

Equating the coefficients of identical powers of x in different parts of the equation, we find

$$a_1=m;\quad a_1+2a_2=ma_1;\quad \ldots;\quad na_n+(n+1)a_{n+1}=ma_n;\quad \ldots$$

Whence for the coefficients of the series we get the expressions

$$a_0=1;\quad a_1=m;\quad a_2=\frac{a_1(m-1)}{2}=\frac{m(m-1)}{2};$$
$$a_3=\frac{a_2(m-2)}{3}=\frac{m(m-1)(m-2)}{2\cdot 3};\quad \ldots;$$
$$a_n=\frac{m(m-1)\ldots[m-n+1]}{1\cdot 2\cdot\ldots n};\quad \ldots$$

These are binomial coefficients.

Putting them into formula (2), we obtain

$$s(x)=1+mx+\frac{m(m-1)}{1\cdot 2}x^2+\ldots$$
$$\ldots+\frac{m(m-1)\ldots[m-(n-1)]}{1\cdot 2\ldots n}x^n+\ldots \tag{3}$$

---

*) We took the absolute term equal to unity by virtue of the initial condition $s(0)=1$.

If m is a positive integer, then beginning with the term containing x^{m+1} all coefficients are equal to zero, and the series is converted into a polynomial. For m fractional or a negative integer, we have an infinite series.

Let us determine the radius of convergence of series (3):

$$u_{n+1}=\frac{m(m-1)\ldots[m-n+1]}{n!}x^n,$$

$$u_n=\frac{m(m-1)\ldots[m-n+2]}{(n-1)!}x^{n-1},$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{m(m-1)\ldots(m-n+1)(n-1)!}{m(m-1)\ldots(m-n+2)\,n!}x\right|=$$

$$=\lim_{n\to\infty}\left|\frac{m-n+1}{n}\right||x|=|x|.$$

Thus, series (3) converges for $|x|<1$.

In the interval $(-1, 1)$, series (3) is a function $s(x)$ that satisfies the differential equation (1) and the condition

$$s(0)=1.$$

Since the differential equation (1) and the condition $s(0)=1$ are satisfied by a unique function, it follows that the sum of the series (3) is identically equal to the function $(1+x)^m$, and we obtain the expansion

$$(1+x)^m=1+mx+\frac{m(m-1)}{1\cdot 2}x^2+\frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3+\ldots \qquad (3')$$

For the particular case $m=-1$, we have

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots \qquad (4)$$

For $m=\frac{1}{2}$ we get

$$\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{2\cdot 4}x^2+\frac{1\cdot 3}{2\cdot 4\cdot 6}x^3-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4+\ldots \qquad (5)$$

For $m=-\frac{1}{2}$ we have

$$\frac{1}{\sqrt{1+x}}=1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4-\ldots \qquad (6)$$

2. We apply the binomial expansion to the expansion of other functions. Expand the following function in a Maclaurin's series:

$$f(x)=\arcsin x.$$

Putting into equation (6) the expression $-x^2$ in place of x, we get

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+$$
$$+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\ldots+\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}x^{2n}+\ldots$$

By the theorem of integration of power series we have, for $|x|<1$:

$$\int_0^x \frac{dx}{\sqrt{1-x^2}}=\arcsin x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\ldots$$
$$\ldots+\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\frac{x^{2n+1}}{2n+1}+\ldots$$

This series converges in the interval $(-1, 1)$. One could prove that the series converges for $x=\pm 1$ as well as that for these values the sum of the series is likewise equal to $\arcsin x$. Then, setting $x=1$, we get a formula for computing π:

$$\arcsin 1=\frac{\pi}{2}=1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7}+\ldots$$

SEC. 20. EXPANSION OF THE FUNCTION ln $(1+x)$ IN A POWER SERIES. COMPUTING LOGARITHMS

Integrating equation (4), Sec. 19, from 0 to x (when $|x|<1$), we obtain

$$\int_0^x \frac{dx}{1+x}=\int_0^x (1-x+x^2-x^3+\ldots)\,dx$$

or

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n+1}\frac{x^n}{n}+\ldots \qquad (1)$$

This equation holds true in the interval $(-1, 1)$.

If in this formula x is replaced by $-x$, then we get the series

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots, \qquad (2)$$

which converges in the interval $(-1, 1)$.

Using the series (1) and (2) we can compute the logarithms of numbers lying between zero and two. We note, without proof, that for $x=1$ the expansion (1) also holds true.

We will now derive a formula for computing the natural logarithms of all integers.

Since in the term-by-term subtraction of two convergent series we get a convergent series (see Sec. 1, Theorem 3), then by subtracting equation (2) from equation (1) term by term, we find

$$\ln(1+x)-\ln(1-x)=\ln\frac{1+x}{1-x}=2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right].$$

Now put $\frac{1+x}{1-x}=\frac{n+1}{n}$; then $x=\frac{1}{2n+1}$.

For any $n>0$ we have $0<x<1$; therefore

$$\ln\frac{1+x}{1-x}=\ln\frac{n+1}{n}=2\left[\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right],$$

whence

$$\ln(n+1)-\ln n=2\left[\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right]. \qquad (3)$$

For $n=1$ we then obtain

$$\ln 2=2\left[\frac{1}{1\cdot 3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\ldots\right].$$

To compute $\ln 2$ to a given degree of accuracy δ, one has to compute the partial sum s_p, choosing the number p of its terms such that the sum of the suppressed terms (that is, the error R_p committed when replacing s by s_p) is less than the admissible error δ. To do this, let us evaluate the error R_p:

$$R_p=2\left[\frac{1}{(2p+1)\,3^{2p+1}}+\frac{1}{(2p+3)\,3^{2p+3}}+\frac{1}{(2p+5)\,3^{2p+5}}+\ldots\right].$$

Since the numbers $2p+3$, $2p+5$, ... are greater than $2p+1$, it follows that by replacing them by $2p+1$ we increase each fraction. Therefore,

$$R_p<2\left[\frac{1}{(2p+1)\,3^{2p+1}}+\frac{1}{(2p+1)\,3^{2p+3}}+\frac{1}{(2p+1)\,3^{2p+5}}+\ldots\right],$$

or

$$R_p<\frac{1}{2p+1}\left[\frac{1}{3^{2p+1}}+\frac{1}{3^{2p+3}}+\frac{1}{3^{2p+5}}+\ldots\right].$$

The series in the brackets is a geometric progression with ratio $\frac{1}{9}$. Computing the sum of this progression we find

$$R_p<\frac{2}{2p+1}\,\frac{\frac{1}{3^{2p+1}}}{1-\frac{1}{9}}=\frac{1}{(2p+1)\,3^{2p-1}4}. \qquad (4)$$

If we now want to compute $\ln 2$ to, for example, seven decimal places, we must choose p such that $R_p<0.0000001$. This can be done by selecting p so that the right side of inequality (4) is less

than 0.0000001. By direct choice we find that it is sufficient to take $p=8$. To seven-decimal accuracy we have

$$\ln 2 \approx s_8 = 2\left[\frac{1}{1\cdot 3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\frac{1}{7\cdot 3^7}+\frac{1}{9\cdot 3^9}+\frac{1}{11\cdot 3^{11}}+\right.$$
$$\left.+\frac{1}{13\cdot 3^{13}}+\frac{1}{15\cdot 3^{15}}\right]=0.6931471.$$

Thus, $\ln 2=0.6931471$. These seven digits are significant digits.

Assuming $n=2$ in formula (3), we obtain

$$\ln 3=\ln 2+2\left[\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\dots\right]=1.098612, \text{ and so forth.}$$

In this way we obtain the **natural** logarithms of any integer.

To get the **common** logarithms of numbers, use the following relationship (see Sec. 8, Ch. II)

$$\log N = M \ln N,$$

where $M=0.434294$. Then, for example, we get $\ln 2=0.6931472$, $\log 2=0.30103$.

SEC. 21. INTEGRATION BY USE OF SERIES (CALCULATING DEFINITE INTEGRALS)

In Chapters X and XI it was noted that there exist definite integrals, which, as functions of the superior limit, are not, in final form, expressible in terms of elementary functions. It is sometimes convenient to compute such integrals by means of series.

Let us consider several examples.

1. Let it be required to compute the integral

$$\int_0^a e^{-x^2}\,dx.$$

Here, the antiderivative of e^{-x^2} is not an elementary function. To evaluate this integral we expand the integrand in a series, replacing x by $-x^2$ in the expansion of e^x [see formula (2), Sec. 17]:

$$e^{-x^2}=1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\dots+(-1)^n\frac{x^{2n}}{n!}+\dots$$

Integrating both sides of this equality from 0 to a, we obtain

$$\int_0^a e^{-x^2}\,dx=\left(\frac{x}{1}-\frac{x^3}{1\cdot 3}+\frac{x^5}{2!\,5}-\frac{x^7}{3!\,7}+\dots\right)\Big|_0^a=$$
$$=\frac{a}{1}-\frac{a^3}{1!\,3}+\frac{a^5}{2!\cdot 5}-\frac{a^7}{3!\cdot 7}+\dots$$

Using this equation, we can calculate the given integral to any degree of accuracy for any a.

2. It is required to evaluate the integral

$$\int_0^a \frac{\sin x}{x}\,dx.$$

Expand the integrand in a series: from the equation

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

we get

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots,$$

the latter series converges for all values of x. Integrating term by term, we obtain

$$\int_0^a \frac{\sin x}{x}\,dx = a - \frac{a^3}{3!\,3} + \frac{a^5}{5!\,5} - \frac{a^7}{7!\,7} + \dots$$

The sum of the series is readily computed to any degree of accuracy for any a.

3. Evaluate the elliptic integral

$$\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\,d\varphi \quad (k<1).$$

Expand the integrand in a binomial series, putting $m=\frac{1}{2}$, $x=-k^2\sin^2\varphi$ [see formula (5), Sec. 19]:

$$\sqrt{1-k^2\sin^2\varphi} = 1 - \frac{1}{2}k^2\sin^2\varphi - \frac{1}{2}\frac{1}{4}k^4\sin^4\varphi - \frac{1}{2}\frac{1}{4}\frac{3}{6}k^6\sin^6\varphi - \dots$$

This series converges for all values of φ and admits term-by-term integration because it majorises on any interval. Therefore,

$$\int_0^{\varphi} \sqrt{1-k^2\sin^2\varphi}\,d\varphi = \varphi - \frac{1}{2}k^2\int_0^{\varphi}\sin^2\varphi\,d\varphi - \frac{1}{2}\frac{1}{4}k^4\int_0^{\varphi}\sin^4\varphi\,d\varphi -$$

$$-\frac{1}{2}\frac{1}{4}\frac{3}{6}k^6\int_0^{\varphi}\sin^6\varphi\,d\varphi - \dots$$

The integrals on the right are computed in elementary fashion. For $\varphi=\frac{\pi}{2}$ we have

$$\int_0^{\frac{\pi}{2}} \sin^{2n}\varphi\, d\varphi=\frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n}\,\frac{\pi}{2}$$

(see Sec. 6, Ch. XI) and, hence,

$$\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\, d\varphi=\frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2 k^2-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{k^4}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{k^6}{5}-\ldots\right].$$

SEC. 22. INTEGRATING DIFFERENTIAL EQUATIONS BY MEANS OF SERIES

If the integration of a differential equation does not reduce to quadratures, one resorts to approximate methods of integrating the equation. One of these methods is representing the equation in a Taylor's series; the sum of a finite number of terms of this series will be approximately equal to the desired particular solution.

To take an example, let it be required to find the solution of a second-order differential equation,

$$y''=F(x,\ y,\ y'), \tag{1}$$

that satisfies the initial conditions

$$(y)_{x=x_0}=y_0,\ (y')_{x=x_0}=y_0'. \tag{2}$$

Suppose that the solution $y=f(x)$ exists and may be given in the form of a Taylor's series (we will not discuss the conditions under which this occurs):

$$y=f(x)=f(x_0)+\frac{x-x_0}{1}f'(x_0)+\frac{(x-x_0)^2}{1\cdot 2}f''(x_0)+\ldots \tag{3}$$

We have to find $f(x_0)$, $f'(x_0)$, $f''(x_0)$, ..., i.e., the values of the derivatives of the particular solution when $x=x_0$. But this can be done by means of equation (1) and conditions (2).

Indeed, from conditions (2) it follows that

$$f(x_0)=y_0,\ f'(x_0)=y_0';$$

from equation (1) we have

$$f''(x_0)=(y'')_{x=x_0}=F(x_0,\ y_0,\ y_0').$$

Differentiating both sides of (1) with respect to x, we get

$$y''' = F'_x(x, y, y') + F'_y(x, y, y')y' + F'_{y'}(x, y, y')y''; \quad (4)$$

and substituting the value $x = x_0$ into the right side, we find

$$f'''(x_0) = (y''')_{x=x_0}.$$

Differentiating the relationship (4) once again, we find

$$f^{IV}(x_0) = (y^{IV})_{x=x_0}$$

and so on.

We put these values of the derivatives into (3). For those values of x for which this series converges, this series represents the solution of the equation.

Example 1. Find the solution of the equation

$$y'' = -yx^2,$$

which satisfies the initial conditions

$$(y)_{x=0} = 1, \quad (y')_{x=0} = 0.$$

Solution. We have

$$f(0) = y_0 = 1; \quad f'(0) = y'_0 = 0.$$

From the given equation we find $(y'')_{x=0} = f''(0) = 0$; further,

$$y''' = -y'x^2 - 2xy, \qquad (y''')_{x=0} = f'''(0) = 0,$$

$$y^{IV} = -x^2y'' - 4xy' - 2y, \qquad (y^{IV})_{x=0} = -2$$

and, generally, differentiating k times both sides of the equation by the Leibniz formula, we find (Sec. 22, Ch. III)

$$y^{(k+2)} = -y^{(k)}x^2 - 2ky^{(k-1)}x - k(k-1)y^{(k-2)}.$$

Putting $x = 0$, we have

$$y_0^{(k+2)} = -k(k-1)y_0^{k-2}$$

or, setting $k + 2 = n$,

$$y_0^n = -(n-3)(n-2)y_0^{(n-4)}.$$

Whence

$$y_0^{IV} = -1\cdot 2, \quad y_0^{(8)} = -5\cdot 6y_0^{IV} = (-1)^2(1\cdot 2)(5\cdot 6),$$

$$y_0^{(12)} = -9\cdot 10y_0^{(8)} = (-1)^3(1\cdot 2)(5\cdot 6)(9\cdot 10),$$

$$y_0^{4k} = (-1)^k(1\cdot 2)(5\cdot 6)(9\cdot 10)\ldots[(4k-3)(4k-2)].$$

In addition,

$$y_0^{(5)} = 0, \quad y_0^{(9)} = 0, \quad \ldots, \quad y_0^{(4k+1)} = 0,$$

$$y_0^{(6)} = 0, \quad y_0^{(10)} = 0, \quad \ldots, \quad y_0^{(4k+2)} = 0,$$

$$y_0^{(7)} = 0, \quad y_0^{(11)} = 0, \quad \ldots, \quad y_0^{(4k+3)} = 0.$$

Thus, only those derivatives whose order is a multiple of four do not become zero.

Putting the values of the derivatives that we have found into a Maclaurin's series, we get the solution of the equation

$$y=1-\frac{x^4}{4!}1\cdot 2+\frac{x^8}{8!}(1\cdot 2)(5\cdot 6)-\frac{x^{12}}{12!}(1\cdot 2)(5\cdot 6)(9\cdot 10)+\dots$$

$$\dots+(-1)^k\frac{x^{4k}}{(4k)!}(1\cdot 2)(5\cdot 6)\dots[(4k-3)(4k-2)]+\dots$$

By means of d'Alembert's test we can verify that this series converges for all values of x; hence, it is the solution of the equation.

If the equation is linear, it is more convenient to seek the coefficients of expansion of the particular solution by the method of undetermined coefficients. To do this, we put the series

$$y=a_0+a_1x+a_2x^2+\dots+a_nx^n+\dots$$

into the differential equation and equate the coefficients of identical powers of x on different sides of the equation.

Example 2. Find the solution of the equation

$$y''=2xy'+4y$$

that satisfies the initial conditions

$$(y)_{x=0}=0,\ (y')_{x=0}=1.$$

Solution. We set

$$y=a_0+a_1x+a_2x^2+a_3x^3+\dots+a_nx^n+\dots$$

On the basis of the initial conditions we find

$$a_0=0,\ a_1=1.$$

Hence,

$$y=x+a_2x^2+a_3x^3+\dots+a_nx^n+\dots$$

$$y'=1+2a_2x+3a_3x^2+\dots+na_nx^{n-1}+\dots$$

$$y''=2a_2+3\cdot 2a_3x+\dots+n(n-1)a_nx^{n-2}+\dots$$

Putting these expressions into the given equation and equating the coefficients of identical powers of x, we obtain

$$2a_2=0,\quad \text{whence } a_2=0;$$

$$3\cdot 2a_3=2+4,\quad \text{whence } a_3=1;$$

$$4\cdot 3a_4=4a_2+4a_2,\quad \text{whence } a_4=0;$$

$$\dots\dots\dots\dots$$

$$n(n-1)a_n=(n-2)2a_{n-2}+4a_{n-2},\quad \text{whence } a_n=\frac{2a_{n-2}}{n-1}.$$

Consequently,

$$a_5=\frac{2\cdot 1}{4}=\frac{1}{2!};\qquad a_7=\frac{2\cdot\frac{1}{2}}{6}=\frac{1}{3!};\qquad a_9=\frac{1}{4!};$$

$$a_{2k+1}=\frac{2\cdot\frac{1}{(k-1)!}}{2k}=\frac{1}{k!};$$

$$a_4=0;\quad a_6=0;\quad a_{2k}=0.$$

Substituting the coefficients which we have found, we get the desired solution:

$$y=x+\frac{x^3}{1}+\frac{x^5}{2!}+\frac{x^7}{3!}+\ldots+\frac{x^{2k+1}}{k!}+\ldots$$

The series thus obtained converges for all values of x.

It will be noted that this particular solution may be expressed in terms of the elementary functions: taking x outside the brackets we get (inside the brackets) an expansion of the function e^{x^2}. Hence,

$$y=xe^{x^2}.$$

SEC. 23. BESSEL'S EQUATION

Bessel's equation is a differential equation of the form

$$x^2y''+xy'+(x^2-p^2)y=0\quad (p=\text{const}). \tag{1}$$

The solution of this equation (as also of certain other equations with variable coefficients) should be sought not in the form of a power series, but in the form of a product of some power of x by a power series:

$$y=x^r\sum_{k=0}^{\infty}a_kx^k. \tag{2}$$

The coefficient a_0 may be considered nonzero due to the indefiniteness of the exponent r.

We rewrite the expression (2) in the form

$$y=\sum_{k=0}^{\infty}a_kx^{r+k}$$

and find its derivatives:

$$y'=\sum_{k=0}^{\infty}(r+k)a_kx^{r+k-1};$$

$$y''=\sum_{k=0}^{\infty}(r+k)(r+k-1)a_kx^{r+k-2}.$$

Put these expressions into equation (1):

$$x^2\sum_{k=0}^{\infty}(r+k)(r+k-1)a_kx^{r+k-2}+$$

$$+x\sum_{k=0}^{\infty}(r+k)a_kx^{r+k-1}+(x^2-p^2)\sum_{k=0}^{\infty}a_kx^{r+k}=0.$$

Equating to zero the coefficients of x to the powers r, $r+1$, $r+2, \ldots, r+k$, we get a system of equations:

$$\left.\begin{aligned} &[r(r-1)+r-p^2]\,a_0=0 \text{ or } [r^2-p^2]\,a_0=0,\\ &[(r+1)r+(r+1)-p^2]\,a_1=0 \text{ or } [(r+1)^2-p^2]\,a_1=0,\\ &[(r+2)(r+1)+(r+2)-p^2]\,a_2+a_0=0 \text{ or } [(r+2)^2-p^2]a_2+a_0=0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &[(r+k)(r+k-1)+(r+k)-p^2]a_k+a_{k-2}=0 \text{ or}\\ &\qquad\qquad [(r+k)^2-p^2]\,a_k+a_{k-2}=0. \end{aligned}\right\} \quad (3)$$

Let us consider the latter equation:

$$[(r+k)^2-p^2]\,a_k+a_{k-2}=0. \quad (3')$$

It may be rewritten as follows:

$$[(r+k-p)(r+k+p)]\,a_k+a_{k-2}=0.$$

It is given that $a_0 \neq 0$; hence,

$$r^2-p^2=0,$$

therefore, $r_1=p$ or $r_2=-p$.

Let us first consider the solution for $r_1=p>0$.

From the system of equations (3) we determine all the coefficients $a_1, a_2, \ldots$ in succession; a_0 remains arbitrary. For instance, put $a_0=1$. Then

$$a_k=-\frac{a_{k-2}}{k(2p+k)}.$$

Assigning various values to k, we find

$$\left.\begin{aligned} &a_1=0, \quad a_3=0 \text{ and, generally, } a_{2m+1}=0;\\ &a_2=-\frac{1}{2(2p+2)}; \quad a_4=\frac{1}{2\cdot 4(2p+2)(2p+4)}; \ldots;\\ &a_{2\nu}=(-1)^{\nu+1}\frac{1}{2\cdot 4\cdot 6\ldots 2\nu(2p+2)(2p+4)\ldots(2p+2\nu)}. \end{aligned}\right\} \quad (4)$$

Putting the coefficients found into (2), we obtain

$$y_1=x^p\left[1-\frac{x^2}{2(2p+2)}+\frac{x^4}{2\cdot 4(2p+2)(2p+4)}-\right.$$
$$\left.-\frac{x^6}{2\cdot 4\cdot 6(2p+2)(2p+4)(2p+6)}+\ldots\right]. \quad (5)$$

All the coefficients $a_{2\nu}$ will be determined, since for every k the coefficient of a_k in (3),

$$(r+k)^2-p^2,$$

will be different from zero.

Thus, y_1 is a particular solution of equation (1).

Let us further establish the conditions under which all the coefficients a_k will be determined for the second root $r_2 = -p$ as well. This will occur if for any even integral positive k the following inequalities are fulfilled:

$$(r_2+k)^2 - p^2 \neq 0 \tag{6}$$

or

$$r_2 + k \neq p.$$

But $p = r_1$; hence,

$$r_2 + k \neq r_1.$$

Thus, condition (6) is in this case equivalent to the following

$$r_1 - r_2 \neq k,$$

where k is a positive even integer. But

$$r_1 = p, \quad r_2 = -p,$$

hence

$$r_1 - r_2 = 2p.$$

Thus, if p is not equal to an integer, it is possible to write a second particular solution that is obtained from expression (5) by substituting $-p$ for p:

$$y_2 = x^{-p}\left[1 - \frac{x^2}{2(-2p+2)} + \frac{x^4}{2\cdot 4(-2p+2)(-2p+4)} - \right.$$
$$\left. - \frac{x^6}{2\cdot 4\cdot 6(-2p+2)(-2p+4)(-2p+6)} + \ldots\right]. \tag{5'}$$

The power series (5) and (5') converge for all values of x; this is readily found by d'Alembert's test. It is likewise obvious that y_1 and y_2 are linearly independent.*)

The solution y_1 multiplied by a certain constant is called a *Bessel function of the first kind of order* p and is designated by the symbol J_p. The solution y_2 is denoted by the symbol J_{-p}.

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*) The linear independence of functions is verified as follows. Consider the relation

$$\frac{y_2}{y_1} = x^{-2p}\frac{1 - \frac{x^2}{2(-2p+2)} + \frac{x^4}{2\cdot 4(-2p+2)(-2p+4)} - \ldots}{1 - \frac{x^2}{2(2p+2)} + \frac{x^4}{2\cdot 4(2p+2)(2p+4)} - \ldots}.$$

This relation is not constant, since for $x \to 0$ it approaches infinity. Hence the functions y_1 and y_2 are linearly independent.

Thus, for p not equal to an integer, the general solution of equation (1) has the form

$$y=C_1J_p+C_2J_{-p}.$$

For instance, when $p=\frac{1}{2}$ the series (5) will have the form

$$x^{\frac{1}{2}}\left[1-\frac{x^2}{2\cdot 3}+\frac{x^4}{2\cdot 4\cdot 3\cdot 5}-\frac{x^6}{2\cdot 4\cdot 6\cdot 3\cdot 5\cdot 7}+\dots\right]=$$

$$=\frac{1}{\sqrt{x}}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\right].$$

This solution multiplied by the constant factor $\sqrt{\frac{2}{\pi}}$ is called Bessel's function $J_{\frac{1}{2}}$; we note that the brackets contain a series whose sum is equal to $\sin x$. Hence,

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x.$$

In exactly the same way, using formula (5′), we obtain

$$J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos x.$$

The general integral of (1) for $p=\frac{1}{2}$ is

$$y=C_1J_{\frac{1}{2}}(x)+C_2J_{-\frac{1}{2}}(x).$$

Now let p be an integer which we shall denote by $n\,(n\geqslant 0)$. The solution of (5) will in this case be meaningful and is the first particular solution of (1).

But the solution of (5′) will not be meaningful because one of the factors of the denominator will become zero upon expansion.

For positive integral $p=n$ the Bessel function J_n is determined by the series (5) multiplied into the constant factor $\frac{1}{2^n n!}$ (when $n=0$ we multiply by 1):

$$J_n(x)=\frac{x^n}{2^n n!}\left[1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2\cdot 4(2n+2)(2n+4)}-\right.$$

$$\left.-\frac{x^6}{2\cdot 4\cdot 6(2n+2)(2n+4)(2n+6)}+\dots\right]$$

or

$$J_n(x)=\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!\,(n+\nu)!}\left(\frac{x}{2}\right)^{n+2\nu}. \tag{7}$$

It may be shown that the second particular solution should in this case be sought in the form

$$K_n(x) = J_n(x)\ln x + x^{-n}\sum_{k=0}^{\infty} b_k x^k.$$

Putting this expression into (1), we determine the coefficients b_k. The function $K_n(x)$, with the coefficients thus determined, multiplied by a certain constant is called *Bessel's function of the second kind of order n*.

This is the second solution of (1), which with the first one forms a linearly independent system.

The general integral will be of the form

$$y = C_1 J_n(x) + C_2 K_n(x). \tag{8}$$

We note that

$$\lim_{x \to 0} K_n(x) = \infty.$$

Hence, if we want to consider the final solutions for $x=0$, then we must put $C_2 = 0$ into formula (8).

Example. Find the solution of Bessel's equation, for $p=0$,

$$y'' + \frac{1}{x}y' + y = 0$$

that satisfies the initial conditions: for $x=0$,

$$y=2,\ y'=0.$$

Solution. From (7) we find one particular solution:

$$J_0(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{(\nu!)^2}\left(\frac{x}{2}\right)^{2\nu} = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots$$

Using this solution, we can write a solution that satisfies the given initial conditions, namely:

$$y = 2J_0(x).$$

Note. If we had to find the general integral of this given equation we would seek the second particular solution in the form

$$K_0(x) = J_0 \ln x + \sum_{k=0}^{\infty} b_k x^k.$$

Without giving all the computations, we indicate that the second particular solution, which we denote by $K_0(x)$, is of the form

$$K_0(x) = J_0(x)\ln x + \frac{x^2}{2^2} - \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4\left(1+\frac{1}{2}\right) + \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6\left(1+\frac{1}{2}+\frac{1}{3}\right) - \ldots$$

This function multiplied by some constant factor is called Bessel's function of the second kind of order zero.

Exercises on Chapter XVI

Write the first several terms of the series according to the given general term:

1. $u_n=\frac{1}{n(n+1)}$. 2. $u_n=\frac{n^3}{n+1}$. 3. $u_n=\frac{(n!)^2}{(2n)!}$. 4. $u_n=(-1)^{n+1}\frac{x^n}{n^k}$.
5. $u_n=\sqrt[3]{n^3+1}-\sqrt{n^2+1}$.

Test the following series for convergence: 6. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}+\ldots$ *Ans*. Converges. 7. $\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{20}}+\frac{1}{\sqrt{30}}+\ldots+\frac{1}{\sqrt{10n}}+\ldots$ *Ans*. Diverges. 8. $2+\frac{3}{2}+\frac{4}{3}+\ldots+\frac{n+1}{n}+\ldots$ *Ans.* Diverges. 9. $\frac{1}{\sqrt[3]{7}}+\frac{1}{\sqrt[3]{8}}+\ldots+\frac{1}{\sqrt[3]{n+6}}+\ldots$ *Ans*. Diverges. 10. $\frac{1}{2}+\left(\frac{2}{3}\right)^4+\left(\frac{3}{4}\right)^9+\ldots+\left(\frac{n}{n+1}\right)^{n^2}+\ldots$ *Ans*. Converges. 11. $\frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\frac{4}{17}+\ldots+\frac{n}{n^2+1}+\ldots$ *Ans*. Diverges. 12. $\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\ldots+\frac{1}{1+n^2}+\ldots$ *Ans*. Converges.

Test for convergence the following series with given general terms: 13. $u_n=\frac{1}{n^3}$. *Ans*. Converges. 14. $u_n=\frac{1}{\sqrt[3]{n^2}}$. *Ans*. Diverges. 15. $u_n=\frac{2}{5n+1}$. *Ans*. Diverges. 16. $u_n=\frac{1+n}{3+n^2}$. *Ans*. Diverges. 17. $u_n=\frac{1}{n^2+2n+3}$. *Ans*. Converges. 18. $u_{n-1}=\frac{1}{n\ln n}$. *Ans*. Diverges. 19. Prove the inequality

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}>\ln(n+1)>\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+1}.$$

20. Is the Leibniz theorem applicable to the series

$$\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\ldots+\frac{1}{\sqrt{n}-1}-\frac{1}{\sqrt{n}+1}+\ldots?$$

Ans. It is not applicable because the terms of the series do not decrease monotonically in absolute value. The series diverges.

How many first terms must be taken in the series so that their sum should not differ by more than 10^{-6} of the sum of the corresponding series:

21. $\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}+\ldots+\frac{1}{2^n}-\frac{1}{2^{n+1}}+\ldots$ *Ans*. $n=20$. 22. $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\ldots+\frac{1}{n}-\frac{1}{n+1}+\ldots$ *Ans*. $n=10^6$. 23. $\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\ldots+\frac{1}{n^2}-\frac{1}{(n+1)^2}+\ldots$ *Ans*. $n=10^3$. 24. $\frac{1}{2}-\frac{1}{2\cdot3}+\frac{1}{2\cdot3\cdot4}-\frac{1}{2\cdot3\cdot4\cdot5}+\ldots+\frac{1}{n!}-\frac{1}{(n+1)!}+\ldots$ *Ans*. $n=10$.

Find out which of the following series converges absolutely:

25. $1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}+\ldots+(-1)^{n+1}\frac{1}{(2n-1)^2}+\ldots$ *Ans.* Converges absolutely. 26. $\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{3}\cdot\frac{1}{2^3}-\ldots+(-1)^{n+1}\frac{1}{n}\cdot\frac{1}{2^n}+\ldots$ *Ans.* Converges absolutely. 27. $\frac{1}{\ln 2}-\frac{1}{\ln 3}+\frac{1}{\ln 4}-\frac{1}{\ln 5}+\ldots(-1)^n\frac{1}{\ln n}+\ldots$ *Ans.* Converges conditionally. 28. $-1+\frac{1}{\sqrt[5]{2}}-\frac{1}{\sqrt[5]{3}}+\frac{1}{\sqrt[5]{4}}+\ldots(-1)^n\frac{1}{\sqrt[5]{n}}+\ldots$ *Ans.* Converges conditionally.

Find the sum of the series: 29. $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\ldots+\frac{1}{n(n+1)(n+2)}+\ldots$ *Ans.* $\frac{1}{4}$.

For what values of x do the following series converge:

30. $1+\frac{x}{2}+\frac{x^2}{4}+\ldots+\frac{x^n}{2^n}+\ldots$ *Ans.* $-2<x<2$. 31. $x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+$ $+\ldots(-1)^{n+1}\frac{x^n}{n^2}+\ldots$ *Ans.* $-1\leqslant x\leqslant 1$. 32. $3x+3^4x^4+3^9x^9+\ldots+3^{n^2}x^{n^2}+\ldots$ *Ans.* $|x|<\frac{1}{3}$. 33. $1+\frac{100x}{1\cdot3}+\frac{10{,}000x^2}{1\cdot3\cdot5}+\frac{1{,}000{,}000x^3}{1\cdot3\cdot5\cdot7}+\ldots$ *Ans.* $-\infty<x<\infty$.

34. $\sin x+2\sin\frac{x}{3}+4\sin\frac{x}{9}+\ldots+2^n\sin\frac{x}{3^n}+\ldots$ *Ans.* $-\infty<x<\infty$.

35. $\frac{x}{1+\sqrt{1}}+\frac{x^2}{2+\sqrt{2}}+\ldots+\frac{x^n}{n+\sqrt{n}}+\ldots$ *Ans.* $-1\leqslant x<1$.

36. $x+\frac{2^k}{2!}x^2+\frac{3^k}{3!}x^3+\ldots+\frac{n^k}{n!}x^n+\ldots$ *Ans.* $-\infty<x<\infty$. 37. $x+\frac{2!}{2^2}x^2+$ $+\frac{3!}{3^3}x^3+\ldots+\frac{n!}{n^n}x^n+\ldots$ *Ans.* $-e<x<e$. 38. $x+\frac{2^2}{4!}x^2+\frac{(1\cdot2\cdot3)^2}{6!}x^3+$ $+\ldots+\frac{(n!)^2}{(2n)!}x^n+\ldots$ *Ans.* $-4<x<4$. 39. Find the sum of the series $x+2x^2+\ldots+nx^n+\ldots(|x|<1)$.

Hint. Write the series in the form

$$\begin{array}{l} x+x^2+x^3+x^4+\ldots \\ \quad x^2+x^3+x^4+\ldots \\ \qquad x^3+x^4+\ldots \\ \qquad\quad x^4+\ldots \\ \ldots\ldots\ldots\ldots \end{array} \qquad \textit{Ans. } \frac{x}{(1-x)^2}.$$

Determine which of the following series is majorised on the indicated intervals: 40. $1+\frac{x}{1^2}+\frac{x^2}{2^2}+\ldots+\frac{x^n}{n^2}+\ldots$ $(0\leqslant x\leqslant 1)$. *Ans.* Majorised.

41. $1+\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}+\ldots$ $(0\leqslant x\leqslant 1)$. *Ans.* Not majorised.

42. $\frac{\sin x}{1^2}+\frac{\sin 2x}{2^2}+\frac{\sin 3x}{3^2}+\ldots+\frac{\sin nx}{n^2}+\ldots$ $[0, 2\pi]$. *Ans.* Majorised.

Expanding Functions in Series

43. Expand $\frac{1}{10+x}$ in powers of x and determine the interval of convergence. *Ans.* The series converges for $-10<x<10$.

44. Expand $\cos x$ in powers of $\left(x-\frac{\pi}{4}\right)$. *Ans.* $\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\left(x-\frac{\pi}{4}\right)-\frac{1}{2\sqrt{2}}\left(x-\frac{\pi}{4}\right)^2+\frac{1}{6\sqrt{2}}\left(x-\frac{\pi}{4}\right)^3+\ldots$

45. Expand e^{-x} in powers of x. *Ans.* $1-x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

46. Expand e^x in powers of $(x-2)$. *Ans.* $e^2+e^2(x-2)+\frac{e^2}{2!}(x-2)^2+\frac{e^2}{3!}(x-2)^3+\ldots$

47. Expand x^3-2x^2+5x-7 in powers of $(x-1)$. *Ans.* $-3+4(x-1)+(x-1)^2+(x-1)^3$.

48. Expand the polynomial $x^{10}+2x^9-3x^7-6x^6+3x^4+6x^3-x-2$ in a Taylor's series in powers of $(x-1)$; check to see that this polynomial has the number 1 for a triple root. *Ans.* $f(x)=81(x-1)^3+270(x-1)^4+342(x-1)^5+330(x-1)^6+186(x-1)^7+63(x-1)^8+12(x-1)^9+(x-1)^{10}$.

49. Expand $\cos(x+a)$ in powers of x. *Ans.* $\cos a-x\sin a-\frac{x^2}{2!}\cos a+\frac{x^3}{3!}\sin a+\frac{x^4}{4!}\cos a-\ldots$

50. Expand $\ln x$ in powers of $(x-1)$. *Ans.* $(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\ldots$

51. Expand e^x in a series of powers of $(x+2)$. *Ans.* $e^{-2}\left[1+\sum_{n=1}^{\infty}\frac{(x+2)^n}{n!}\right]$.

52. Expand $\cos^2 x$ in a series of powers of $\left(x-\frac{\pi}{4}\right)$.

Ans. $\frac{1}{2}+\sum_{n=1}^{\infty}(-1)^n\frac{4^{n-1}\left(x-\frac{\pi}{4}\right)^{2n-1}}{(2n-1)!}$ $(|x|<\infty)$.

53. Expand $\frac{1}{x^2}$ in a series of powers of $(x+1)$. *Ans.* $\sum_{n=0}^{\infty}(n+1)(x+1)^n$ $(-2<x<0)$.

54. Expand $\tan x$ in a series of powers of $\left(x-\frac{\pi}{4}\right)$. *Ans.* $1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\ldots$

Write the first four terms of the series expansion, in powers of x, of the following functions: 55. $\tan x$. *Ans.* $x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\ldots$

56. $e^{\cos x}$. *Ans.* $e\left(1-\frac{x^2}{2!}+\frac{4x^4}{4!}-\frac{31x^6}{720}-\ldots\right)$.

57. $e^{\arc\tan x}$. *Ans.* $1+x+\frac{x^2}{2}+\frac{x^3}{6}-\frac{7x^4}{24}+\ldots$

58. $\ln(1+e^x)$. *Ans.* $\ln 2+\frac{x}{2}-\frac{x^2}{8}+\frac{7x^4}{384}+\ldots$

59. $e^{\sin x}$. *Ans.* $1+x+\frac{x^2}{2}-\frac{x^4}{8}+\ldots$

60. $(1+x)^x$. *Ans.* $1+x^2-\frac{x^3}{2}+\frac{5}{6}x^4-\ldots$

61. $\sec x$. *Ans.* $1+\frac{x^2}{2}+\frac{5x^4}{24}+\ldots$

62. $\ln\cos x$. *Ans.* $-\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^6}{45}-\ldots$

63. Expand $\sin kx$ in powers of x. *Ans.* $kx-\frac{(kx)^3}{3!}+\frac{(kx)^5}{5!}-\frac{(kx)^7}{7!}+\ldots$

64. Expand $\sin^2 x$ in powers of x and determine the interval of convergence. *Ans.* $\frac{2x^2}{2!}-\frac{2^3x^4}{4!}+\frac{2^5x^6}{6!}-\ldots+(-1)^{n-1}\frac{2^{2n-1}x^{2n}}{(2n)!}+\ldots$ The series converges for all values of x.

65. Expand $\frac{1}{1+x^2}$ in a series in powers of x. *Ans.* $1-x^2+x^4-x^6+\ldots$

66. Expand $\arctan x$ in a series in powers of x.

Hint. Take advantage of the formula $\arctan x=\int\limits_0^x\frac{dx}{1+x^2}$. *Ans.* $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$ $(-1\leqslant x\leqslant 1)$.

67. Expand $\frac{1}{(1+x)^2}$ in a series of powers of x. *Ans.* $1-2x+3x^2-4x^3+\ldots$ $(-1<x<1)$.

Using the formulas for expansion of the functions e^x, $\sin x$, $\cos x$, $\ln(1+x)$ and $(1+x)^m$ into power series and applying various procedures, expand the following functions in power series and determine the intervals of convergence:

68. $\sinh x$. *Ans.* $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$ $(-\infty<x<\infty)$. 69. $\cosh x$. *Ans.* $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$ $(-\infty<x<\infty)$. 70. $\cos^2 x$. *Ans.* $1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}$ $(-\infty<x<\infty)$. 71. $(1+x)\ln(1+x)$. *Ans.* $x+\sum_{n=2}^{\infty}(-1)^n\frac{x^n}{(n-1)n}$ $(|x|\leqslant 1)$.

72. $(1+x)e^{-x}$. *Ans.* $1+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{n-1}{n!}x^n$ $(-\infty<x<\infty)$. 73. $\frac{1}{4-x^4}$.

Ans. $\sum_{n=0}^{\infty}\frac{x^{4n}}{4^{n+1}}$ $(|x|<\sqrt{2})$. **74.** $\frac{e^x-1}{x}$. *Ans.* $1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots+\frac{x^{n-1}}{n!}+\ldots$ $(-\infty<x<\infty)$. **75.** $\frac{1}{(1-x)^2}$. *Ans.* $\sum_{n=0}^{\infty}(n+1)x^n$ $(|x|<1)$. **76.** $e^x\sin x$. *Ans.* $x+x^2+\frac{2x^3}{3!}-\frac{4x^5}{5!}+\ldots+\sqrt{2^n}\times\sin\frac{n\pi}{4}\frac{x^n}{n!}+\ldots$ $(-\infty<x<\infty)$. **77.** $x+\sqrt{1+x^2}$. *Ans.* $x-\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\frac{x^5}{5}+\ldots+(-1)^{n+1}\frac{1\cdot3\ldots(2n-1)}{2^n\cdot n!}\times\frac{x^{2n+1}}{2n+1}+\ldots$ $(-1\leqslant x\leqslant 1)$. **78.** $\int_0^x\frac{\ln(1+x)}{x}dx$. *Ans.* $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n^2}$ $|x|\leqslant 1)$. **79.** $\int_0^x\frac{\arctan x}{x}dx$. *Ans.* $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)^2}$ $(-1\leqslant x\leqslant 1)$.

80. $\int\frac{\cos x}{x}dx$. *Ans.* $C+\ln|x|+\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{(2n)(2n)!}$ $(-\infty<x<0$ and $0<x<\infty)$. **81.** $\int_0^x\frac{dx}{1-x^9}$. *Ans.* $\sum_{n=1}^{\infty}\frac{x^{9n-8}}{9n-8}$. **82.** Prove the equations

$$\sin(a+x)=\sin a\cos x+\cos a\sin x,$$
$$\cos(a+x)=\cos a\cos x-\sin a\sin x$$

by expanding the left sides in powers of x.

Utilising appropriate series, compute: **83.** $\cos 10°$ to four decimals. *Ans.* 0.9848. **84.** $\sin 1°$ to four decimals. *Ans.* 0.0175.

85. $\sin 18°$ to three decimals. *Ans.* 0.309. **86.** $\sin\frac{\pi}{4}$ to four decimals. *Ans.* 0.7071. **87.** $\arctan\frac{1}{5}$ to four decimals. *Ans.* 0.1973. **88.** $\ln 5$ to three decimals. *Ans.* 1.609. **89.** $\log_{10}5$ to three decimals. *Ans.* 0.699. **90.** $\arcsin 1$ to within 0.0001. *Ans.* 1.5708. **91.** $\sqrt{e}$ to within 0.0001. *Ans.* 1.6487. **92.** $\log e$ to within 0.00001. *Ans.* 0.43429. **93.** $\cos 1$ to within 0.00001. *Ans.* 0.5403.

Using a Maclaurin series expansion of the function $f(x)=\sqrt[n]{a^n+x}$, compute to within 0.001: **94.** $\sqrt[3]{30}$. *Ans.* 3.107. **95.** $\sqrt{70}$. *Ans.* 4.121. **96.** $\sqrt[3]{500}$. *Ans.* 7.937. **97.** $\sqrt[5]{250}$. *Ans.* 3.017. **98.** $\sqrt{84}$. *Ans.* 9.165. **99.** $\sqrt[3]{2}$. *Ans.* 1.2598.

Expanding the integrand in a series, compute the integrals:

100. $\int_0^1\frac{\sin x}{x}dx$ to five decimal places. *Ans.* 0.94608. **101.** $\int_0^1 e^{-x^2}dx$ to four decimals. *Ans.* 0.7468. **102.** $\int_0^{\frac{\pi}{4}}\sin(x^2)\,dx$ to four decimals. *Ans.* 0.1571.

103. $\int\limits_0^{\frac{1}{2}} e^{\sqrt{x}}\,dx$ to two decimals. *Ans.* 0.81. **104.** $\int\limits_0^{0.5} \frac{\text{arc tan}\,x}{x}\,dx$ to three decimal places. *Ans.* 0.487. **105.** $\int\limits_0^1 \cos\sqrt{x}\,dx$ to within 0.001. *Ans.* 0.764. **106.** $\int\limits_0^{\frac{1}{4}} \ln(1+\sqrt{x})\,dx$ to within 0.001. *Ans.* 0.071. **107.** $\int\limits_0^1 e^{-\frac{x^2}{4}}\,dx$ to within 0.0001. *Ans.* 0.9226. **108.** $\int\limits_0^{\frac{1}{5}} \frac{\sin x}{\sqrt{1-x}}\,dx$ to within 0.0001. *Ans.* 0.0214.

109. $\int\limits_0^{0.5} \frac{dx}{1+x^4}$ to within 0.001. *Ans.* 0.494. **110.** $\int\limits_0^1 \frac{\ln(1+x)}{x}\,dx$. *Ans.* $\frac{\pi^2}{12}$.

Note. When solving this exercise and the two following ones it is well to bear in mind the equations: $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$, $\sum\limits_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$ which will be established in Sec. 2, Ch. XVII.

111. $\int\limits_0^1 \frac{\ln(1-x)}{x}\,dx$. *Ans.* $\frac{\pi^2}{6}$.

112. $\int\limits_0^1 \ln\frac{1+x}{1-x}\frac{dx}{x}$. *Ans.* $\frac{\pi^2}{4}$.

Integrating Differential Equations by Means of Series

113. Find the solution of the equation $y''=xy$ that satisfies the initial conditions for $x=0$, $y=1$, $y'=0$.

Hint. Look for the solution in the form of a series. *Ans.* $1+\frac{x^3}{2\cdot3}+\frac{x^6}{2\cdot3\cdot5\cdot6}+\ldots+\frac{x^{3k}}{2\cdot3\cdot5\cdot6\ldots(3k-1)\,3k}+\ldots$

114. Find the solution of the equation $y''+xy'+y=0$ that satisfies the initial conditions for $x=0$, $y=0$, $y'=1$. *Ans.* $x-\frac{x^3}{3}+\frac{x^5}{1\cdot3\cdot5}-\ldots\frac{(-1)^{n+1}x^{2n-1}}{1\cdot3\cdot5\ldots(2n-1)}$.

115. Find the general solution of the equation

$$x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=0.$$

Hint. Seek the solution in the form

$$y=x^p\,(A_0+A_1x+A_2x^2+\ldots).$$

Ans. $C_1x^{\frac{1}{2}}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\dots\right]+C_2x^{-\frac{1}{2}}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\right]=$
$=C_1\frac{\sin x}{\sqrt{x}}+C_2\frac{\cos x}{\sqrt{x}}$.

116. Find the solution of the equation $xy''+y'+xy=0$ that satisfies the initial conditions for $x=0$, $y=1$, $y'=0$. *Ans.* $1-\frac{x^2}{2^2}+\frac{x^4}{(1\cdot2)^2 2^4}-\frac{x^6}{(1\cdot2\cdot3)^2 2^6}+$ $+\dots(-1)^k\frac{x^{2k}}{(k!)^2 2^{2k}}+\dots$

Hint. The two latter differential equations are particular cases of the Bessel equation

$$x^2y''+xy'+(x^2-n^2)y=0$$

for $n=\frac{1}{2}$ and $n=0$.

117. Find the general solution of the equation

$$4xy''+2y'+y=0.$$

Hint. Seek the solution in the form of a series $x^p(a_0+a_1x+a_2x^2+\dots)$. *Ans.* $C_1\cos\sqrt{x}+C_2\sin\sqrt{x}$.

118. Find the solution of the equation $(1-x^2)y''-xy'=0$ that satisfies the initial conditions $y'=1$ when $x=0$ and $y=0$. *Ans.* $x+\frac{1}{2}\frac{x^3}{3}+\frac{1}{2}\frac{3}{4}\frac{x^5}{5}+$ $+\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7}+\dots$

119. Find the solution of the equation $(1+x^2)y''+2xy'=0$ that satisfies the initial conditions $y'=1$ when $x=0$ and $y=0$. *Ans.* $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots$

120. Find the solution of the equation $y''=xyy'$ that satisfies the initial conditions $y'=1$ when $x=0$ and $y=1$. *Ans.* $1+x+\frac{x^3}{3!}+\frac{2x^4}{4!}+\frac{3x^5}{5!}+\dots$

121. Find the solution of the equation $(1-x)y'=1+x-y$ that satisfies the initial conditions $y=0$ when $x=0$, and indicate the interval of convergence of the series obtained. *Ans.* $x+\frac{x^2}{1\cdot2}+\frac{x^3}{2\cdot3}+\frac{x^4}{3\cdot4}+\dots\quad(-1\leqslant x\leqslant1)$.

122. Find the solution of the equation $xy''+y=0$ that satisfies the initial conditions $y'=1$ when $x=0$ and $y=0$, and indicate the interval of convergence. *Ans.* $x-\frac{x^2}{(1!)^2 2}+\frac{x^3}{(2!)^2 3}-\frac{x^4}{(3!)^2 4}+\dots\quad(-\infty<x<\infty)$.

123. Find the solution of the equation $y''+\frac{2}{x}y'+y=0$ that satisfies the initial conditions $y'=1$ when $x=0$, $y=1$. *Ans.* $\frac{\sin x}{x}$.

124. Find the solution of the equation $y''+\frac{1}{x}y'+y=0$ that satisfies the initial conditions $y'=0$ when $x=0$ and $y=1$, and indicate the interval of convergence of the series obtained. *Ans.* $1-\frac{x^2}{2!}+\frac{x^4}{2^2\cdot4^2}-\frac{x^6}{2^2\cdot4^2\cdot6^2}+\dots$ $(|x|<\infty)$.

Find the first three terms of the expansion in a power series of the solutions of the following differential equations for the given initial conditions **125.** $y'=x^2+y^2$; for $x=0$, $y=1$. *Ans.* $1+x+x^2+\frac{4x^3}{3}+\dots$ **126.** $y''=e^y+x$; for $x=0$, $y=1$, $y'=0$. *Ans.* $1+\frac{ex^2}{2}+\frac{x^3}{6}+\dots$ **127.** $y'=\sin y-\sin x$; for $x=0$, $y=0$. *Ans.* $-\frac{x^2}{2}-\frac{x^3}{6}-\dots$

Find several terms of the series expansion of solutions of differential equations under the indicated initial conditions: **128.** $y''=yy'-x^2$ when $x=0$, $y=0$ and $y'=0$. *Ans.* $1+x+\frac{x^2}{2!}+\frac{2x^3}{3!}+\frac{8x^4}{4!}+\frac{14x^5}{5!}+\dots$ **129.** $y'=y^2+x^3$ when $x=0$ and $y=\frac{1}{2}$. *Ans.* $\frac{1}{2}+\frac{1}{4}x+\frac{1}{8}x^2+\frac{1}{16}x^3+\frac{9}{32}x^4+\dots$ **130.** $y'=x^2-y^2$ when $x=0$ and $y=0$. *Ans.* $\frac{1}{3}x^3-\frac{1}{7\cdot 9}x^7+\frac{2}{7\cdot 11\cdot 27}x^{11}-\dots$ **131.** $y'=x^2y^2-1$ when $x=0$ and $y=1$. *Ans.* $1-x+\frac{x^3}{3}-\frac{x^4}{2}+\frac{x^5}{5}-\dots$ **132.** $y'=e^y+xy$ when $x=0$ and $y=0$. *Ans.* $x+\frac{x^2}{2}+\frac{2x^3}{3}+\frac{11x^4}{2\cdot 3\cdot 4}+\dots$

CHAPTER XVII

FOURIER SERIES

SEC. 1. DEFINITION. STATEMENT OF THE PROBLEM

A functional series of the form

$$\frac{a_0}{2}+a_1\cos x+b_1\sin x+a_2\cos 2x+b_2\sin 2x+\ldots,$$

or, more compactly, a series of the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx), \tag{1}$$

is called a *trigonometric series*. The constants a_0, a_n and b_n $(n=1, 2,\ldots)$ are called *coefficients of the trigonometric series*.

If series (1) converges, then its sum is a periodic function $f(x)$ with a period 2π, since $\sin nx$ and $\cos nx$ are periodic functions with period 2π.

Thus,

$$f(x)=f(x+2\pi).$$

Let us pose the following problem.

Given a function $f(x)$ which is periodic and has a period 2π. *Under what conditions for $f(x)$ is it possible to find a trigonometric series convergent to the given function?*

That is the problem that we shall solve in this chapter.

Determining the coefficients of a series from Fourier's formulas. Let the periodic function $f(x)$ with period 2π be such that it may be represented as a trigonometric series convergent to a given function in the interval $(-\pi, \pi)$; i. e., that it is the sum of this series:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{2}$$

Suppose that the integral of the function on the left-hand side of this equation is equal to the sum of the integrals of the terms of the series (2). This will be the case, for example, if we assume that the numerical series made up of the coefficients of the given

trigonometric series converges absolutely; that is, that the following positive number series converges:

$$\left|\frac{a_0}{2}\right|+|a_1|+|b_1|+|a_2|+|b_2|+\ldots+|a_n|+|b_n|+\ldots \qquad (3)$$

Then series (1) is majorised and, consequently, it may be integrated termwise in the interval from $-\pi$ to π. Let us take advantage of this for computing the coefficient a_0.

Integrate both sides of (2) from $-\pi$ to $+\pi$:

$$\int_{-\pi}^{\pi} f(x)\,dx=\int_{-\pi}^{\pi}\frac{a_0}{2}\,dx+\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n\cos nx\,dx+\int_{-\pi}^{\pi} b_n\sin nx\,dx\right).$$

Evaluate separately each integral on the right side:

$$\int_{-\pi}^{\pi}\frac{a_0}{2}\,dx=\pi a_0;$$

$$\int_{-\pi}^{\pi} a_n\cos nx\,dx=a_n\int_{-\pi}^{\pi}\cos nx\,dx=\frac{a_n\sin nx}{n}\Big|_{-\pi}^{\pi}=0;$$

$$\int_{-\pi}^{\pi} b_n\sin nx\,dx=b_n\int_{-\pi}^{\pi}\sin nx\,dx=b_n\frac{\cos nx}{n}\Big|_{-\pi}^{\pi}=0.$$

Consequently,

$$\int_{-\pi}^{\pi} f(x)\,dx=\pi a_0,$$

whence

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. \qquad (4)$$

To calculate the other coefficients of the series we shall need certain definite integrals, which we will consider first.

If n and k are integers, then we have the following equations: if $n \neq k$, then

$$\left.\begin{aligned}&\int_{-\pi}^{\pi}\cos nx\cos kx\,dx=0;\\&\int_{-\pi}^{\pi}\cos nx\sin kx\,dx=0;\\&\int_{-\pi}^{\pi}\sin nx\sin kx\,dx=0;\end{aligned}\right\} \qquad (I)$$

but if $n=k$, then

$$\left.\begin{aligned}&\int_{-\pi}^{\pi}\cos^2 kx\,dx=\pi;\\&\int_{-\pi}^{\pi}\sin kx\cos kx dx=0;\\&\int_{-\pi}^{\pi}\sin^2 kx\,dx=\pi.\end{aligned}\right\}\qquad\text{(II)}$$

To take an example, evaluate the first integral of group (I). Since

$$\cos nx\cos kx=\frac{1}{2}[\cos(n+k)x+\cos(n-k)x],$$

it follows that

$$\int_{-\pi}^{\pi}\cos nx\cos kx\,dx=\frac{1}{2}\int_{-\pi}^{\pi}\cos(n+k)x\,dx+\frac{1}{2}\int_{-\pi}^{\pi}\cos(n-k)x\,dx=0.$$

The other formulas of (I)*) are obtained in similar fashion. The integrals of group (II) are computed directly (see Ch. X).

Now we can compute the coefficients a_k and b_k of series (2).

To find the coefficient a_k for some definite value $k\neq 0$, multiply both sides of (2) by $\cos kx$:

$$f(x)\cos kx=\frac{a_0}{2}\cos kx+\sum_{n=1}^{\infty}(a_n\cos nx\cos kx+b_n\cos nx\cos kx).\qquad(2')$$

The resulting series on the right may be majorised, since its terms do not exceed (in absolute value) the terms of the convergent positive series (3). We can therefore integrate it termwise on any interval.

Integrate (2′) from $-\pi$ to π:

$$\int_{-\pi}^{\pi}f(x)\cos kx\,dx=\frac{a_0}{2}\int_{-\pi}^{\pi}\cos kx\,dx+$$
$$+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos kx\,dx+b_n\int_{-\pi}^{\pi}\sin nx\cos kx\,dx\right).$$

*) By means of the formulas

$$\cos nx\sin kx={}^1/_2\,[\sin(n+k)x-\sin(n-k)x],$$
$$\sin nx\sin kx={}^1/_2\,[-\cos(n+k)x+\cos(n-k)x].$$

Taking into account formulas (II) and (I), we see that all the integrals on the right are equal to zero, with the exception of the integral with coefficient a_k. Hence,

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = a_k \int_{-\pi}^{\pi} \cos^2 kx\,dx = a_k\pi,$$

whence

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx. \tag{5}$$

Multiplying both sides of (2) by $\sin kx$ and again integrating from $-\pi$ to π, we find

$$\int_{-\pi}^{\pi} f(x)\sin kx\,dx = b_k \int_{-\pi}^{\pi} \sin^2 kx\,dx = b_k\pi,$$

whence

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx. \tag{6}$$

The coefficients determined from formulas (4), (5) and (6) are called *Fourier coefficients* of the function $f(x)$, and the trigonometric series (1) with such coefficients is called *a Fourier series* of the function $f(x)$.

Let us now revert to the question posed at the beginning of this section: What properties must a function have so that the Fourier series constructed for it should converge and so that the sum of the constructed Fourier series should equal the values of the given function at corresponding points? We shall here state a theorem that will yield sufficient conditions for representing a function $f(x)$ by a Fourier series.

Definition. A function $f(x)$ is called **piecewise monotonic** on the interval $[a, b]$ if this interval may be divided by a finite number of points $x_1, x_2, \ldots, x_{n-1}$ into subintervals (a, x_1), $(x_1, x_2), \ldots,$ (x_{n-1}, b) such that the function is monotonic (that is, either nonincreasing or nondecreasing) on each of the subintervals.

From the definition it follows that if the function $f(x)$ is piecewise monotonic and bounded on the interval $[a, b]$, then it can have only discontinuities of the first kind. Indeed, if $x=c$ is a point of discontinuity of the function $f(x)$, then by virtue of the monotonicity of the function there exist the limits

$$\lim_{x\to c-0} f(x) = f(c-0), \quad \lim_{x\to c+0} f(x) = f(c+0),$$

i. e., the point c is a discontinuity of the first kind (Fig. 356).

We now state the following theorem.

Theorem. *If a periodic function $f(x)$ with period 2π is piecewise monotonic and bounded on the interval $[-\pi, \pi]$, then the Fourier series constructed for this function converges at all points. The sum of the resultant series $s(x)$ is equal to the value of $f(x)$ at the discontinuities of the function. At the discontinuities of $f(x)$, the sum of the series is equal to the arithmetical mean of the limits of $f(x)$ on the right and on the left: that is, if $x=c$ is a discontinuity of the function $f(x)$, then*

$$s(x)_{x=c}=\frac{f(c-0)+f(c+0)}{2}.$$

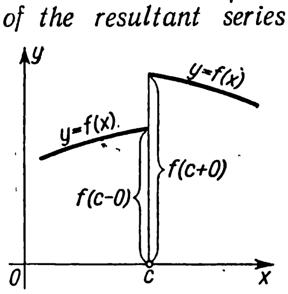


Fig. 356.

From this theorem it follows that the class of functions that may be represented by Fourier series is rather broad. That is why Fourier series have found extensive applications in various divisions of mathematics. Particularly effective use is made of Fourier series in mathematical physics and its applications to specific problems of mechanics and physics (see Ch. XVIII).

We give this theorem without proof. In Secs. 8-10 we will prove another sufficient condition for the expandability of a function in a Fourier series, which condition in a certain sense deals with a narrower class of functions.

SEC 2. EXPANSIONS OF FUNCTIONS IN FOURIER SERIES

The following are some instances of the expansion of functions in Fourier series.

Example 1. A periodic function $f(x)$ with period 2π is defined as follows:

$$f(x)=x, \quad -\pi<x\leqslant\pi.$$

This function is piecewise monotonic and bounded (Fig. 357). Hence, it admits expansion in a Fourier series.

By formula (4), Sec. 1, we find

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx=\frac{1}{\pi}\,\frac{x^2}{2}\Big|_{-\pi}^{\pi}=0.$$

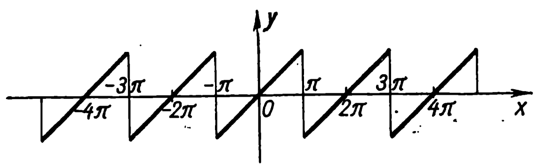


Fig. 357.

Applying formula (5), Sec. 1, and integrating by parts, we find

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos kx\,dx=\frac{1}{\pi}\left[x\frac{\sin kx}{k}\Big|_{-\pi}^{\pi}-\frac{1}{k}\int_{-\pi}^{\pi}\sin kx\,dx\right]=0.$$

By formula (6), Sec. 1, we have

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin kx\,dx=\frac{1}{\pi}\left[-x\frac{\cos kx}{k}\Big|_{-\pi}^{\pi}+\frac{1}{k}\int_{-\pi}^{\pi}\cos kx\,dx\right]=(-1)^{k+1}\frac{2}{k}.$$

Thus, we get the series

$$f(x)=2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots(-1)^{k+1}\frac{\sin kx}{k}+\ldots\right].$$

This equation occurs at all points except points of discontinuity. At each discontinuity, the sum of the series is equal to the arithmetical mean of its limits on the right and left, which is zero.

Example 2. A periodic function $f(x)$ with period 2π is defined as follows:

$$f(x)=-x \quad \text{when } -\pi\leqslant x\leqslant 0,$$
$$f(x)=x \quad \text{when } 0<x\leqslant\pi$$

[or $f(x)=|x|$] (Fig. 358). This function is also piecewise monotonic and bounded on the interval $-\pi\leqslant x\leqslant\pi$.

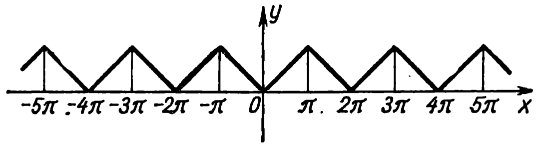


Fig. 358.

Let us determine its Fourier coefficients:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\,dx+\int_{0}^{\pi}x\,dx\right]=\pi,$$

$$a_k=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos kx\,dx+\int_{0}^{\pi}x\cos kx\,dx\right]=$$

$$=\frac{1}{\pi}\left[-\frac{x\sin kx}{k}\Big|_{-\pi}^{0}+\frac{1}{k}\int_{-\pi}^{0}\sin kx\,dx+\frac{x\sin kx}{k}\Big|_{0}^{\pi}-\frac{1}{k}\int_{0}^{\pi}\sin kx\,dx\right]=$$

$$=\frac{1}{\pi k}\left[-\frac{\cos kx}{k}\Big|_{-\pi}^{0}+\frac{\cos kx}{k}\Big|_{0}^{\pi}\right]=$$

$$=\frac{2}{\pi k^2}(\cos k\pi-1)=\begin{cases}0 \text{ for } k \text{ even,}\\ -\frac{4}{\pi k^2} \text{ for } k \text{ odd;}\end{cases}$$

$$b_k = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\sin kx\,dx + \int_{0}^{\pi} x\sin kx\,dx\right] = 0.$$

We thus obtain the series

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots + \frac{\cos(2p+1)x}{(2p+1)^2} + \ldots\right].$$

This series converges at all points, and its sum is equal to the given function.

Example 3. A periodic function $f(x)$ with period 2π is defined as follows:

$$f(x) = -1 \quad \text{for } -\pi < x < 0,$$
$$f(x) = 1 \quad \text{for } 0 \leqslant x \leqslant \pi.$$

This function (Fig. 359) is piecewise monotonic and bounded on the interval $-\pi \leqslant x \leqslant \pi$.

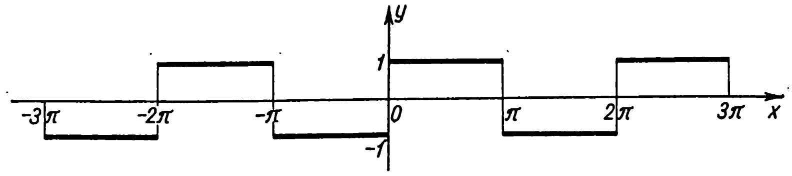


Fig. 359.

Let us compute its Fourier coefficients:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\,dx + \int_{0}^{\pi} dx\right] = 0;$$

$$a_k = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos kx\,dx + \int_{0}^{\pi}\cos kx\,dx\right] = -1\left.\frac{\sin kx}{k}\right|_{-\pi}^{0} + \left.\frac{\sin kx}{k}\right|_{0}^{\pi} = 0;$$

$$b_k = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin kx\,dx + \int_{0}^{\pi}\sin kx\,dx\right] = \frac{1}{\pi}\left[\left.\frac{\cos kx}{k}\right|_{-\pi}^{0} - \left.\frac{\cos kx}{k}\right|_{0}^{\pi}\right] =$$

$$= \frac{2}{\pi k}[1 - \cos\pi] = \begin{cases} 0 & \text{for } k \text{ even}, \\ \dfrac{4}{\pi k} & \text{for } k \text{ odd}. \end{cases}$$

Consequently, for the function at hand the Fourier series has the form

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin(2p+1)x}{2p+1} + \ldots\right].$$

This equation holds at all points with the exception of discontinuities.

Fig. 360 illustrates how the partial sums S_n of the series represent more and more accurately the function $f(x)$ as $n \to \infty$.

Example 4. A periodic function $f(x)$ with period 2π is defined as follows:

$$f(x) = x^2, \quad -\pi \leqslant x \leqslant \pi \text{ (Fig. 361).}$$

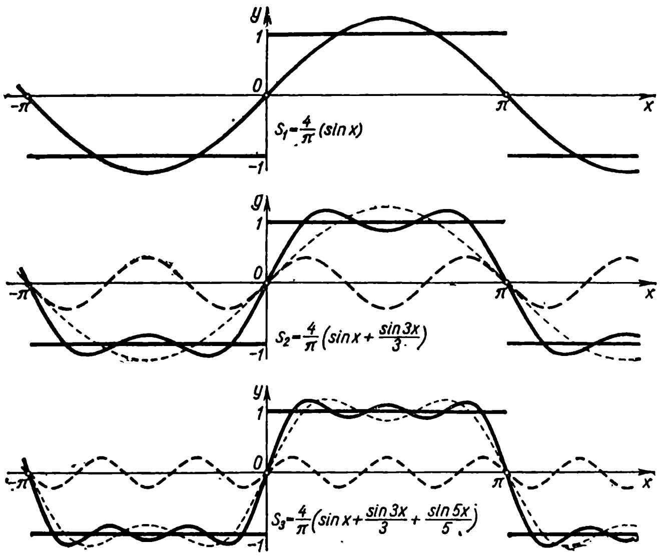


Fig. 360.

Determine its Fourier coefficients.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx=\frac{1}{\pi}\frac{x^3}{3}\Big|_{-\pi}^{\pi}=\frac{2\pi^2}{3};$$

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos kx\,dx=\frac{1}{\pi}\left[\frac{x^2\sin kx}{k}\Big|_{-\pi}^{\pi}-\frac{2}{k}\int_{-\pi}^{\pi} x\sin kx\,dx\right]=$$

$$=-\frac{2}{\pi k}\left[-\frac{x\cos kx}{k}\Big|_{-\pi}^{\pi}+\frac{1}{k}\int_{-\pi}^{\pi}\cos kx\,dx\right]=\frac{4}{\pi k^2}[\pi\cos k\pi]=$$

$$=\begin{cases}\frac{4}{k^2}\text{ for } k \text{ even},\\ -\frac{4}{k^2}\text{ for } k \text{ odd};\end{cases}$$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin kx\,dx=+\frac{1}{\pi}\left[-\frac{x^2\cos kx}{k}\Big|_{-\pi}^{\pi}+\frac{2}{k}\int_{-\pi}^{\pi} x\cos kx\,dx\right]=$$

$$=\frac{2}{\pi k}\left[\frac{x\sin kx}{k}\Big|_{-\pi}^{\pi}-\frac{1}{k}\int_{-\pi}^{\pi}\sin kx\,dx\right]=0.$$

Thus, the Fourier series of the given function has the form

$$x^2=\frac{\pi^2}{3}-4\left(\frac{\cos x}{1}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\dots\right).$$

Since the function is piecewise monotonic, bounded and continuous, this equation is fulfilled at all points.

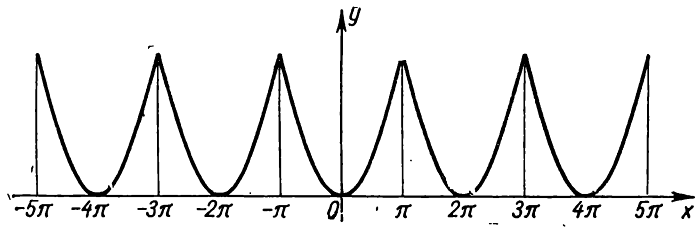


Fig. 361.

Putting $x=\pi$ in the equality obtained, we get

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Example 5. A periodic function $f(x)$ with period 2π is defined as follows:

$$f(x)=0 \quad \text{for} \quad -\pi\leqslant x\leqslant 0,$$
$$f(x)=x \quad \text{for} \quad 0<x\leqslant\pi \text{ (Fig. 362).}$$

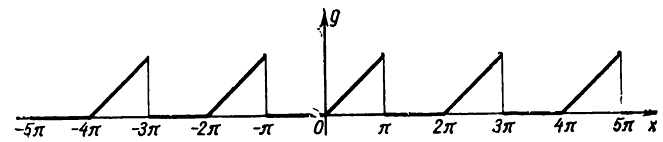


Fig. 362.

Determine the Fourier coefficients:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\,dx+\int_{0}^{\pi}x\,dx\right]=\frac{1}{\pi}\frac{\pi^2}{2}=\frac{\pi}{2};$$

$$a_k=\frac{1}{\pi}\int_{0}^{\pi}x\cos kx\,dx=\frac{1}{\pi}\left[\frac{x\sin kx}{k}\Big|_0^{\pi}-\frac{1}{k}\int_0^{\pi}\sin kx\,dx\right]=$$

$$=\frac{1}{\pi k}\frac{\cos kx}{k}\Big|_0^{\pi}=\begin{cases}-\dfrac{2}{\pi k^2} & \text{for } k \text{ odd,}\\ 0 & \text{for } k \text{ even;}\end{cases}$$

$$b_k=\frac{1}{\pi}\int_0^{\pi} x\sin kx\,dx=\frac{1}{\pi}\left[-\frac{x\cos kx}{k}\Big|_0^{\pi}+\frac{1}{k}\int_0^{\pi}\cos kx\,dx\right]=$$

$$=-\frac{\pi}{\pi k}\cos k\pi=\begin{cases}\ \ \frac{1}{k}\ \text{for } k \text{ odd},\\ -\frac{1}{k}\ \text{for } k \text{ even}.\end{cases}$$

The Fourier series will thus have the form

$$f(x)=\frac{\pi}{4}-\frac{2}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\dots\right)+\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\dots\right).$$

At the discontinuities of the function $f(x)$, the sum of the series is equal to the arithmetical mean of its limits on the right and left $\left(\text{in this case, to the number } \frac{\pi}{2}\right)$.

Putting $x=0$ in the equality obtained, we get

$$\frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

SEC. 3. REMARK ON THE EXPANSION OF A PERIODIC FUNCTION IN A FOURIER SERIES

We note the following property of a periodic function $\psi(x)$ with period 2π:

$$\int_{-\pi}^{\pi}\psi(x)\,dx=\int_{\lambda}^{\lambda+2\pi}\psi(x)\,dx,$$

no matter what the number λ.

Indeed, since

$$\psi(\xi-2\pi)=\psi(\xi)$$

it follows that, putting $x=\xi-2\pi$, we can write (for all c and d):

$$\int_c^d\psi(x)\,dx=\int_{c+2\pi}^{d+2\pi}\psi(\xi-2\pi)\,d\xi=\int_{c+2\pi}^{d+2\pi}\psi(\xi)\,d\xi=\int_{c+2\pi}^{d+2\pi}\psi(x)\,dx.$$

In particular, taking $c=-\pi$, $d=\lambda$, we get

$$\int_{-\pi}^{\lambda}\psi(x)\,dx=\int_{\pi}^{\lambda+2\pi}\psi(x)\,dx,$$

therefore,

$$\int_{\lambda}^{\lambda+2\pi} \psi(x)\,dx = \int_{\lambda}^{-\pi} \psi(x)\,dx + \int_{-\pi}^{\pi} \psi(x)\,dx + \int_{\pi}^{\lambda+2\pi} \psi(x)\,dx =$$

$$= \int_{\lambda}^{-\pi} \psi(x)\,dx + \int_{-\pi}^{\pi} \psi(x)\,dx + \int_{-\pi}^{\lambda} \psi(x)\,dx = \int_{-\pi}^{\pi} \psi(x)\,dx.$$

This property means that the *integral of a periodic function* $\psi(x)$ *over any interval whose length is equal to the period always has the same value.* This fact is readily illustrated geometrically: the cross-hatched areas in Fig. 363 are equal.

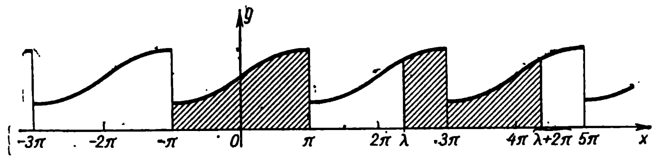


Fig. 363.

From the property that has been proved it follows that when computing Fourier coefficients we can replace the interval of integration $(-\pi, \pi)$ by the interval of integration $(\lambda, \lambda+2\pi)$, that is, we can put

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\cos nx\,dx,\\ b_n &= \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\sin nx\,dx, \end{aligned}\right\} \qquad (1)$$

where λ is any number.

This follows from the fact that the function $f(x)$ is, by hypothesis, periodic with period 2π; hence, both the functions $f(x)\cos nx$ and $f(x)\sin nx$ are periodic functions with period 2π. We now illustrate how this property simplifies the process of finding coefficients in certain cases.

Example. Let it be required to expand in a Fourier series the function $f(x)$ with period 2π, which is given on the interval $0 \leqslant x \leqslant 2\pi$ by the equation

$$f(x) = x.$$

The graph of $f(x)$ is shown in Fig. 364. On the interval $[-\pi, \pi]$ this function is represented by two formulas: $f(x) = x + 2\pi$ on the interval $[-\pi, 0]$ and $f(x) = x$ on the interval $[0, \pi]$. Yet, on $[0, 2\pi]$ it is far more simply

represented by a single formula $f(x)=x$. Therefore, to expand this function in a Fourier series it is better to make use of formulas (1), setting $\lambda=0$:

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi} x\,dx=2\pi;$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx=\frac{1}{\pi}\left[\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right]_0^{2\pi}=0;$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx=\frac{1}{\pi}\left[-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{2\pi}=-\frac{2}{n}.$$

Consequently,

$$f(x)=\pi-2\sin x-\frac{2}{2}\sin 2x-\frac{2}{3}\sin 3x-\frac{2}{4}\sin 4x-\frac{2}{5}\sin 5x-\ldots$$

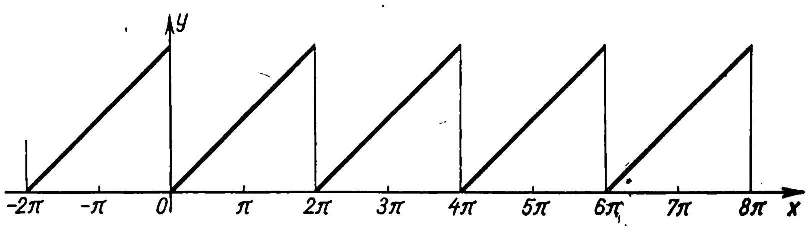


Fig. 364.

This series yields the given function at all points with the exception of points of discontinuity (i.e., except the points $x=0, 2\pi, 4\pi, \ldots$). At these points the sum of the series is equal to the half sum of the limiting values of the function $f(x)$ on the right and on the left (to the number π, in this case).

SEC. 4. FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

From the definition of an even and odd function it follows that if $\psi(x)$ is an **even** function, then

$$\int_{-\pi}^{\pi}\psi(x)\,dx=2\int_0^{\pi}\psi(x)\,dx.$$

Indeed,

$$\int_{-\pi}^{\pi}\psi(x)\,dx=\int_{-\pi}^{0}\psi(x)\,dx+\int_0^{\pi}\psi(x)\,dx=\int_0^{\pi}\psi(-x)\,dx+\int_0^{\pi}\psi(x)\,dx=$$

$$=\int_0^{\pi}\psi(x)\,dx+\int_0^{\pi}\psi(x)\,dx=2\int_0^{\pi}\psi(x)\,dx,$$

since by the definition of an even function $\psi(-x)=\psi(x)$.

It may similarly be proved that if $\varphi(x)$ is an **odd** function, then

$$\int_{-\pi}^{\pi} \varphi(x)\,dx = \int_{0}^{\pi} \varphi(-x)\,dx + \int_{0}^{\pi} \varphi(x)\,dx = -\int_{0}^{\pi} \varphi(x)\,dx + \int_{0}^{\pi} \varphi(x)\,dx = 0.$$

If an **odd** function $f(x)$ is expanded in a Fourier series, then the product $f(x)\cos kx$ is also an odd function, while $f(x)\sin kx$ is an even function; hence,

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0; \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = 0, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin kx\,dx. \end{aligned}\right\} \qquad (1)$$

Thus the Fourier series or an **odd** function contains **"only sines"** (see Example 1, Sec. 2).

If an **even** function is expanded in a Fourier series, the product $f(x)\sin kx$ is an odd function, while $f(x)\cos kx$ is an even function and, hence,

$$\left.\begin{aligned} a_0 &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx, \\ a_k &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos kx\,dx, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = 0. \end{aligned}\right\} \qquad (2)$$

Thus, the Fourier series of an **even** function contains **"only cosines"** (see Example 2, Sec. 2).

The formulas obtained permit simplifying computations when seeking Fourier coefficients in cases when the given function is even or odd. It is obvious that not every periodic function is even or odd (see Example 5, Sec. 2).

Example. Let it be required to expand in a Fourier series the even function $f(x)$ which has a period of 2π and on the interval $[0, \pi]$ is given by the equation

$$y = x,$$

We have already expanded this function in a Fourier series in Example 2, Sec. 2. (Fig. 358). Let us again compute the Fourier series of this function, taking advantage of the fact that the given function is even.

By virtue of formulas (2) $b_k=0$ for any k;

$$a_0=\frac{2}{\pi}\int_0^\pi x\,dx=\pi,\quad a_k=\frac{2}{\pi}\int_0^\pi x\cos kx\,dx=$$

$$=\frac{2}{\pi}\left[\frac{x\sin kx}{k}+\frac{\cos kx}{k^2}\right]_0^\pi=\frac{2}{\pi k^2}[(-1)^k-1]=\begin{cases}0 \text{ for } k \text{ even,}\\ -\frac{4}{\pi k^2} \text{ for } k \text{ odd.}\end{cases}$$

We obtained the same coefficients as in Example 2, Sec. 2, but this time by a short cut.

SEC. 5. THE FOURIER SERIES FOR A FUNCTION WITH PERIOD $2l$

Let $f(x)$ be a periodic function with period $2l$, generally speaking, different from 2π. Expand it in a Fourier series.

Make a substitution by the formula

$$x=\frac{l}{\pi}t.$$

Then the function $f\left(\frac{l}{\pi}t\right)$ will be a periodic function of t with period 2π.

It may be expanded in a Fourier series on the interval $-\pi\leqslant x\leqslant\pi$:

$$f\left(\frac{l}{\pi}t\right)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kt+b_k\sin kt), \tag{1}$$

where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{l}{\pi}t\right)dt,\ a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{l}{\pi}t\right)\cos kt\,dt,$$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{l}{\pi}t\right)\sin kt\,dt.$$

Now let us return to the original variable x:

$$x=\frac{l}{\pi}t,\quad t=x\frac{\pi}{l},\quad dt=\frac{\pi}{l}dx.$$

We will then have

$$\left.\begin{aligned} a_0 &= \frac{1}{l}\int_{-l}^{l} f(x)\,dx, \quad a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos k\frac{\pi}{l}x\,dx, \\ b_k &= \frac{1}{l}\int_{-l}^{l} f(x)\sin k\frac{\pi}{l}x\,dx. \end{aligned}\right\} \tag{2}$$

Formula (1) takes the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi}{l}x + b_k\sin\frac{k\pi}{l}x\right), \tag{3}$$

where the coefficients a_0, a_k, b_k are computed from formulas (2). This is the Fourier series for a periodic function with period $2l$.

We note that all the theorems that hold for Fourier series of periodic functions with period 2π hold also for Fourier series of

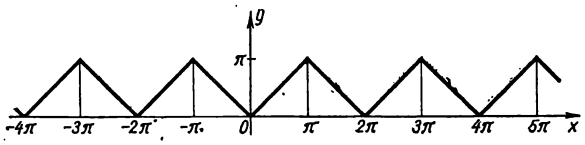


Fig. 365.

periodic functions with some other period $2l$. In particular, the sufficient condition for expansion of a function in a Fourier series (see end of Sec. 1) holds true, as do also the remark on the possibility of computing coefficients of the series by integrating over any interval whose length is equal to the period (see Sec. 3), and the remark on the possibility of simplifying computation of coefficients of the series if the function is even or odd (Sec. 4).

Example. Expand in a Fourier series the periodic function $f(x)$ with period $2l$ which on the interval $[-l, l]$ is given by the equation $f(x) = |x|$ (Fig. 365).

Solution. Since the function at hand is even, it follows that

$$b_k = 0; \quad a_0 = \frac{2}{l}\int_0^l x\,dx = l;$$

$$a_k = \frac{2}{l}\int_0^l x\cos\frac{k\pi x}{l}\,dx = \frac{2l}{\pi^2}\int_0^{\pi} x\cos kx\,dx = \begin{cases} 0 & \text{for } k \text{ even,} \\ -\dfrac{4l}{\pi^2k^2} & \text{for } k \text{ odd.} \end{cases}$$

Hence, the expansion is of the form

$$|x|=\frac{l}{2}-\frac{4l}{\pi^2}\left[\frac{\cos\frac{\pi}{l}x}{1}+\frac{\cos\frac{3\pi}{l}x}{3^2}+\ldots+\frac{\cos\frac{(2p+1)\pi}{l}x}{(2p+1)^2}+\ldots\right].$$

SEC. 6. ON THE EXPANSION OF A NONPERIODIC FUNCTION IN A FOURIER SERIES

Let there be given, on some interval $[a, b]$ a piecewise monotonic function $f(x)$ (Fig. 366). We shall show that this function $f(x)$ may be represented in the form of a sum of a Fourier series at the points of its discontinuity. To do this, let us consider an arbitrary periodic piecewise monotonic function $f_1(x)$ with period $2\mu \geqslant |b-a|$, which coincides with the function $f(x)$ on the interval $[a, b]$. [We have redefined the function $f(x)$.]

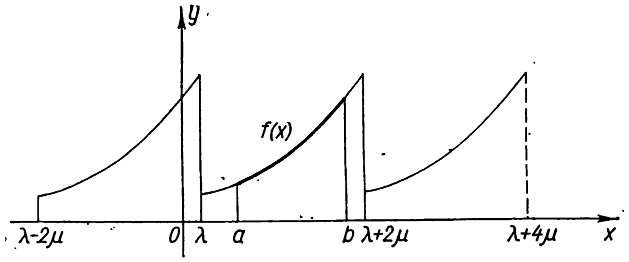


Fig. 366.

Expand $f_1(x)$ in a Fourier series. At all points of the interval $[a, b]$ (with the exception of points of discontinuity) the sum of this series coincides with the given function $f(x)$; in other words, we expanded the function $f(x)$ in a Fourier series on the interval $[a, b]$.

Let us now consider the following important case. Let a function $f(x)$ be given on the interval $[0, l]$. Redefining this function in arbitrary fashion on the interval $[-l, 0]$ (retaining piecewise monotonicity), we can expand it in a Fourier series. In particular, if we redefine this function so that when $-l \leqslant x < 0$, $f(x)=f(-x)$, we will get an even function (Fig. 367). [In this case we say that the function $f(x)$ is "continued in even fashion".] This function is expanded in a Fourier series that contains only cosines. Thus, we have expanded in cosines the function $f(x)$ given on the interval $[0, l]$.

But if we redefine the function $f(x)$ when $-l \leqslant x < 0$ as follows: $f(x) = -f(-x)$, then we get an odd function which may be expanded in sines (Fig. 368). [The function $f(x)$ is "continued in odd fashion".]

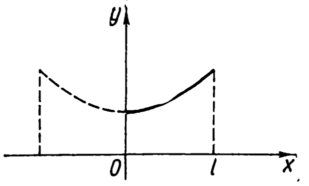


Fig. 367.

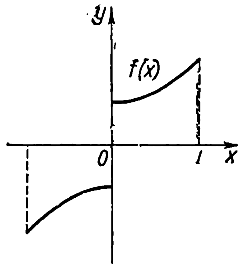


Fig. 368.

Thus, if on the interval $[0, l]$ there is given some piecewise monotonic function $f(x)$, it may be expanded in a Fourier series both in cosines and in sines.

Example 1. Let it be required to expand the function $f(x) = x$ in a series in sines on the interval $[0, \pi]$.

Solution. Continuing this function in odd fashion (Fig. 357), we get the series

$$x = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right]$$

(see Example 1, Sec. 2).

Example 2. Expand the function $f(x) = x$ in a series in cosines on the interval $[0, \pi]$.

Solution. Continuing this function in even fashion, we get

$$f(x) = |x|, \quad -\pi < x \leqslant \pi$$

(Fig. 358). Expanding it in a series we find

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

(see Example 2, Sec. 2). And so on the interval $[0, \pi]$ we have the equation

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right].$$

SEC. 7. MEAN APPROXIMATION OF A GIVEN FUNCTION BY A TRIGONOMETRIC POLYNOMIAL

Representing a function by an infinite series (Fourier's, Taylor's and so forth) has the following meaning in practice: the finite sum obtained in terminating the series with the nth term

is an **approximate expression** of the function being expanded. This approximate expression may be made as accurate as desired by choosing a sufficiently large value of n. However, the character of the approximate representation may differ.

For instance, the sum of the first n terms of a Taylor's series s_n coincides with the function at hand at one point, and at this point has derivatives up to the nth order that coincide with the derivatives of the function under consideration. An nth degree Lagrange polynomial (see Sec. 9, Ch. VII) coincides with the function under consideration at $n+1$ points.

Let us see what the character is of an approximate representation of a periodic function $f(x)$ by trigonometric polynomials of the form

$$s_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n} a_k \cos kx + b_k \sin kx,$$

where $a_0, a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ are Fourier coefficients; that is, by the sum of the first n terms of a Fourier series. We first make several remarks.

Suppose we regard some function $y=f(x)$ on the interval $[a, b]$ and want to evaluate the error when replacing this function by another function $\varphi(x)$. For the measure of error we can, for instance, take max $|f(x)-\varphi(x)|$ on the interval $[a, b]$, which is the so-called *maximum deviation* of $\varphi(x)$ from $f(x)$. But it is sometimes more natural to take for the measure of error the so-called *root mean square deviation* δ, which is defined by the equation

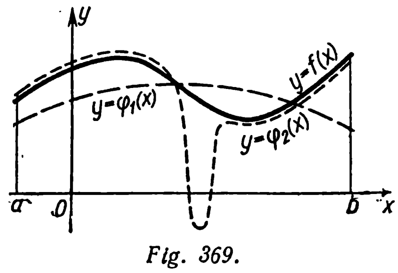


Fig. 369.

$$\delta^2=\frac{1}{(b-a)}\int_a^b [f(x)-\varphi(x)]^2\,dx.$$

Fig. 369 illustrates the difference between the root mean square deviation and the maximum deviation.

Let the solid line depict the function $y=f(x)$, the dashed lines the approximations $\varphi_1(x)$ and $\varphi_2(x)$. The maximum deviation of the curve $y=\varphi_1(x)$ is less than of the curve $y=\varphi_2(x)$, but the root mean square deviation of the first curve is greater than the second because the curve $y=\varphi_2(x)$ is considerably different from

the curve $y=f(x)$ only on a narrow section and for this reason characterises the curve $y=f(x)$ better than the first.

Now let us return to our problem.

Let there be given a periodic function $f(x)$ with period 2π. From among all the trigonometric polynomials of order n

$$\frac{\alpha_0}{2}+\sum_{k=1}^{n}(\alpha_k \cos kx+\beta_k \sin kx)$$

it is required to find (by choice of the coefficients α_k and β_k) that polynomial for which the root mean square deviation defined by the equation

$$\delta_n^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x)-\frac{\alpha_0}{2}-\sum_{k=1}^{n}(\alpha_k \cos kx+\beta_k \sin kx)\right]^2 dx,$$

has the smallest value.

The problem reduces to finding the minimum of the function $2n+1$ of the variables $\alpha_0, \alpha_1, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$.

Expanding the square under the integral sign and integrating termwise, we get

$$\delta_n^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{f^2(x)-2f(x)\left[\frac{\alpha_0}{2}+\sum_{k=1}^{n}(\alpha_k \cos kx+\beta_k \sin kx)\right]+\right.$$
$$\left.+\left[\frac{\alpha_0}{2}+\sum_{k=1}^{n}(\alpha_k \cos kx+\beta_k \sin kx)\right]^2\right\}dx=$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f^2(x)\,dx-\frac{\alpha_0}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx-\frac{1}{\pi}\sum_{k=1}^{n}\alpha_k\int_{-\pi}^{\pi}f(x)\cos kx\,dx+$$
$$+\frac{1}{\pi}\sum_{k=1}^{n}\beta_k\int_{-\pi}^{\pi}f(x)\sin kx\,dx+\frac{1}{2\pi}\frac{\alpha_0^2}{4}\int_{-\pi}^{\pi}dx+\frac{1}{2\pi}\sum_{k=1}^{n}\alpha_k^2\int_{-\pi}^{\pi}\cos^2 kx\,dx+$$
$$+\frac{1}{2\pi}\sum_{k=1}^{n}\beta_k^2\int_{-\pi}^{\pi}\sin^2 kx\,dx+\frac{1}{2\pi}\alpha_0\sum_{k=1}^{n}\alpha_k\int_{-\pi}^{\pi}\cos kx\,dx+$$
$$+\frac{1}{2\pi}\alpha_0\sum_{k=1}^{n}\beta_k\int_{-\pi}^{\pi}\sin kx\,dx+\frac{1}{\pi}\sum_{\substack{k=1\\k\neq j}}^{n}\sum_{j=1}^{n}\alpha_k\alpha_j\int_{-\pi}^{\pi}\cos kx\cos jx\,dx+$$
$$+\frac{1}{\pi}\alpha_0\sum_{k=1}^{n}\sum_{j=1}^{n}\alpha_k\beta_j\int_{-\pi}^{\pi}\cos kx\sin jx\,dx+\frac{1}{\pi}\sum_{\substack{k=1\\k\neq j}}^{n}\sum_{j=1}^{n}\beta_k\beta_j\int_{-\pi}^{\pi}\sin kx\sin jx\,dx.$$

We note that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = a_0;\quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = a_k;$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = b_k$$

are the Fourier coefficients of the function $f(x)$.

Further, by formulas (I) and (II), Sec. 1, we have: for $k=j$

$$\int_{-\pi}^{\pi}\cos^2 kx\,dx = \pi,\quad \int_{-\pi}^{\pi}\sin^2 kx\,dx = \pi,$$

$$\int_{-\pi}^{\pi}\sin kx\cos jx\,dx = 0;$$

for $k \neq j$

$$\int_{-\pi}^{\pi}\cos kx\cos jx\,dx = 0,\quad \int_{-\pi}^{\pi}\sin kx\sin jx\,dx = 0.$$

Thus, we obtain

$$\delta_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx - \frac{\alpha_0 a_0}{2} - \sum_{k=1}^{n}(\alpha_k a_k + \beta_k b_k) + \frac{\alpha_0^2}{4} + \frac{1}{2}\sum_{k=1}^{n}(\alpha_k^2 + \beta_k^2).$$

Adding and subtracting the sum

$$\frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{n}(a_k^2 + b_k^2),$$

we will have

$$\delta_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx - \frac{a_0^2}{4} - \frac{1}{2}\sum_{k=1}^{n}(a_k^2 + b_k^2) + \frac{1}{4}(\alpha_0 - a_0)^2 +$$

$$+ \frac{1}{2}\sum_{k=1}^{n}[(\alpha_k - a_k)^2 + (\beta_k - b_k)^2]. \qquad (1)$$

The first three terms of this sum are independent of the choice of coefficients $\alpha_0, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$. The remaining terms

$$\frac{1}{4}(\alpha_0 - a_0)^2 + \frac{1}{2}\sum_{k=1}^{n}[(\alpha_k - a_k)^2 + (\beta_k - b_k)^2]$$

are nonnegative. Their sum reaches the least value (equal to zero) if we put $\alpha_0 = a_0$, $\alpha_1 = a_1$, ..., $\alpha_n = a_n$, $\beta_1 = b_1$, ..., $\beta_n = b_n$. With this choice of coefficients $\alpha_0, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ the trigonometric polynomial

$$\frac{a_0}{2} + \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx)$$

will least of all differ from the function $f(x)$ in the sense that in such a choice of coefficients the square deviation δ_n^2 will be least.

We have thus proved the theorem:

Of all trigonometric polynomials of order n, that polynomial has the least root mean square deviation from the function $f(x)$, the coefficients of which polynomial are the Fourier coefficients of the function $f(x)$.

The least square deviation is

$$\delta_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx - \frac{a_0^2}{4} - \frac{1}{2}\sum_{k=1}^{n} (a_k^2 + b_k^2). \tag{2}$$

Since $\delta_n^2 \geqslant 0$, it follows that for any n we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx \geqslant \frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{n} (a_k^2 + b_k^2).$$

Hence, the series on the right converges (when $n \to \infty$), and we can write

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx \geqslant \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2). \tag{3}$$

This relation is called *Bessel's inequality*.

We note without proof that for any bounded and piecewise monotonic function the root mean square deviation obtained upon replacing the given function by the nth partial sum of the Fourier series tends to zero as $n \to \infty$, that is, $\delta_n^2 \to 0$ as $n \to \infty$. But then from formula (2) there follows the equation

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx, \tag{3'}$$

which is called the *Lyapunov equation*. (We note that A. M. Lyapunov proved this equation even for a broader class of function than that which we here consider.)

From what has been proved it follows that for a function which satisfies the Lyapunov equation (in particular, for any bounded piecewise monotonic function), the corresponding Fourier series yields a root mean square deviation equal to zero.

Note. Let us establish a property of Fourier coefficients that will be needed in the future. We first introduce a definition.

A function $f(x)$ is called *piecewise continuous* on the interval $[a, b]$ if it has a definite number of discontinuities of the first kind on this interval (or is everywhere continuous).

We shall prove the following proposition.

If a function $f(x)$ is piecewise continuous on the interval $[-\pi, \pi]$, then its Fourier coefficients approach zero as $n \to \infty$; that is,

$$\lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} b_n = 0. \tag{4}$$

Proof. If the function $f(x)$ is piecewise continuous on the interval $[-\pi, \pi]$, then the function $f^2(x)$ too is piecewise continuous on this interval. Then $\int\limits_{-\pi}^{\pi} f^2(x)\,dx$ exists and is a finite number*). In this case, from the Bessel inequality (3) it follows that the series $\sum\limits_{n=1}^{\infty}(a_n^2 + b_n^2)$ converges. But if the series converges then its general term approaches zero; in this case, $\lim\limits_{n\to 0}(a_n^2 + b_n^2) = 0$. Whence we get equations (4) directly. Thus, the following equations are valid for a piecewise continuous and bounded function:

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0,$$

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0.$$

If a function $f(x)$ is periodic with period 2π, then the latter equations may be written as follows (for any a):

$$\lim_{n\to\infty}\int_{a}^{a+2\pi} f(x)\cos nx\,dx = 0; \qquad \lim_{n\to\infty}\int_{a}^{a+2\pi} f(x)\sin nx\,dx = 0.$$

* This integral may be presented as the sum of definite integrals of continuous functions over the subintervals into which the interval $[-\pi, \pi]$ is subdivided.

We note that these equations continue to hold if in the integrals we take any arbitrary interval of integration $[a, b]$, which is to say that the integrals

$$\int_a^b f(x)\cos nx\,dx \quad \text{and} \quad \int_a^b f(x)\sin nx\,dx$$

approach zero when n increases without bound if $f(x)$ is a bounded and piecewise continuous function.

Indeed, taking $b-a<2\pi$ for definiteness, we consider the auxiliary function $\varphi(x)$ with period 2π defined as follows:

$$\varphi(x)=f(x) \qquad \text{when } a\leqslant x\leqslant b$$
$$\varphi(x)=0 \qquad \text{when } b<x\leqslant a+2\pi.$$

Then

$$\int_a^b f(x)\cos nx\,dx=\int_a^{a+2\pi}\varphi(x)\cos nx\,dx,$$
$$\int_a^b f(x)\sin nx\,dx=\int_a^{a+2\pi}\varphi(x)\sin nx\,dx.$$

Since $\varphi(x)$ is a bounded and piecewise continuous function, the integrals on the right approach zero as $n\to\infty$. Hence, the integrals on the left approach zero as well. Thus, the proposition is proved; that is,

$$\lim_{n\to\infty}\int_a^b f(x)\cos nx\,dx=0;\quad \lim_{n\to\infty}\int_a^b f(x)\sin nx\,dx=0 \tag{5}$$

for any numbers a and b and any piecewise continuous function $f(x)$ bounded on $[a, b]$.

SEC. 8. THE DIRICHLET INTEGRAL

In this section we shall derive a formula that expresses the nth partial sum of a Fourier series in terms of a certain integral. This formula will be needed in the subsequent sections.

Consider the nth partial sum of a Fourier series for the periodic function $f(x)$ with period 2π:

$$s_n(x)=\frac{a_0}{2}+\sum(a_k\cos kx+b_k\sin kx),$$

where

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos kt\,dt,\qquad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin kt\,dt.$$

Putting these expressions into the formula for $s_n(x)$, we obtain

$$s_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt+$$
$$+\sum_{k=1}^{n}\left[\frac{\cos kx}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt+\frac{\sin kx}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt\right],$$

or bringing $\cos kx$ and $\sin kx$ under the integral sign (which is possible since $\cos kx$ and $\sin kx$ are independent of the variable of integration and, hence, can be regarded as constants), we get

$$s_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt+$$
$$+\frac{1}{\pi}\sum_{k=1}^{n}\left[\int_{-\pi}^{\pi} f(t)\cos kx\cos kt\,dt+\int_{-\pi}^{\pi} f(t)\sin kx\sin kt\,dt\right].$$

Now taking $\frac{1}{\pi}$ outside the brackets and replacing the sum of integrals by the integral of the sum, we obtain

$$s_n(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}\left\{\frac{f(t)}{2}+\sum_{k=1}^{n}[f(t)\cos kx\cos kt+f(t)\sin kx\sin kt]\right\}dt,$$

or

$$s_n(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2}+\sum_{k=1}^{n}(\cos kt\cos kx+\sin kt\sin kx)\right]dt=$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2}+\sum_{k=1}^{n}\cos k(t-x)\right]dt. \qquad (1)$$

Transform the expression in the brackets. Let

$$\sigma_n(z)=\frac{1}{2}+\cos z+\cos 2z+\ldots+\cos nz;$$

then

$$2\sigma_n(z)\cos z=\cos z+2\cos z\cos z+2\cos z\cos 2z+\ldots$$
$$\ldots+2\cos z\cos nz=\cos z+(1+\cos 2z)+(\cos z+\cos 3z)+$$
$$+(\cos 2z+\cos 4z)+\ldots+[\cos(n-1)z+\cos(n+1)z]=$$
$$=1+2\cos z+2\cos 2z+\ldots+2\cos(n-1)z+\cos nz+\cos(n+1)z$$

or

$$2\sigma_n(z)\cos z = 2\sigma_n(z) - \cos nz + \cos(n+1)z,$$

$$\sigma_n(z) = \frac{\cos nz - \cos(n+1)z}{2(1-\cos z)}.$$

But

$$\cos nz - \cos(n+1)z = 2\sin(2n+1)\frac{z}{2}\sin\frac{z}{2},$$

$$1-\cos z = 2\sin^2\frac{z}{2}.$$

Hence,

$$\sigma_n(z) = \frac{\sin(2n+1)\frac{z}{2}}{2\sin\frac{z}{2}}.$$

Thus, equation (1) may be rewritten as

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(2n+1)\frac{t-x}{2}}{2\sin\frac{t-x}{2}}\,dt.$$

Since the integrand is periodic (with period 2π), it follows that the integral retains its value on any interval of integration of length 2π. We can therefore write

$$s_n(x) = \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(t)\frac{\sin(2n+1)\frac{t-x}{2}}{2\sin\frac{t-x}{2}}\,dt.$$

Introducing a new variable α, we put

$$t-x=\alpha, \qquad t=x+\alpha.$$

Then we get the formula

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\alpha)\frac{\sin(2n+1)\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}\,d\alpha. \tag{2}$$

The integral on the right is *Dirichlet's integral*.

In this formula put $f(x)\equiv 1$; then $a_0=2$, $a_k=0$, $b_k=0$ when $k>0$; hence, $s_n(x)=1$ for any n and we get the identity

$$1 = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(2n+1)\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}\,d\alpha, \tag{3}$$

which we will need later on.

SEC. 9. THE CONVERGENCE OF A FOURIER SERIES AT A GIVEN POINT

Assume that the function $f(x)$ is piecewise continuous on the interval $[-\pi, \pi]$.

Multiplying both sides of the identity (3) of the preceding section by $f(x)$ and bringing $f(x)$ under the integral sign, we get the equation

$$f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin(2n+1)\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}\,d\alpha.$$

Subtract the terms of the latter equation from the corresponding terms of (2) of the preceding section; we get

$$s_n(x)-f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+\alpha)-f(x)]\frac{\sin(2n+1)\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}\,d\alpha.$$

Thus, the convergence of a Fourier series to the value of a function $f(x)$ at a given point depends on whether the integral on the right approaches zero as $n\to\infty$.

Let us break up this integral into two integrals:

$$s_n(x)-f(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+\alpha)-f(x)]\frac{\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}\sin n\alpha\,d\alpha+$$

$$+\frac{1}{2}\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+\alpha)-f(x)]\cos n\alpha\,d\alpha,$$

taking advantage of the fact that $\sin(2n+1)\frac{\alpha}{2}=\sin n\alpha\,\cos\frac{\alpha}{2}+$ $+\cos n\alpha\,\sin\frac{\alpha}{2}$. Break up the first of the integrals on the right of the latter equation into three integrals:

$$s_n(x)-f(x)=\frac{1}{\pi}\int_{-\delta}^{\delta}[f(x+\alpha)-f(x)]\frac{\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}\sin n\alpha\,d\alpha+$$

$$+\frac{1}{\pi}\int_{-\pi}^{-\delta}[f(x+\alpha)-f(x)]\frac{\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}+\sin n\alpha\,d\alpha+$$

$$+\frac{1}{\pi}\int_{\delta}^{\pi}[f(x+\alpha)-f(x)]\frac{\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}\sin n\alpha\,d\alpha+\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x+\alpha)-f(x)]\frac{1}{2}\cos n\alpha\,d\alpha.$$

Put $\Phi_1(\alpha)=\frac{f(x+\alpha)-f(x)}{2}$. Since $f(x)$ is a bounded piecewise continuous function, it follows that $\Phi_1(\alpha)$ is also a bounded and piecewise continuous periodic function of α. Hence, the latter integral approaches zero as $n \to \infty$, since it is a Fourier coefficient of this function. The function

$$\Phi_2(\alpha)=[f(x+\alpha)-f(x)]\frac{\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}$$

is bounded when $-\pi \leqslant \alpha < -\delta$ and $\delta \leqslant \alpha \leqslant \pi$ and

$$|\Phi_2(\alpha)| \leqslant [M+M]\frac{1}{2\sin\frac{\varepsilon}{2}},$$

where M is the upper limit of the quantity $|f(x)|$. Also, the function $\Phi_2(\alpha)$ is likewise piecewise continuous. Hence, by formulas (5) of Sec. 7, the second and third integrals approach zero as $n \to \infty$.

We can thus write

$$\lim_{n\to\infty}[s_n(x)-f(x)]=\lim_{n\to\infty}\frac{1}{\pi}\int_{-\delta}^{\delta}[f(x+\alpha)-f(x)]\frac{\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}\sin n\alpha\,d\alpha. \quad (1)$$

In the expression on the right, the integration is performed over the interval $-\delta \leqslant \alpha \leqslant \delta$; consequently, the integral is dependent on the values of the function $f(x)$ only in the interval from $x-\delta$ to $x+\delta$. An important proposition thus follows from the latter equation: *the convergence of a Fourier series at a given point x depends only on the behaviour of the function f(x) in an arbitrarily small neighbourhood of this point.*

Therein lies the so-called *principle of localisation in the study of Fourier series. If two functions $f_1(x)$ and $f_2(x)$ coincide in the neighbourhood of some point x, then their Fourier series simultaneously either converge or diverge at this point.*

SEC. 10. CERTAIN SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF A FOURIER SERIES

In the preceding section it was shown that if the function $f(x)$ is piecewise continuous in the interval $[-\pi, \pi]$, then the convergence of a Fourier series at the given point x_0 to a value of the function $f(x_0)$ depends on the behaviour of the function in a

certain arbitrary small neighbourhood $[x_0-\delta,\ x_0+\delta]$ with centre at the point x_0.

Let us now prove that if in the neighbourhood of the point x_0 the function $f(x)$ is such that there exist finite limits

$$\lim_{\alpha\to-0}\frac{f(x_0+\alpha)-f(x_0)}{\alpha}=k_1, \qquad (1)$$

$$\lim_{\alpha\to+0}\frac{f(x_0+\alpha)-f(x_0)}{\alpha}=k_2, \qquad (2)$$

while the function is continuous at the very point x_0 (Fig. 370), then the Fourier series converges at this point to a corresponding value of the function $f(x)$ *).

Proof. Let us consider the function $\Phi_2(\alpha)$ defined in the preceding section:

$$\Phi_2(\alpha)=[f(x_0+\alpha)-f(x_0)]\,\frac{\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}};$$

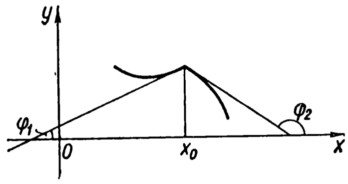


Fig. 370.

since the function $f(x)$ is piecewise continuous on the interval $[-\pi,\ \pi]$ and is continuous at the point x_0, it is therefore continuous in some neighbourhood $[x_0-\delta,\ x_0+\delta]$ of the point x_0.. For this reason, the function $\Phi_2(\alpha)$ is continuous at all points where $\alpha\neq 0$ and $|\alpha|\leqslant\delta$. When $\alpha=0$ the function $\Phi_2(\alpha)$ is not defined.

Let us find the limits $\lim\limits_{\alpha\to 0-0}\Phi_2(\alpha)$ and $\lim\limits_{\alpha\to 0+0}\Phi_2(\alpha)$, making use of conditions (1) and (2):

$$\lim_{\alpha\to 0-0}\Phi_2(\alpha)=\lim_{\alpha\to 0-0}[f(x_0+\alpha)-f(x_0)]\frac{\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}=$$

$$=\lim_{\alpha\to 0-0}\frac{f(x_0+\alpha)-f(x_0)}{\alpha}\,\frac{\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\cos\frac{\alpha}{2}=$$

$$=\lim_{\alpha\to 0-0}\frac{f(x_0+\alpha)-f(x_0)}{\alpha}\lim_{\alpha\to 0-0}\frac{\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\lim_{\alpha\to 0-0}\cos\frac{\alpha}{2}=k_1\cdot 1\cdot 1=k_1.$$

*) If conditions (1) and (2) are fulfilled, then we say that the function $f(x)$ has, at the point x, a derivative on the right and a derivative on the left. Fig. 370 shows a function where $k_1=\tan\varphi_1$, $k_2=\tan\varphi_2$, $k_1\neq k_2$. If $k_1=k_2$, that is, if the derivatives on the right and left are equal, then the function will be differentiable at the given point.

Thus, if we redefine the function $\Phi_2(\alpha)$ by putting $\Phi_2(0)=k_1$, then it will be continuous on the interval $[-\delta, 0]$, and, hence, bounded as well. Similarly we prove that

$$\lim_{\alpha\to 0+0}\Phi_2(\alpha)=k_2.$$

Consequently, the function $\Phi_2(\alpha)$ is bounded and continuous on the interval $[0, \delta]$. Thus, on the interval $[-\delta, \delta]$ the function $\Phi_2(\alpha)$ is bounded and piecewise continuous. Now let us return to equation (1), Sec. 9 (denoting x in terms of x_0),

$$\lim_{n\to\infty}[s_n(x_0)-f(x_0)]=\lim_{n\to\infty}\frac{1}{\pi}\int_{-\delta}^{\delta}[f(x_0+\alpha)-f(x_0)]\frac{\cos\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}\sin n\alpha\, d\alpha$$

or

$$\lim_{n\to\infty}[s_n(x_0)-f(x_0)]=\lim_{n\to\infty}\frac{1}{\pi}\int_{-\delta}^{\delta}\Phi_2(\alpha)\sin n\alpha\, d\alpha.$$

From formulas (5) of Sec. 7 we conclude that the limit on the right is equal to zero, and therefore

$$\lim_{n\to\infty}[s_n(x_0)-f(x_0)]=0$$

or

$$\lim_{n\to\infty}s_n(x_0)=f(x_0).$$

The theorem is proved.

This theorem differs from the theorem stated in Sec. 1 in that in the latter case it was required, for convergence of the Fourier series at a point x_0 to the value of the function $f(x_0)$, that the point x_0 should be a point of continuity on the interval $[-\pi, \pi]$, whereas the function should be piecewise monotonic; here, however, it is required that the function at the point x_0 should be a point of continuity and that the conditions (1) and (2) be fulfilled, while throughout the interval $[-\pi, \pi]$ the function should be piecewise continuous and bounded. It is obvious that these conditions are different.

Note 1. If a piecewise continuous function is differentiable at the point x_0, it is obvious that conditions (1) and (2) are fulfilled. Here, $k_1=k_2$. Hence, at points where the function $f(x)$ is differentiable, the Fourier series converges to a value of the function at the corresponding point.

Note 2: a) The function considered in Example 2, Sec. 2 (Fig. 358), satisfies conditions (1) and (2) at the points $0, \pm 2\pi, \pm 4\pi, \ldots$ At all the other points it is differentiable. Consequent-

ly, a Fourier series constructed for it converges to the value of this function at each point.

b) The function considered in Example 4, Sec. 2 (Fig. 361), satisfies conditions (1) and (2) at the points $\pm\pi$, $\pm 3\pi$, $\pm 5\pi$. It is differentiable at all points. It is represented by a Fourier series at each point.

c) The function considered in Example 1, Sec. 2 (Fig. 357), is discontinuous at the points $\pm\pi$, $\pm 3\pi$, $\pm 5\pi$. At all other points it is differentiable. Hence, at all points, with the exception of points of discontinuity, the Fourier series corresponding to it converges to the value of the function at the corresponding points. At the discontinuities, the sum of the Fourier series is equal to the arithmetical mean limit of the function on the right and on the left (in this case, zero).

SEC. 11. PRACTICAL HARMONIC ANALYSIS

The theory of expanding functions in Fourier series is called *harmonic analysis*. We shall now make several remarks about approximate computation of the coefficients of a Fourier series, that is to say, about practical harmonic analysis.

As we know, the Fourier coefficients of a function $f(x)$ with period 2π are defined by the formulas

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx; \qquad a_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx;$$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx.$$

In many practical cases, the function $f(x)$ is represented either in tabular form (when the functional relation is obtained by experiment) or in the form of a curve which is plotted by some kind of instrument. In these cases the Fourier coefficients are calculated by means of approximate methods of integration (see Sec. 8, Ch. XI).

Let us consider the interval $-\pi\leqslant x\leqslant\pi$ of length 2π. This can always be done by proper choice of scale on the x-axis.

Divide the interval $[-\pi, \pi]$ into n equal parts by the points

$$x_0=-\pi,\ x_1,\ x_2,\ \ldots,\ x_n=\pi.$$

Then the subinterval will be

$$\Delta x=\frac{2\pi}{n}.$$

We denote the values of the function $f(x)$ at the points x_0, x_1, x_2, ..., x_n (respectively) in terms of

$$y_0, y_1, y_2, \ldots, y_n.$$

These values are determined either from a table or from the graph of the given function (by measuring the corresponding ordinates).

Then, taking advantage, for example, of the rectangular formula [see formula (1), Sec. 8, Ch. XI], we determine the Fourier coefficients:

$$a_0 = \frac{2}{n}\sum_{i=1}^{n} y_i, \quad a_k = \frac{2}{n}\sum_{i=1}^{n} y_i \cos kx_i, \quad b_k = \frac{2}{n}\sum_{i=1}^{n} y_i \sin kx_i.$$

Diagrams have been devised that simplify computation of Fourier coefficients (see, for instance, V. I. Smirnov, "Course of Higher Mathematics", Vol. II; A. M. Lopshits, "Models for Harmonic Analysis"). We cannot deal here with the details but we can note that there are instruments (harmonic analysers) which permit approximating the values of Fourier coefficients from the graph of the function.

SEC. 12. FOURIER INTEGRAL

Let a function $f(x)$ be defined in an infinite interval $(-\infty, \infty)$ and absolutely integrable over it; that is, there exists an integral

$$\int_{-\infty}^{\infty} |f(x)|\,dx = Q. \tag{1}$$

Further, let the function $f(x)$ be such that it is expandable into a Fourier series in any interval $(-l, +l)$:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi}{l}x + b_k \sin\frac{k\pi}{l}x, \tag{2}$$

where

$$a_k = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{k\pi}{l}t\,dt, \qquad b_k = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{k\pi}{l} + dt. \tag{3}$$

Putting into series (2) the expressions of the coefficients a_k and b_k from formulas (3), we can write

$$f(x)=\frac{1}{2l}\int_{-l}^{l} f(t)\,dt+\frac{1}{l}\sum_{k=1}^{\infty}\left(\int_{-l}^{l} f(t)\cos\frac{k\pi}{l}t\,dt\right)\cos\frac{k\pi}{l}x+$$

$$+\left(\int_{-l}^{l} f(t)\sin\frac{k\pi}{l}t\,dt\right)\sin\frac{k\pi}{l}\,x=$$

$$=\frac{1}{2l}\int_{-l}^{l} f(t)\,dt+\frac{1}{l}\sum_{k=1}^{\infty}\int_{-l}^{l} f(t)\left[\cos\frac{k\pi}{l}t\cos\frac{k\pi}{l}x+\sin\frac{k\pi}{l}\,t\sin\frac{k\pi}{l}x\right]dt$$

or

$$f(x)=\frac{1}{2l}\int_{-l}^{l} f(t)\,dt+\frac{1}{l}\sum_{k=1}^{\infty}\int_{-l}^{l} f(t)\cos\frac{k\pi(t-x)}{l}\,dt. \tag{4}$$

Let us investigate what form expansion (4) will take when passing to the limit as $l\to\infty$.

We introduce the following notation:

$$\alpha_1=\frac{\pi}{l},\quad \alpha_2=\frac{2\pi}{l},\ \ldots,\ \alpha_k=\frac{k\pi}{l},\ \ldots\ \text{and}\ \Delta\alpha_k=\frac{\pi}{l}. \tag{5}$$

Substituting into (4), we get

$$f(x)=\frac{l}{2l}\int_{-l}^{l} f(t)\,dt+\frac{1}{\pi}\sum_{k=1}^{\infty}\left(\int_{-l}^{l} f(t)\cos\alpha_k(t-x)\,dt\right)\Delta\alpha_k. \tag{6}$$

As $l\to\infty$, the first term on the right approaches zero. Indeed,

$$\left|\frac{1}{2l}\int_{-l}^{l} f(t)\,dt\right|\leqslant\frac{1}{2l}\int_{-l}^{l}|f(t)|\,dt<\frac{1}{2l}\int_{-\infty}^{\infty}|f(t)|\,dt=\frac{1}{2l}Q\to 0.$$

For any fixed l, the expression in the parentheses is a function of α_k [see formula (5)], which takes on values from $\frac{\pi}{l}$ to ∞. We will show, without proof, that if the function $f(x)$ is piecewise monotonic on every finite interval, is bounded on an infinite interval and satisfies condition (1), then as $l\to+\infty$ formula (6) takes the form

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\right)d\alpha. \tag{7}$$

The expression on the right is known as the *Fourier integral* of the function $f(x)$. Equation (7) occurs for all points where the function is continuous. At points of discontinuity we have the equation

$$\frac{1}{\pi}\int_0^\infty\left(\int_{-\infty}^\infty f(t)\cos\alpha(t-x)\,dx\right) = \frac{f(x+0)+f(x-0)}{2}. \qquad (7')$$

Let us transform the integral on the right of (7) by expanding $\cos\alpha(t-x)$:

$$\cos\alpha(t-x)=\cos\alpha t\cos\alpha x+\sin\alpha t\sin\alpha x.$$

Putting this expession into formula (7) and taking $\cos\alpha x$ and $\sin\alpha x$ outside the integral signs, where the integration is performed with respect to the variable t, we get

$$f(x)=\frac{1}{\pi}\int_0^\infty\left(\int_{-\infty}^\infty f(t)\cos\alpha t\,dt\right)\cos\alpha x\,d\alpha+$$
$$+\frac{1}{\pi}\int_0^\infty\left(\int_{-\infty}^\infty f(t)\sin\alpha t\,dt\right)\sin\alpha x\,d\alpha. \qquad (8)$$

Each of the integrals in brackets with respect to t exists, since the function $f(t)$ is absolutely integrable in the interval $(-\infty,\infty)$, and therefore the functions $f(t)\cos\alpha t$ and $f(t)\sin\alpha t$ are also absolutely integrable.

Let us consider particular cases of formula (8).

1. Let $f(x)$ be even. Then $f(t)\cos\alpha t$ is an even function, while $f(t)\sin\alpha t$ is odd and we have

$$\int_{-\infty}^\infty f(t)\cos\alpha t\,dt=2\int_0^\infty f(t)\cos\alpha t\,dt,$$

$$\int_{-\infty}^\infty f(t)\sin\alpha t\,dt=0.$$

Formula (8) in this case takes the form

$$f(x)=\frac{2}{\pi}\int_0^\infty\left(\int_0^\infty f(t)\cos\alpha t\,dt\right)\cos\alpha x\,d\alpha. \qquad (9)$$

2. Let $f(x)$ be odd. Analysing the character of the integrals in formula (8) in this case, we obtain

$$f(x)=\frac{2}{\pi}\int_0^\infty\left(\int_0^\infty f(t)\sin\alpha t\,dt\right)\sin\alpha x\,d\alpha. \tag{10}$$

If $f(x)$ is defined only in the interval $(0, \infty)$, then for $x>0$ it may be represented by either formula (9) or (10). In the first case we redefine it in the interval $(-\infty, 0)$ in even fashion; in the latter case, in odd fashion.

Let it be noted once again that at the points of discontinuity we should write the following expression in place of $f(x)$ in the left-hand members of (9) and (10):

$$\frac{f(x+0)+f(x-0)}{2}.$$

Let us return to formula (8). The integrals in brackets are functions of α. We introduce the following notation:

$$A(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\alpha t\,dt,$$

$$B(\alpha)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\alpha t\,dt.$$

Then formula (8) may be rewritten as follows:

$$f(x)=\int_0^\infty [A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x]\,d\alpha. \tag{11}$$

We say the formula (11) yields an expansion of the function $f(x)$ into harmonics with a frequency α that continuously varies from 0 to ∞. The law of distribution of amplitudes and initial phases as dependent upon the frequency α is expressed in terms of the functions $A(\alpha)$ and $B(\alpha)$.

Let us return to formula (9). We set

$$F(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\alpha t\,dt; \tag{12}$$

then formula (9) takes the form

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F(\alpha)\cos\alpha x\,d\alpha. \tag{13}$$

The function $F(\alpha)$ is called the *Fourier cosine transform* of the function $f(x)$.

If in (12) we consider $F(\alpha)$ as given and $f(t)$ as the unknown function, then it is an *integral equation* of the function $f(t)$. Formula (13) gives the solution of this equation.

On the basis of formula (10) we can write the following equations:

$$\Phi(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\alpha t\,dt, \tag{14}$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \Phi(x)\sin\alpha x\,d\alpha. \tag{15}$$

The function $\Phi(\alpha)$ is called the *Fourier sine transform*.

Example. Let

$$f(x)=e^{-\beta x}\quad(\beta>0,\ x\geqslant 0).$$

From (12) we determine the Fourier cosine transform:

$$F(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\beta t}\cos\alpha t\,dt=\sqrt{\frac{2}{\pi}}\frac{\beta}{\beta^2+\alpha^2}.$$

From (14) we determine the Fourier sine transform:

$$\Phi(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\beta t}\sin\alpha t\,dt=\sqrt{\frac{2}{\pi}}\frac{\alpha}{\beta^2+\alpha^2}.$$

From formulas (13) and (15) we find the reciprocal relationships:

$$\frac{2\beta}{\pi}\int_0^\infty\frac{\cos\alpha x}{\beta^2+\alpha^2}\,d\alpha=e^{-\beta x}\qquad(x\geqslant 0),$$

$$\frac{2}{\pi}\int_0^\infty\frac{\alpha\sin\alpha x}{\beta^2+\alpha^2}\,d\alpha=e^{-\beta x}\qquad(x>0).$$

SEC. 13. THE FOURIER INTEGRAL IN COMPLEX FORM

In the Fourier integral [formula (7), Sec. 12], the brackets contain an even function of α; hence, it is defined for negative values of α as well. On the basis of the foregoing, formula (7) can be rewritten as follows:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(t)\cos\alpha(t-x)\,dt\right)d\alpha. \tag{1}$$

Let us now consider the following expression, which is identically equal to zero:

$$\int_{-M}^{M}\left(\int_{-\infty}^{\infty} f(t)\sin\alpha(t-x)\,dt\right)d\alpha=0.$$

The expression on the left is identically equal to zero because the function of α in the brackets is an odd function, and an integral of an odd function from $-M$ to $+M$ is equal to zero. It is obvious that

$$\lim_{M\to\infty}\int_{-M}^{M}\left(\int_{-\infty}^{\infty} f(t)\sin\alpha(t-x)\,dt\right)d\alpha=0$$

or

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\sin\alpha(t-x)\,dt\right)d\alpha=0. \tag{2}$$

Note. It is necessary to point to the following. A convergent integral with infinite limits is defined as follows:

$$\int_{-\infty}^{\infty}\varphi(\alpha)\,d\alpha=\int_{-\infty}^{c}\varphi(\alpha)\,d\alpha+\int_{c}^{\infty}\varphi(\alpha)\,d\alpha=$$

$$=\lim_{M\to\infty}\int_{-M}^{c}\varphi(\alpha)\,d\alpha+\lim_{M\to\infty}\int_{c}^{M}\varphi(\alpha)\,d\alpha \tag{*}$$

provided that each of the limits to the right exists (see Sec. 7, Ch. XI). But in equation (2) we wrote

$$\int_{-\infty}^{\infty}\varphi(\alpha)\,d\alpha=\lim_{M\to\infty}\int_{-M}^{M}\varphi(\alpha)\,d\alpha. \tag{**}$$

Obviously, it may happen that the limit (**) exists, while the limits on the right side of equation (*) do not exist. The expression on the right of (**) is called the *principal value of the integral*. Thus, in equation (2) we consider the principal value of the improper (outer) integral. The subsequent integrals of this section will be written in this sense.

Let us multiply the terms of (2) by $\frac{i}{2\pi}$ and add them to the corresponding terms of (1); we then get

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\left(\cos\alpha(t-x)+i\sin\alpha(t-x)\right)dt\right]d\alpha$$

or

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\,e^{i\alpha(t-x)}\,dt\right]d\alpha. \tag{3}$$

This is the *Fourier integral in complex form*. Formula (3) may be rewritten as follows:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{i\alpha t}\,dt\right)e^{-i\alpha x}\,d\alpha.$$

On the basis of this latter equation we can write

$$F^*(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{i\alpha t}\,dt, \tag{4}$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F^*(\alpha)\,e^{-i\alpha x}\,d\alpha. \tag{5}$$

The function $F^*(\alpha)$ defined by formula (4) is called the *Fourier transform* of the function $f(t)$. The function $f(x)$ defined by formula (5) is called the *Fourier inverse transform* of the function $F^*(\alpha)$ (the transforms differ in the sign in front of i).

Exercises on Chapter XVII

1. Expand the following function in a Fourier series in the interval $(-\pi, \pi)$:

$$f(x)=2x \text{ for } 0\leqslant x\leqslant \pi,$$
$$f(x)=x \text{ for } -\pi< x\leqslant 0.$$

Ans. $\frac{1}{4}\pi-\frac{2}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right)+3\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots\right)$.

2. Taking advantage of the expansion of the function $f(x)=1$ in the interval $(0, \pi)$ in the sines of multiple arcs, calculate the sum of the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$ *Ans.* $\frac{\pi}{4}$.

3. Utilising the expansion of the function $f(x)=x^2$ in a Fourier series, compute the sum of the series $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$ *Ans.* $\frac{\pi^2}{12}$.

4. Expand the function $f(x)=\frac{\pi^2}{12}-\frac{x^2}{4}$ in a Fourier series in the interval $(-\pi, \pi)$. *Ans.* $\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+\ldots$

5. Expand the following function in a Fourier series in the interval $(-\pi, \pi)$

$$f(x) = -\frac{(\pi+x)}{2} \text{ for } -\pi \leqslant x < 0,$$

$$f(x) = \frac{1}{2}(\pi - x) \text{ for } 0 \leqslant x < \pi.$$

Ans. $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots$

6. Expand in a Fourier series, in the interval $(-\pi, \pi)$, the function

$$f(x) = -x \text{ for } -\pi < x \leqslant 0,$$
$$f(x) = 0 \quad \text{for } 0 < x \leqslant \pi.$$

Ans. $\frac{\pi}{4} - \frac{2}{\pi}\sum_{m=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n}$.

7. Expand in a Fourier series, in the interval $(-\pi, \pi)$, the function

$$f(x) = 1 \quad \text{for } -\pi < x \leqslant 0,$$
$$f(x) = -2 \text{ for } 0 < x \leqslant \pi.$$

Ans. $-\frac{1}{2} - \frac{6}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}$.

8. Expand the function $f(x) = x^2$, in the interval $(0, \pi)$, in a series of sines.

Ans. $\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left\{\frac{\pi^2}{n} + \frac{2}{n^3}[(-1)^n - 1]\right\}\sin nx$.

9. Expand the function $y = \cos 2x$ in a series of sines in the interval $(0, \pi)$.

Ans. $-\frac{4}{\pi}\left[\frac{\sin x}{2^2-1} + \frac{3\sin 3x}{2^2-3^2} + \frac{5\sin 5x}{2^2-5^2} + \ldots\right]$.

10. Expand the function $y = \sin x$ in a series of cosines in the interval $(0, \pi)$.

Ans. $\frac{4}{\pi}\left[\frac{1}{2} + \frac{\cos 2x}{1-2^2} + \frac{\cos 4x}{1-4^2} + \ldots\right]$.

11. Expand the function $y = e^x$ in a Fourier series in the interval $(-l, l)$.

Ans. $$\frac{e^l - e^{-l}}{2l} + l(e^l - e^{-l})\sum_{n=1}^{\infty}\frac{(-1)^n \sin\frac{n\pi x}{l}}{l^2 + n^2\pi^2} +$$
$$+ \pi(e^l - e^{-l})\sum_{n=1}^{\infty}\frac{(-1)^{n-1} n \sin\frac{n\pi x}{l}}{l^2 + n^2\pi^2}.$$

12. Expand the function $f(x) = 2x$ in a series of sines in the interval $(0, 1)$.

Ans. $1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n}$,

13. Expand the function $f(x)=x$ in a series of sines in the interval $(0,l)$.

Ans. $\frac{2l}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\frac{n\pi x}{l}}{n}$.

14. Expand the function

$$(x)=\begin{cases} x \text{ for } 0<x\leqslant 1, \\ 2-x \text{ for } 1<x<2 \end{cases}$$

in the interval (0, 2): a) in a series of sines; b) in a series of cosines.

Ans. a) $\frac{8}{\pi^2}\sum_{n=0}^{\infty}(-1)^n\frac{\sin\frac{(2n+1)\pi x}{2}}{(2n+1)^2}$; b) $\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x}{(2n+1)^2}$.

CHAPTER XVIII

EQUATIONS OF MATHEMATICAL PHYSICS

SEC. 1. BASIC TYPES OF EQUATIONS OF MATHEMATICAL PHYSICS

The basic equations of mathematical physics (for the case of functions of two independent variables) are the following second-order partial differential equations.

I. Wave Equation:

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}. \tag{1}$$

This equation is invoked in the study of processes of transversal vibrations of a string, the longitudinal vibrations of rods, electric oscillations in conductors, the torsional oscillations of shafts, gas vibrations, and so forth. This equation is the simplest of the class of *hyperbolic equations.*

II. Fourier Equation for Heat Conduction:

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}. \tag{2}$$

This equation is invoked in the study of processes of the propagation of heat, the filtration of liquids and gases in a porous medium (for example, the filtration of oil and gas in subterranean sandstones), some problems in probability theory, etc. This equation is the simplest of the class of *parabolic equation.*

III. Laplace's Equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \tag{3}$$

This equation is invoked in the study of problems dealing with electric and magnetic fields, stationary thermal states, problems in hydrodynamics, diffusion, and so on. This equation is the simplest in the class of *elliptic equations.*

In equations (1), (2), and (3), the unknown function u depends on two variables. Also considered are appropriate equations of functions with a larger number of variables. Thus, the wave

equation in three independent variables is of the form

$$\frac{\partial^2 u}{\partial t^2}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \qquad (1')$$

the heat-conduction equation in three independent variables is of the form

$$\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \qquad (2')$$

the Laplace equation in three independent variables has the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0. \qquad (3')$$

SEC. 2. DERIVATION OF THE EQUATION OF OSCILLATION OF A STRING. FORMULATION OF THE BOUNDARY-VALUE PROBLEM. DERIVATION OF EQUATIONS OF ELECTRIC OSCILLATIONS IN WIRES

In mathematical physics a string is understood to be a flexible and elastic thread. The tensions that arise in a string at any instant of time are directed along a tangent to its profile. Let a string of length l be, at the initial instant, directed along a segment of the x-axis from 0 to l. Assume that the ends of the string are fixed at the points $x=0$ and $x=l$. If the string is deflected from its original position and then let loose; or if without deflecting the string we impart to its points a certain velocity at the initial time, or if we deflect the string and impart a velocity to its points, then the points of the string will perform certain motions; we say that the string is set into oscillation. The problem is to determine the shape of the string at any instant of time and to determine the law of motion of every point of the string as a function of time.

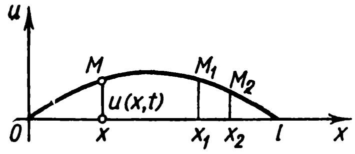


Fig. 371.

Let us consider small deflections of the points of the string from the initial position. We may suppose that the motion of the points of the string is perpendicular to the x-axis and in a single plane. On this assumption, the process of oscillation of the string is described by a single function $u(x, t)$, which yields the amount that a point of the string with abscissa x has moved at time t (Fig. 371).

Since we consider small deflections of the string in the (x, u)-plane, we shall assume that the length of an element of string

M_1M_2 is equal to its projection on the x-axis, that is,*) $M_1M_2 = x_2 - x_1$. We also assume that the tension of the string at all points is the same; we denote it by T.

Consider an element of the string MM' (Fig. 372). Forces T act at the ends of this element along tangents to the string. Let the tangents form with the x-axis angles φ and $\varphi+\Delta\varphi$. Then the projection on the u-axis of forces acting on the element MM' will be equal to $T\sin(\varphi+\Delta\varphi) - T\sin\varphi$. Since the angle φ is small, we can put $\tan\varphi \approx \sin\varphi$, and we will have

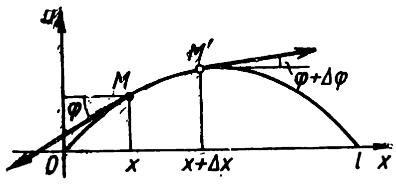


Fig. 372.

$$T\sin(\varphi+\Delta\varphi) - T\sin\varphi \approx$$

$$\approx T\tan(\varphi+\Delta\varphi) - T\tan\varphi = T\left[\frac{\partial u(x+\Delta x,\ t)}{\partial x} - \frac{\partial u(x,\ t)}{\partial x}\right] =$$

$$= T\,\frac{\partial^2 u(x+\theta\Delta x,\ t)}{\partial x^2}\,\Delta x \approx T\,\frac{\partial^2 u(x,\ y)}{\partial x^2}\,\Delta x,$$

$$0<\theta<1$$

(here, we applied the Lagrange theorem to the expression in the square brackets).

In order to obtain the equation of motion, we must equate to the force of inertia the external forces applied to the element. Let ϱ be the linear density of the string. Then the mass of the element of the string will be $\varrho\Delta x$. The acceleration of the element is $\frac{\partial^2 u}{\partial t^2}$. Hence, by d'Alembert's principle we will have

$$\varrho\Delta x\,\frac{\partial^2 u}{\partial t^2} = T\,\frac{\partial^2 u}{\partial x^2}\,\Delta x.$$

Cancelling out Δx and denoting $\frac{T}{\varrho} = a^2$, we get the equation of motion:

$$\frac{\partial^2 u}{\partial t^2} = a^2\,\frac{\partial^2 u}{\partial x^2}. \qquad (1)$$

This is the *wave equation*, the equation of vibrations of a string. Equation (1) by itself is not sufficient for a complete definition

*) This assumption is equivalent to neglecting $u_x'^2$ as compared with 1. Indeed,

$$M_1M_2 = \int_{x_1}^{x_2}\sqrt{1+u_x'^2}\,dx = \int_{x_1}^{x_2}\left(1+\frac{1}{2}u_x'^2 - \dots\right)dx \approx \int_{x_1}^{x_2}dx = x_2 - x_1.$$

of the motion of a string. The desired function $u(x, t)$ must also satisfy *boundary conditions* that indicate what occurs at the ends of the string ($x=0$ and $x=l$) and *initial conditions*, which describe the state of the string at the initial time ($t=0$). The boundary and initial conditions are referred to collectively as *boundary-value conditions*.

For example, as we assumed, let the ends of the string at $x=0$ and $x=l$ be fixed. Then for any t the following equalities must hold:

$$u(0, t)=0, \tag{2'}$$

$$u(l, t)=0. \tag{2''}$$

These equations are the *boundary conditions* for our problem.

At $t=0$ the string has a definite shape, that which we gave it. Let this shape be defined by a function $f(x)$. We should then have

$$u(x, 0)=u|_{t=0}=f(x). \tag{3'}$$

Further, at the initial instant the velocity at each point of the string must be given; it is defined by the function $\varphi(x)$. Thus, we should have

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=\varphi(x). \tag{3''}$$

The conditions (3′) and (3″) are the *initial conditions*.

Note. For a special case we may have $f(x)\equiv 0$ or $\varphi(x)\equiv 0$. But if $f(x)\equiv 0$ and $\varphi(x)\equiv 0$, then the string will be in a state of rest; hence, $u(x, t)\equiv 0$.

As has already been pointed out, the problem of electric oscillations in wires likewise leads to equation (1). Let us show this to be the case. The electric current in a wire is characterised by the current flow $i(x, t)$ and the voltage $v(x, t)$, which are dependent on the coordinate x of the point of the wire and on the time t. Regarding an element of wire Δx, we can write that the voltage drop on the element Δx is equal to $v(x, t)-v(x+\Delta x, t)\approx \approx -\frac{\partial v}{\partial x}\Delta x$. This voltage drop consists of the ohmic drop, equal to $iR\Delta x$, and the inductive drop, equal to $\frac{\partial i}{\partial t}L\Delta x$. Thus,

$$-\frac{\partial v}{\partial x}\Delta x=iR\Delta x+\frac{\partial i}{\partial t}L\Delta x, \tag{4}$$

where R and L are the resistance and the coefficient of self-induction reckoned per unit length of wire. The minus sign indicates

that the current flow is in a direction opposite to the build-up of v. Cancelling out Δx, we get the equation

$$\frac{\partial v}{\partial x}+iR+L\frac{\partial i}{\partial t}=0. \tag{5}$$

Further, the difference between the current leaving element Δx and entering it during time Δt will be

$$i(x,\ t)-i(x+\Delta x,\ t)\approx-\frac{\partial i}{\partial x}\Delta x\Delta t.$$

It is taken up in charging the element $\left(\text{this is equal to } C\Delta x\frac{\partial v}{\partial t}\Delta t\right)$ and in leakage through the lateral surface of the wire due to imperfect insulation, equal to $Av\Delta x\Delta t$ (here A is the leak coefficient). Equating these expressions and cancelling out $\Delta x\Delta t$, we get the equation

$$\frac{\partial i}{\partial x}+C\frac{\partial v}{\partial t}+Av=0. \tag{6}$$

Equations (5) and (6) are generally called *telegraph equations*.

From the system of equations (5) and (6) we can obtain an equation that contains only the desired function $i(x,\ t)$, and an equation containing only the desired function $v(x,\ t)$. Differentiate the terms of equation (6) with respect to x; differentiate the terms of (5) with respect to t and multiply them by C. Subtracting, we get

$$\frac{\partial^2 i}{\partial x^2}+A\frac{\partial v}{\partial x}-CR\frac{\partial i}{\partial x}-CL\frac{\partial^2 i}{\partial t^2}=0.$$

Substituting into the latter equation the expression $\frac{\partial v}{\partial x}$ from (5), we get

$$\frac{\partial^2 i}{\partial x^2}+A\left(-iR-L\frac{\partial i}{\partial t}\right)-CR\frac{\partial i}{\partial x}-CL\frac{\partial^2 i}{\partial t^2}=0$$

or

$$\frac{\partial^2 i}{\partial x^2}=CL\frac{\partial^2 i}{\partial t^2}+(CR+AL)\frac{\partial i}{\partial t}+ARi. \tag{7}$$

Similarly, we obtain an equation for determining $v(x,\ t)$:

$$\frac{\partial^2 v}{\partial x^2}=CL\frac{\partial^2 v}{\partial t^2}+(CR+AL)\frac{\partial v}{\partial t}+ARv. \tag{8}$$

If we neglect the leakage through the insulation ($A=0$) and the resistance ($R=0$), then equations (7) and (8) pass into the

wave equations

$$a^2 \frac{\partial^2 i}{\partial x^2} = \frac{\partial^2 i}{\partial t^2}, \quad a^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2},$$

where $a^2 = \frac{1}{CL}$. The physical conditions dictate the formulation of the boundary and initial conditions of the problem.

SEC. 3. SOLUTION OF THE EQUATION OF OSCILLATIONS OF A STRING BY THE METHOD OF SEPARATION OF VARIABLES (THE FOURIER METHOD)

The method of separation of variables (or the Fourier method), which we shall now discuss, is typical of the solution of many problems of mathematical physics. Let it be required to find the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

which satisfies the boundary-value conditions

$$u(0, t) = 0, \tag{2}$$

$$u(l, t) = 0, \tag{3}$$

$$u(x, 0) = f(x), \tag{4}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \varphi(x). \tag{5}$$

We shall seek a particular solution (not identically equal to zero) of equation (1) that satisfies the boundary conditions (2) and (3), in the form of a product of two functions $X(x)$ and $T(t)$, of which the former is dependent only on x, and the latter, only on t:

$$u(x, t) = X(x) T(t). \tag{6}$$

Substituting into equation (1), we get $X(x) T''(t) = a^2 X''(x) T(t)$, and dividing the terms of the equation by $a^2 XT$,

$$\frac{T''}{a^2 T} = \frac{X''}{X}. \tag{7}$$

The left member of this equation is a function that does not depend on x, the right member is a function that does not depend on t. Equation (7) is possible only when the left and right members are not dependent either on x or on t, that is, are equal to a constant number. We denote it by $-\lambda$, where $\lambda > 0$ (later on we will consider the case $\lambda < 0$). Thus,

$$\frac{T''}{a^2 T} = \frac{X''}{X} = -\lambda.$$

From these equations we get two equations:

$$X'' + \lambda X = 0, \tag{8}$$

$$T'' + a^2\lambda T = 0. \tag{9}$$

The general solutions of these equations are (see Ch. XIII, Sec. 21)

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x, \tag{10}$$

$$T(x) = C\cos a\sqrt{\lambda}t + D\sin a\sqrt{\lambda}t, \tag{11}$$

where A, B, C, and D are arbitrary constants.

Substituting the expressions $X(x)$ and $T(t)$ into (6), we get

$$u(x, t) = (A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x)(C\cos a\sqrt{\lambda}t + D\sin a\sqrt{\lambda}t).$$

Now choose the constants A and B so that the conditions (2) and (3) are satisfied. Since $T(t) \not\equiv 0$ (otherwise we would have $u(x, t) \equiv 0$, which contradicts the hypothesis), the function $X(x)$ must satisfy the conditions (2) and (3); that is, we must have $X(0) = 0$, $X(l) = 0$. Putting the values $x = 0$ and $x = l$ into (10), we obtain, on the basis of (2) and (3),

$$0 = A\cdot 1 + B\cdot 0,$$

$$0 = A\cos\sqrt{\lambda}l + B\sin\sqrt{\lambda}l = 0.$$

From the first equation we find $A = 0$. From the second it follows that

$$B\sin\sqrt{\lambda}l = 0.$$

$B \neq 0$, since otherwise we would have $X \equiv 0$ and $u \equiv 0$, which contradicts the hypothesis. Consequently, we must have

$$\sin\sqrt{\lambda}l = 0,$$

whence

$$\sqrt{\lambda} = \frac{n\pi}{l} \quad (n = 1, 2, \ldots) \tag{12}$$

(we do not take the value $n = 0$, since then we would have $X \equiv 0$ and $u \equiv 0$). And so we have

$$X = B\sin\frac{n\pi}{l}x. \tag{13}$$

These values of λ are called *eigenvalues* of the given boundary-value problem. The functions $X(x)$ corresponding to them are called *eigenfunctions*.

Note. If in place of $-\lambda$ we took the expression $+\lambda = k^2$, then equation (8) would take the form

$$X'' - k^2X = 0.$$

The general solution of this equation is

$$X = Ae^{kx} + Be^{-kx}.$$

A nonzero solution in this form cannot satisfy the boundary conditions (2) and (3).

Knowing $\sqrt{\lambda}$ we can [utilising (11)] write

$$T(t) = C \cos \frac{an\pi}{l} t + D \sin \frac{an\pi}{l} t \quad (n = 1, 2, \ldots). \tag{14}$$

For each value of n, hence for every λ, we put the expressions (13) and (14) into (6) and obtain a solution of equation (1) that satisfies the boundary conditions (2) and (3). We denote this solution by $u_n(x, t)$:

$$u_n(x, t) = \sin \frac{n\pi}{l} x \left(C_n \cos \frac{an\pi}{l} t + D_n \sin \frac{an\pi}{l} t \right). \tag{15}$$

For each value of n we can take the constants C and D and thus write C_n and D_n (the constant B is included in C_n and D_n). Since equation (1) is linear and homogeneous, the sum of the solutions is also a solution, and therefore the function represented by the series

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t)$$

or

$$u(x, t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{an\pi}{l} t + D_n \sin \frac{an\pi}{l} t \right) \sin \frac{n\pi}{l} x \tag{16}$$

will likewise be a solution of the differential equation (1), which will satisfy the boundary conditions (2) and (3). Series (16) will obviously be a solution of equation (1) only if the coefficients C_n and D_n are such that this series converges and that the series resulting from a double term-by-term differentiation with respect to x and to t converge as well.

The solution (16) should also satisfy the initial conditions (4) and (5). We shall try to do this by choosing the constants C_n and D_n. Substituting into (16) $t = 0$, we get [see condition (4)]:

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l} x. \tag{17}$$

If the function $f(x)$ is such that in the interval $(0, l)$ it may be expanded in a Fourier series (see Sec. 1, Ch. XVII), the condition (17) will be fulfilled if we put

$$C_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx. \tag{18}$$

We then differentiate the terms of (16) with respect to t and substitute $t=0$. From condition (5) we get the equality

$$\varphi(x)=\sum_{n=1}^{\infty} D_n \frac{an\pi}{l} \sin\frac{n\pi}{l} x.$$

We define the Fourier coefficients of this series:

$$D_n \frac{an\pi}{l}=\frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi}{l} x\,dx$$

or

$$D_n=\frac{2}{an\pi}\int_0^l \varphi(x)\sin\frac{n\pi}{l} x\,dx. \tag{19}$$

Thus, we have proved that the series (16), where the coefficients C_n and D_n are defined by formulas (18) and (19) [if it admits double termwise differentiation], is a function $u(x, t)$, which is the solution of equation (1) and satisfies the boundary and initial conditions (2) to (5).

Note. Solving the problem at hand for the wave equation by a different method, we can prove that the series (16) is a solution even when it does not admit termwise differentiation. In this case the function $f(x)$ must be twice differentiable and $\varphi(x)$ must be once differentiable*).

SEC. 4. THE EQUATION FOR PROPAGATION OF HEAT IN A ROD. FORMULATION OF THE BOUNDARY-VALUE PROBLEM

Let us consider a homogeneous rod of length l. We assume that the lateral surface of the rod is impenetrable to heat transfer and that the temperature is the same at all points of any cross-sectional area of the rod. Let us study the process of propagation of heat in the rod.

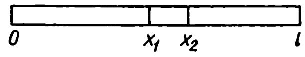


Fig. 373.

We place the x-axis so that one end of the rod coincides with the point $x=0$, the other with the point $x=l$ (Fig. 373). Let $u(x, t)$ be the temperature in the cross section of the rod with abscissa x at time t. Experiment tells us that the rate of propa-

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* These conditions are dealt with in detail in "Equations of Mathematical Physics", A. N. Tikhonov and A. A. Samarsky, Gostekhizdat, 1954 (Russian edition).

gation of heat (that is, the quantity of heat passing through a cross section with abscissa x in unit time) is given by the formula

$$q=-k\frac{\partial u}{\partial x}S \tag{1}$$

where S is the cross-sectional area of the rod and k is the coefficient of heat conduction*).

Let us examine an element of rod contained between cross sections with abscissas x_1 and x_2 $(x_2-x_1=\Delta x)$. The quantity of heat passing through the cross section with abscissa x_1 during time Δt will be equal to

$$\Delta Q_1=-k\frac{\partial u}{\partial x}\bigg|_{x=x_1}S\Delta t, \tag{2}$$

and the same for the cross section with abscissa x_2:

$$\Delta Q_2=-k\frac{\partial u}{\partial x}\bigg|_{x=x_2}S\Delta t. \tag{3}$$

The influx of heat $\Delta Q_1-\Delta Q_2$ into the rod element during time Δt will be

$$\Delta Q_1-\Delta Q_2=\left[-k\frac{\partial u}{\partial x}\bigg|_{x=x_1}S\Delta t\right]-\left[-k\frac{\partial u}{\partial x}\bigg|_{x=x_2}S\Delta t\right]\approx$$
$$\approx k\frac{\partial^2 u}{\partial x^2}\Delta xS\Delta t \tag{4}$$

(we applied the Lagrange theorem to the difference $\frac{\partial u}{\partial x}\Big|_{x=x_1}-$ $-\frac{\partial u}{\partial x}\Big|_{x=x_2}\Big)$. This influx of heat during time Δt was spent in raising the temperature of the rod element by Δu:

$$\Delta Q_1-\Delta Q_2=c\varrho\Delta xS\Delta u$$

or

$$\Delta Q_1-\Delta Q_2\approx c\varrho\Delta xS\frac{\partial u}{\partial x}\Delta t, \tag{5}$$

where c is the thermal capacity of the substance of the rod and ϱ is the density of the substance ($\varrho\Delta xS$ is the mass of an element of rod).

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*) The rate of propagation of heat, or the rate of the thermal flux, is determined by

$$q=\lim\frac{\Delta Q}{\Delta t},$$

where ΔQ is the quantity of heat that has passed through a cross section S during a time Δt.

Equating expressions (4) and (5) of one and the same quantity of heat $\Delta Q_1 - \Delta Q_2$, we get

$$k \frac{\partial^2 u}{\partial x^2} \Delta x S \Delta t = c\varrho \Delta x S \frac{\partial u}{\partial t} \Delta t$$

or

$$\frac{\partial u}{\partial t} = \frac{k}{c\varrho} \frac{\partial^2 u}{\partial x^2}.$$

Denoting $\frac{k}{c\varrho} = a^2$, we finally get

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \qquad (6)$$

This is the equation for the propagation of heat (*the equation of heat conduction*) in a homogeneous rod.

For the solution of equation (6) to be definite, the function $u(x, t)$ must satisfy the boundary-value conditions corresponding to the physical conditions of the problem. For the solution of equation (6), the boundary-value conditions may differ. The conditions which correspond to the so-called *first boundary-value problem* for $0 \leqslant t \leqslant T$ are as follows:

$$u(x, 0) = \varphi(x), \qquad (7)$$

$$u(0, t) = \psi_1(t), \qquad (8)$$

$$u(l, t) = \psi_2(t). \qquad (9)$$

Physically, condition (7) (*the initial condition*) corresponds to the fact that for $t = 0$ a temperature is given in various cross sections of the rod equal to $\varphi(x)$. Conditions (8) and (9) (*the boundary conditions*) correspond to the fact that at the ends of the rod, $x = 0$ and $x = l$, a temperature is maintained equal to $\psi_1(t)$ and $\psi_2(t)$, respectively.

It is proved that the equation (6) has only one solution in the region $0 \leqslant x \leqslant l$, $0 \leqslant t \leqslant T$, which satisfies the conditions (7), (8), and (9).

SEC. 5. HEAT PROPAGATION IN SPACE

Let us further consider the process of propagation of heat in three-dimensional space. Let $u(x, y, z, t)$ be the temperature at a point with coordinates (x, y, z) at time t. Experiment states that the rate of heat passage through an area Δs, that is, the quantity of heat passing through in unit time is governed by the formula [similar to formula (1) of the preceding section]

$$\Delta Q = -k \frac{\partial u}{\partial n} \Delta s, \qquad (1)$$

where k is the coefficient of heat conductivity of the medium under consideration, which we regard as homogeneous and isotropic, $\boldsymbol{n}$ is the unit vector directed normally to the area Δs in the direction of motion of the heat. Taking advantage of Sec. 14, Ch. VIII, we can write

$$\frac{\partial u}{\partial n}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma,$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the vector $\boldsymbol{n}$, or

$$\frac{\partial u}{\partial n}=\boldsymbol{n}\operatorname{grad}u.$$

Substituting the expression $\frac{\partial u}{\partial n}$ into formula (1), we get

$$\Delta Q=-k\boldsymbol{n}\operatorname{grad}u\,\Delta s.$$

The quantity of heat passing in time Δt through the elementary area Δs will be

$$\Delta Q\Delta t=-k\boldsymbol{n}\operatorname{grad}u\,\Delta t\,\Delta s.$$

Now let us return to the problem posed at the beginning of the section. In the medium at hand we pick out a small volume V bounded by the surface S. The quantity of heat passing through the surface S will be

$$Q=-\Delta t\iint_S k\boldsymbol{n}\operatorname{grad}u\,ds, \tag{2}$$

where $\boldsymbol{n}$ is the unit vector directed along the external normal to the surface S. It is obvious that formula (2) yields the quantity of heat entering the volume V (or leaving the volume V) during time Δt. The quantity of heat entering V is spent in raising the temperature of the substance of this volume.

Let us consider an elementary volume Δv. Let its temperature rise by Δu in time Δt. Obviously, the quantity of heat expended on raising the temperature of the element Δv will be

$$c\Delta v\varrho\,\Delta u\approx c\Delta v\varrho\frac{\partial u}{\partial t}\Delta t,$$

where c is the heat capacity of the substance and ϱ is the density. The total quantity of heat consumed in raising the temperature in the volume V during time Δt will be

$$\Delta t\iiint_V c\varrho\frac{\partial u}{\partial t}dv.$$

But this is the heat that has entered the volume V during the time Δt; it is defined by formula (2). Thus, we have the equality

$$\Delta t \iint_S k\boldsymbol{n} \operatorname{grad} u \, ds = \Delta t \iiint_V c\varrho \frac{\partial u}{\partial t} \, dv.$$

Cancelling out Δt, we get

$$\iint_S k\boldsymbol{n} \operatorname{grad} u \, ds = \iiint_V c\varrho \frac{\partial u}{\partial t} \, dv. \tag{3}$$

The surface integral on the left-hand side of this equation we transform by the Ostrogradsky formula (see Sec. 8, Ch. XV), assuming $\boldsymbol{F} = k \operatorname{grad} u$:

$$\iint_S (k \operatorname{grad} u) \boldsymbol{n} \, ds = \iiint_V \operatorname{div} (k \operatorname{grad} u) dv.$$

Replacing the double integral on the left of (3) by a triple integral, we get

$$\iiint_V \operatorname{div} (k \operatorname{grad} u) \, dv = \iiint_V c\varrho \frac{\partial u}{\partial t} \, dv$$

or

$$\iiint_V \left[\operatorname{div} (k \operatorname{grad} u) - c\varrho \frac{\partial u}{\partial t}\right] dv = 0. \tag{4}$$

Applying the mean-value theorem to the triple integral on the left (see Sec. 12, Ch. XIV), we get

$$\left[\operatorname{div} (k \operatorname{grad} u) - c\varrho \frac{\partial u}{\partial t}\right]_{x=x_1,\ y=y_1,\ z=z_1} = 0, \tag{5}$$

where the point $P(x, y, z)$ is some point of the volume V.

Since we can pick out an arbitrary volume V in three-dimensional space where propagation of heat is taking place, and since we assume that the integrand in (4) is continuous, equality (5) will be fulfilled at each point of the space. Thus,

$$c\varrho \frac{\partial u}{\partial t} = \operatorname{div} (k \operatorname{grad} u). \tag{6}$$

But

$$k \operatorname{grad} u = k \frac{\partial u}{\partial x} \boldsymbol{i} + k \frac{\partial u}{\partial y} \boldsymbol{j} + k \frac{\partial u}{\partial z} \boldsymbol{k}$$

(see Sec. 14, Ch. VIII) and

$$\operatorname{div} (k \operatorname{grad} u) = \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial u}{\partial z}\right)$$

(see Sec. 9, Ch. XV). Substituting into (6), we obtain

$$c\varrho\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k\,\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\,\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\,\frac{\partial u}{\partial z}\right). \qquad (7)$$

If k is a constant, then

$$\operatorname{div}(k\operatorname{grad}u)=k\operatorname{div}(\operatorname{grad}u)=k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

and equation (6) then yields

$$c\varrho\frac{\partial u}{\partial t}=k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{dz^2}\right)$$

or, putting $\frac{k}{c\varrho}=a^2$,

$$\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right). \qquad (8)$$

Equation (8) is briefly written

$$\frac{\partial u}{\partial t}=a^2\Delta u,$$

where $\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$ is the Laplace operator. Equation (8) is the *equation of heat conduction in space.* To find its unique solution that corresponds to the problem posed here, it is necessary to specify the boundary-value conditions.

Let there be a body Ω with a surface σ. In this body we consider the process of propagation of heat. At the initial time the temperature of the body is specified, which means that the solution is known for $t=0$ (*the initial condition*):

$$u(x,\ y,\ z,\ 0)=\varphi(x,\ y,\ z). \qquad (9$$

In addition to that we must know the temperature at any point M of the surface σ of the body at any time t (*the boundary condition*):

$$u(M,\ t)=\psi(M,\ t). \qquad (10)$$

(Other boundary conditions are possible too.)

If the desired function $u(x,\ y,\ z,\ t)$ is independent of z, which corresponds to the temperature being independent of z, we obtain the equation

$$\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \qquad (11)$$

which is the *equation of heat propagation in a plane.*

If we consider heat propagation in a flat region D with boundary C, then the boundary conditions, like (9) and (10), are

formulated as follows:

$$u(x, y, 0)=\varphi(x, y),$$
$$u(M, t)=\psi(M, t),$$

where φ and ψ are specified functions and M is a point on the boundary C.

But if the function u does not depend either on z or on y, then we get the equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$$

which is the equation of heat propagation in a rod.

SEC. 6. SOLUTION OF THE FIRST BOUNDARY-VALUE PROBLEM FOR THE HEAT-CONDUCTIVITY EQUATION BY THE METHOD OF FINITE DIFFERENCES

When we solve partial differential equations by the method of finite differences, the derivatives, as in the case of ordinary differential equations, are replaced by appropriate differences (see Fig. 374):

$$\frac{\partial u(x, t)}{\partial x}\approx\frac{u(x+h, t)-u(x, t)}{h}, \tag{1}$$

$$\frac{\partial^2 u(x, t)}{\partial x^2}\approx\frac{1}{h}\left[\frac{u(x+h, t)-u(x, t)}{h}-\frac{u(x, t)-u(x-h, t)}{h}\right]$$

or

$$\frac{\partial^2 u(x, t)}{\partial x^2}=\frac{u(x+h, t)-2u(x, t)+u(x-h, t)}{h^2}; \tag{2}$$

similarly,

$$\frac{\partial u(x, t)}{\partial t}=\frac{u(x, t+l)-u(x, t)}{l}. \tag{3}$$

The first boundary-value problem for the heat-conductivity equation is stated (see Sec. 4) as follows. It is required to find the solution of the equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \tag{4}$$

that satisfies the boundary-value conditions

$$u(x, 0)=\varphi(x), \qquad 0\leqslant x\leqslant L, \tag{5}$$
$$u(0, t)=\psi_1(t), \qquad 0\leqslant t\leqslant T, \tag{6}$$
$$u(l, t)=\psi_2(t), \qquad 0\leqslant t\leqslant T, \tag{7}$$

that is, we have to find the solution $u(x, t)$ in a rectangle bounded by the straight lines $t=0$, $x=0$, $x=L$, $t\leqslant T$, if the values

of the desired function are given on three of its sides: $t=0$, $x=0$, $x=L$ (Fig. 375). We cover our region with a grid formed by the straight lines

$$x=ih, \quad i=1, 2, \ldots,$$
$$t=kl, \quad k=1, 2, \ldots,$$

and approximate the values at the nodes of the grid, that is, at the points of intersection of these lines. Introducing the notation

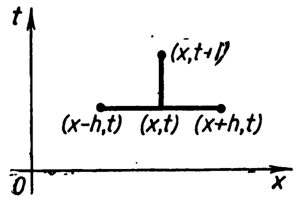


Fig. 374.

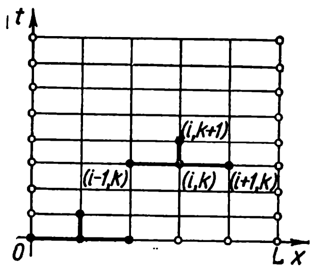


Fig. 375.

$u(ih, kl)=u_{i,k}$, we write [in place of equation (4)] a corresponding difference equation for the point (ih, kl). In accord with (3) and (2), we get

$$\frac{u_{i,k+1}-u_{i,k}}{l}=a^2\frac{u_{i+1,k}-2u_{i,k}+u_{i-1,k}}{h^2}. \tag{8}$$

We determine $u_{i,k+1}$:

$$u_{i,k+1}=\left(1-\frac{2a^2l}{h^2}\right)u_{i,k}+a^2\frac{l}{h^2}(u_{i+1,k}+u_{i-1,k}). \tag{9}$$

From (9) it follows that if we know three values in the kth row: $u_{i,k}$, $u_{i+1,k}$, $u_{i-1,k}$, we can determine the value $u_{i,k+1}$ in the $(k+1)$st row. We know all the values on the straight line $t=0$ [see formula (5)]. By formula (9) determine the values at all the interior points of the segment $t=l$. We know the values of the end points of this segment by virtue of (6) and (7). In this way, row by row, we determine the values of the desired solution at all nodes of the grid.

It is proved that from formula (9) we can obtain an approximate value of the solution not for an arbitrary relationship between the steps h and l, but only if $l\leqslant\frac{h^2}{2a^2}$. Formula (9) is greatly simplified if the step l along the t-axis is chosen so that

$$1-\frac{2a^2l}{h^2}=0$$

or

$$l=\frac{h^2}{2a^2}.$$

In this case, (9) takes the form

$$u_{i,\,k+1}=\frac{1}{2}(u_{i+1,\,k}+u_{i-1,\,k}). \qquad (10)$$

This formula is particularly convenient for computations (Fig. 376). This method gives the solution at the nodes of the grid. Solutions between the nodes may be obtained, for example, by extrapolation, by drawing a plane through every three points in the space (x, t, u). Let us denote by $u_h(x, t)$ a solution obtained by formula (10) and this extrapolation. It is proved that

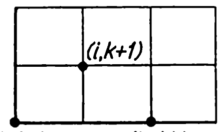


Fig. 376.

$$\lim_{h\to 0} u_h(x, t)=u(x, t),$$

where $u(x, t)$ is the solution of our problem. It is also proved *) that

$$|u_h(x, t)-u(x, t)|<Mh^2,$$

where M is a constant independent of h.

SEC. 7. PROPAGATION OF HEAT IN AN UNBOUNDED ROD

Let the temperature be given at various sections of an unbounded rod at an initial instant of time. It is required to determine the temperature distribution in the rod at subsequent instants of time. (Physical problems reduce to that of heat propagation in an unbounded rod when the rod is so long that the temperature in the interior points of the rod at the instants of time under consideration are but slightly dependent on the conditions at the ends of the rod.)

If the rod coincides with the x-axis, the problem is stated mathematically as follows. Find the solution to the equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

*) This question is dealt with in more detail in D. Yu. Panov's "Reference on Numerical Solution of Partial Differential Equations", Gostekhizdat, 1951; Lothar Collatz, "Numerische Behandlung von Differentialgluchungen", 1955.

in the region $-\infty < x < \infty$, $0 < t$ which satisfies the initial condition

$$u(x, 0) = \varphi(x). \tag{2}$$

To find the solution, we apply the method of separation of variables (see Sec. 3); that is, we shall seek a particular solution of equation (1) in the form of a product of two functions:

$$u(x, t) = X(x)T(t). \tag{3}$$

Putting this into equation (1) we have $X(x)T'(t) = a^2X''(x)T(t)$ or

$$\frac{T'}{a^2T} = \frac{X''}{X} = -\lambda^2. \tag{4}$$

Neither of these relations can be dependent either on x or on t; therefore, we equate them to a constant, *) $-\lambda^2$. From (4) we get two equations:

$$T' + a^2\lambda^2T = 0, \tag{5}$$

$$X'' + \lambda^2X = 0. \tag{6}$$

Solving them we find

$$T = Ce^{-a^2\lambda^2t},$$

$$X = A\cos\lambda x + B\sin\lambda x.$$

Substituting into (3), we obtain

$$u_\lambda(x, t) = e^{-a^2\lambda^2t}[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x] \tag{7}$$

[the constant C is included in $A(\lambda)$ and in $B(\lambda)$].

For each value of λ we obtain a solution of the form (7). For each value of λ the arbitrary constants A and B have definite values. We can therefore consider A and B functions of λ. The sum of the solutions of form (7) is likewise a solution [since equation (1) is linear]:

$$\sum_\lambda e^{-a^2\lambda^2t}[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x].$$

Integrating expression (7) with respect to the parameter λ between 0 and ∞, we also get a solution

$$u(x, t) = \int_0^\infty e^{-a^2\lambda^2t}[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda, \tag{8}$$

*) Since from the meaning of the problem $T(t)$ must be bounded for any t, if $\varphi(x)$ is bounded, it follows that $\frac{T'}{T}$ must be negative. And so we write $-\lambda^2$.

if $A(\lambda)$ and $B(\lambda)$ are such that this integral, its derivative with respect to t and the second derivative with respect to x exist and are obtained by differentiation of the integral with respect to t and x. We choose $A(\lambda)$ and $B(\lambda)$ such that the solution $u(x, t)$ satisfies the condition (2). Putting $t=0$ in (8), we get [on the basis of condition (2)]:

$$u(x, 0)=\varphi(x)=\int_0^\infty [A(\lambda)\cos\lambda x+B(\lambda)\sin\lambda x]\,d\lambda. \tag{9}$$

Suppose that the function $\varphi(x)$ is such that it may be represented by the Fourier integral (see Sec. 12, Ch. XVII):

$$\varphi(x)=\frac{1}{\pi}\int_0^\infty\left(\int_{-\infty}^\infty \varphi(\alpha)\cos\lambda(\alpha-x)\,d\alpha\right)d\lambda$$

or

$$\varphi(x)=\frac{1}{\pi}\int_0^\infty\left[\left(\int_{-\infty}^\infty \varphi(\alpha)\cos\lambda\alpha\,d\alpha\right)\cos\lambda x+\right.$$
$$\left.+\left(\int_{-\infty}^\infty \varphi(\alpha)\sin\lambda\alpha\,d\alpha\right)\sin\lambda x\right]d\lambda. \tag{10}$$

Comparing the right sides of (9) and (10), we get

$$\left.\begin{aligned} A(\lambda)&=\frac{1}{\pi}\int_{-\infty}^\infty \varphi(\alpha)\cos\lambda\alpha\,d\alpha,\\ B(\lambda)&=\frac{1}{\pi}\int_{-\infty}^\infty \varphi(\alpha)\sin\lambda\alpha\,d\alpha.\end{aligned}\right\} \tag{11}$$

Putting the expressions thus found of $A(\lambda)$ and $B(\lambda)$ into (8), we obtain

$$u(x, t)=\frac{1}{\pi}\int_0^\infty e^{-a^2\lambda^2 t}\left[\left(\int_{-\infty}^\infty \varphi(\alpha)\cos\lambda\alpha\,d\alpha\right)\cos\lambda x+\right.$$
$$\left.+\left(\int_{-\infty}^{+\infty} \varphi(\alpha)\sin\lambda\alpha\,d\alpha\right)\sin\lambda x\right]d\lambda=$$
$$=\frac{1}{\pi}\int_0^\infty e^{-a^2\lambda^2 t}\left[\int_{-\infty}^\infty \varphi(\alpha)(\cos\lambda\alpha\cos\lambda x+\sin\lambda\alpha\sin\lambda x)\,d\alpha\right]d\lambda=$$
$$=\frac{1}{\pi}\int_0^\infty e^{-a^2\lambda^2 t}\left(\int_{-\infty}^\infty \varphi(\alpha)\cos\lambda(\alpha-x)\,d\alpha\right)d\lambda$$

or, changing the order of integration, we finally get

$$u(x,\ t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\varphi(\alpha)\left(\int_{0}^{\infty}e^{-a^2\lambda^2 t}\cos\lambda(\alpha-x)\,b\lambda\right)\right]d\alpha. \qquad (12)$$

This is the solution of the problem.

Let us transform formula (12). Compute the integral in the parentheses:

$$\int_{0}^{\infty}e^{-a^2\lambda^2 t}\cos\lambda(\alpha-x)\,d\lambda=\frac{1}{a\sqrt{t}}\int_{0}^{\infty}e^{-z^2}\cos\beta z\,dz. \qquad (13)$$

The integral is transformed by substitution:

$$a\lambda\sqrt{t}=z,\quad \frac{\alpha-x}{a\sqrt{t}}=\beta. \qquad (14)$$

We denote

$$K(\beta)=\int_{0}^{\infty}e^{-z^2}\cos\beta z\,dz. \qquad (15)$$

Differentiating, *) we get

$$K'(\beta)=-\int_{0}^{\infty}e^{-z^2}z\sin\beta z\,dz.$$

Integrating by parts, we find

$$K'(\beta)=\frac{1}{2}\left[e^{-z^2}\sin\beta z\right]_{0}^{\infty}-\frac{\beta}{2}\int_{0}^{\infty}e^{-z^2}\cos\beta z\,dz$$

or

$$K'(\beta)=-\frac{\beta}{2}K(\beta).$$

Integrating this differential equation, we obtain

$$K(\beta)=Ce^{-\frac{\beta^2}{4}}. \qquad (16)$$

Determine the constant C. From (15) it follows that

$$K(0)=\int_{0}^{\infty}e^{-z^2}dz=\frac{\sqrt{\pi}}{2}$$

*) Differentiation here is easily justified.

(see Sec. 5, Ch. XIV). Hence, in (16) we must have

$$C=\frac{\sqrt{\pi}}{2}.$$

And so

$$K(\beta)=\frac{\sqrt{\pi}}{2}e^{-\frac{\beta^2}{4}}. \tag{17}$$

Put the value (17) of the integral (15) into (13):

$$\int_0^\infty e^{-a^2\lambda^2 t}\cos\lambda(\alpha-x)\,d\lambda=\frac{1}{a\sqrt{t}}\,\frac{\sqrt{\pi}}{2}e^{-\frac{\beta^2}{4}}.$$

In place of β we substitute its expression (14) and finally get the value of the integral (13):

$$\int_0^\infty e^{-a^2\lambda^2 t}\cos\lambda(\alpha-x)\,d\lambda=\frac{1}{2a}\sqrt{\frac{\pi}{t}}\,e^{-\frac{(\alpha-x)^2}{4a^2t}}. \tag{18}$$

Putting this expression of the integral into the solution (12), we finally get

$$u(x, t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi(\alpha)e^{-\frac{(\alpha-x)^2}{4a^2t}}\,d\alpha. \tag{19}$$

This formula, called the *Poisson integral*, is the solution to the problem of heat propagation in an unbounded rod.

Note. It may be proved that the function $u(x, t)$, defined by integral (19), is a solution of equation (1) and satisfies condition (2) if the function $\varphi(x)$ is bounded on an infinite interval $(-\infty, \infty)$.

Let us establish the physical meaning of formula (19). We consider the function

$$\varphi^*(x)=\begin{cases}0 & \text{for } -\infty<x<x_0,\\ \varphi(x) & \text{for } x_0\leqslant x\leqslant x_0+\Delta x,\\ 0 & \text{for } x_0+\Delta x<x<\infty.\end{cases} \tag{20}$$

Then the function

$$u^*(x, t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\varphi^*(\alpha)\,e^{-\frac{(\alpha-x)^2}{4a^2t}}\,d\alpha \tag{21}$$

is the solution to equation (1), which solution takes on the value

$\varphi^*(x)$ when $t=0$. Taking (20) into consideration, we can write

$$u^*(x,\ t)=\frac{1}{2a\sqrt{\pi t}}\int_{x_0}^{x_0+\Delta x}\varphi(\alpha)e^{-\frac{(\alpha-x)^2}{4a^2t}}\,d\alpha.$$

Applying the mean-value theorem to the latter integral, we get

$$u^*(x,\ t)=\frac{\varphi(\xi)\,\Delta x}{2a\sqrt{\pi t}}e^{-\frac{(\xi-x)^2}{4a^2t}},\quad x_0<\xi<x_0+\Delta x. \tag{22}$$

Formula (22) gives the value of temperature at a point in the rod at any time if for $t=0$ the temperature in the rod is everywhere $u^*=0$, with the exception of the interval $[x_0,\ x_0+\Delta x]$, where it is $\varphi(x)$. The sum of temperatures of form (22) is what yields the solution of (19). It will be noted that if ϱ is the linear density of the rod, c the heat capacity of the material, then the quantity of heat in the element $[x_0,\ x_0+\Delta x]$ for $t=0$ will be

$$\Delta Q\approx\varphi(\xi)\,\Delta x\,\varrho c. \tag{23}$$

Let us now consider the function

$$\frac{1}{2a\sqrt{\pi t}}e^{-\frac{(\xi-x)^2}{4a^2t}}. \tag{24}$$

Comparing it with the right side of (22) and taking into account (23), we may say that it yields the temperature at any point of the rod at any instant of time t if for $t=0$ there was an instantaneous heat source with quantity of heat $Q=c\varrho$ in the cross section ξ (the limiting case as $\Delta x\to 0$).

SEC. 8. PROBLEMS THAT REDUCE TO INVESTIGATING SOLUTIONS OF THE LAPLACE EQUATION. STATING BOUNDARY-VALUE PROBLEMS

In this section we shall consider certain problems that reduce to the solution of the *Laplace equation:*

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}=0. \tag{1}$$

As already pointed out, the left side of equation (1),

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}\equiv\Delta u$$

is called the *Laplacian operator*. The functions u which satisfy the Laplace equation are called *harmonic functions*.

I. A stationary (steady-state) distribution of temperature in a homogeneous body. Let there be a homogeneous body Ω bounded

by a surface σ. In Sec. 7 it was shown that the temperature at various points of the body statisfies equation (8):

$$\frac{\partial u}{\partial t}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right).$$

If the process is steady-state, that is, if the temperature is not dependent on the time, but only on the coordinates of the points of the body, then $\frac{\partial u}{\partial t}=0$ and, consequently, the temperature satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0. \tag{1}$$

To determine the temperature in the body uniquely from this equation, one has to know the temperature of the surface σ. Thus, for equation (1), the boundary-value problem is formulated as follows.

To find the function $u(x, y, z)$ that satisfies equation (1) inside the volume Ω and that takes on specified values at each point M of the surface σ:

$$u|_\sigma=\psi(M). \tag{2}$$

This problem is called the *Dirichlet problem* or the *first boundary-value problem* of equation (1).

If the temperature on the surface of the body is not known, but the heat flux at every point of the surface is, which is proportional to $\frac{\partial u}{\partial n}$ (see Sec. 5), then in place of the boundary-value condition (2) on the surface σ we will have the condition

$$\left.\frac{\partial u}{\partial n}\right|_\sigma=\psi^*(M). \tag{3}$$

The problem of finding the solution to (1) that satisfies the boundary-value condition (3) is called the *Neumann problem* or the *second boundary-value problem.*

If we consider the temperature distribution in a two-dimensional region D bounded by a contour C, then the function u will depend on two variables x and y and will satisfy the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \tag{4}$$

which is called the Laplace equation in a plane. The boundary-value conditions (2) and (3) must be fulfilled on the contour C.

II. The potential flow of a fluid. Equation of continuity. Let there be a flow of liquid inside a volume Ω bounded by a surface σ (in a particular case, Ω may also be unbounded). Let ϱ

be the density of the liquid. We denote the velocity of the liquid by

$$v = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}, \tag{5}$$

where v_x, v_y, v_z are the projections of the vector $\boldsymbol{v}$ on the coordinate axes. In the body Ω pick out a small volume ω, bounded by the surface S. The following quantity of liquid will pass through each element Δs of the surface S in a time Δt:

$$\Delta Q = \boldsymbol{vn}\, \Delta s \varrho\, \Delta t,$$

where $\boldsymbol{n}$ is the unit vector directed along the outer normal to the surface S. The total quantity of liquid Q entering the volume ω (or flowing out of the volume ω) is expressed by the integral

$$Q = \Delta t \iint_S \varrho\, \boldsymbol{vn}\, ds \tag{6}$$

(see Secs. 5 and 6, Ch. XV). The quantity of liquid in the volume ω at time t was

$$\iiint_\omega \varrho d\omega.$$

During time Δt the quantity of liquid will change (due to changes in density) by the amount

$$Q = \iiint_\omega \Delta\varrho \Delta\omega \approx \Delta t \iiint_\omega \frac{\partial \varrho}{\partial t} d\omega. \tag{7}$$

Assuming that there are no sources in the volume ω, we conclude that this change is brought about by an influx of liquid to an amount that is determined by equation (6). Equating the right sides of (6) and (7) and cancelling out Δt, we get

$$-\iint_S \varrho \boldsymbol{vn}\, ds = +\iiint_\omega \frac{\partial \varrho}{\partial t} d\omega. \tag{8}$$

We transform the iterated integral on the left by Ostrogradsky's formula (Sec. 8, Ch. XV). Then (8) will assume the form

$$-\iiint_\omega \operatorname{div}(\varrho \boldsymbol{v})\, d\omega = \iiint_\omega \frac{\partial \varrho}{\partial t} d\omega$$

or

$$\iiint_\omega \left(\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) \right) d\omega = 0.$$

Since the volume ω is arbitrary and the integrand is continuous we obtain

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0 \tag{9}$$

or

$$\frac{\partial \varrho}{\partial t}+\frac{\partial}{\partial x}(\varrho v_x)+\frac{\partial}{\partial y}(\varrho v_y)+\frac{\partial}{\partial z}(\varrho v_z)=0. \qquad (9')$$

This is the *equation of continuous flow of a compressible liquid.*

Note. In certain problems, for instance when considering the movement of oil or gas in a subterranean porous medium to a well, it may be taken that

$$\boldsymbol{v}=-\frac{k}{\varrho}\operatorname{grad} p,$$

where p is the pressure and k is the coefficient of permeability and

$$\frac{\partial \varrho}{\partial t}\approx\lambda\frac{\partial p}{\partial t},$$

$\lambda=\text{const}$. Substituting into the continuity equation (9), we get

$$\lambda\frac{\partial p}{\partial t}-\operatorname{div}(k\operatorname{grad} p)=0$$

or

$$\lambda\frac{\partial p}{\partial t}=\frac{\partial}{\partial x}\left(k\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial p}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial p}{\partial z}\right). \qquad (10)$$

If k is a constant, then this equation takes on the form

$$\frac{\partial p}{\partial t}=\frac{k}{\lambda}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}\right), \qquad (11)$$

and we arrive at Fourier's equation.

Let us return to equation (9). If the liquid is noncompressible, then $\varrho=\text{const}$, $\frac{\partial \varrho}{\partial t}=0$, and (9) becomes

$$\operatorname{div}(\boldsymbol{v})=0. \qquad (12)$$

If the motion is potential, that is, if the vector $\boldsymbol{v}$ is a gradient of some function φ:

$$\boldsymbol{v}=\operatorname{grad}\varphi,$$

then equation (12) takes the form

$$\operatorname{div}(\operatorname{grad}\varphi)=0$$

or

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0; \qquad (13)$$

that is, the potential function of the velocity φ must satisfy the Laplace equation.

In many problems, as, for example, those dealing with filtration, we can put

$$\boldsymbol{v} = -k_1 \operatorname{grad} p,$$

where p is the pressure and k_1 is a constant; we then get the Laplace equation for the determination of the pressure:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0. \tag{13'}$$

The boundary-value conditions for equation (13) or (13') may be the following:

1. On the surface σ are specified the values of the desired function p—pressure [condition (2)]. This is the Dirichlet problem.
2. On the surface σ are specified the values of the normal derivative $\frac{\partial p}{\partial n}$; the flow through the surface is specified [condition (3)]. This is the Neumann problem.
3. On parts of the surface σ are specified the values of the desired function p—pressure, and on parts of the surface are specified the values of the normal derivative $\frac{\partial p}{\partial n}$—the flow through the surface. This is the Dirichlet-Neumann problem.

If the motion is two-dimensional-parallel—that is, the function φ (or p) does not depend on z—then we get the Laplace equation in a two-dimensional region D with boundary C:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0. \tag{14}$$

Boundary-value conditions of type (2), the Dirichlet problem, or of type (3), the Neumann problem, are specified on the contour C.

III. The potential of a steady-state electric current. Let a homogeneous medium fill some volume V, and let an electric current pass through it whose density at each point is given by the vector $\boldsymbol{J}(x, y, z) = J_x\boldsymbol{i} + J_y\boldsymbol{j} + J_z\boldsymbol{k}$. Suppose that the current density is independent of the time t. Further assume that there are no current sources in the volume under consideration. Thus, the flux or a vector $\boldsymbol{J}$ through any closed surface S lying inside the volume V will be equal to zero:

$$\iint\limits_S \boldsymbol{J}\boldsymbol{n}\, ds = 0,$$

where $\boldsymbol{n}$ is a unit vector directed along the outer normal to the surface.

From Ostrogradsky's formula we conclude that

$$\operatorname{div} \boldsymbol{J}=0. \tag{15}$$

The electric force $\boldsymbol{E}$ in the conducting medium at hand is, on the basis of Ohm's generalised law,

$$\boldsymbol{E}=\frac{\boldsymbol{J}}{\lambda} \tag{16}$$

or

$$\boldsymbol{J}=\lambda \boldsymbol{E},$$

where λ is the conductivity of the medium, which we shall consider constant.

From the general electromagnetic-field equations it follows that if the process is stationary, then the vector field $\boldsymbol{E}$ is irrotational, that is, rot $\boldsymbol{E}=0$. Then, like the case we had when considering the velocity field of a liquid, the vector field is potential (see Sec. 9, Ch. XV). There is a function φ such that

$$\boldsymbol{E}=\operatorname{grad} \varphi. \tag{17}$$

From (16) we get

$$\boldsymbol{J}=\lambda \operatorname{grad} \varphi. \tag{18}$$

From (15) and (18) we have

$$\lambda \operatorname{div} (\operatorname{grad} \varphi)=0$$

or

$$\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}+\frac{\partial^2 \varphi}{\partial z^2}=0. \tag{19}$$

We get the Laplace equation.

Solving this equation for appropriate boundary-value conditions, we find the function φ, and from formulas (18) and (17) we find the current $\boldsymbol{J}$ and the electric force $\boldsymbol{E}$.

SEC. 9. THE LAPLACE EQUATION IN CYLINDRICAL COORDINATES. SOLUTION OF THE DIRICHLET PROBLEM FOR A RING WITH CONSTANT VALUES OF THE DESIRED FUNCTION ON THE INNER AND OUTER CIRCUMFERENCES

Let $u(x, y, z)$ be a harmonic function of three variables. Then

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0. \tag{1}$$

We introduce the cylindrical coordinates (r, φ, z):

$$x=r \cos \varphi, \quad x=r \sin \varphi, \quad z=z,$$

whence

$$r=\sqrt{x^2+y^2}, \quad \varphi=\arctan \frac{y}{x}, \quad z=z. \tag{2}$$

Replacing the independent variables x, y, and z by r, φ, and z, we arrive at the function u^*:

$$u(x, y, z) = u^*(r, \varphi, z).$$

Let us find the equation that will be satisfied by $u^*(r, \varphi, z)$ as a function of the arguments r, φ, and z; we have

$$\frac{\partial u}{\partial x} = \frac{\partial u^*}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u^*}{\partial \varphi}\frac{\partial \varphi}{\partial x},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u^*}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial u^*}{\partial r}\frac{\partial^2 r}{\partial x^2} + 2\frac{\partial^2 u^*}{\partial r\,\partial \varphi}\frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial^2 u^*}{\partial \varphi^2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{\partial u^*}{\partial \varphi}\frac{\partial^2 \varphi}{\partial x^2}; \quad (3)$$

similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u^*}{\partial r^2}\left(\frac{\partial r}{\partial y}\right)^2 + \frac{\partial u^*}{\partial r}\frac{\partial^2 r}{\partial y^2} + 2\frac{\partial^2 u^*}{\partial r\,\partial \varphi}\frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial y} + \frac{\partial^2 u^*}{\partial \varphi^2}\left(\frac{\partial \varphi}{\partial y}\right)^2 + \frac{\partial u^*}{\partial \varphi}\frac{\partial^2 \varphi}{\partial y^2}, \quad (4)$$

besides,

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u^*}{\partial z^2}. \quad (5)$$

We find the expressions for

$$\frac{\partial r}{\partial x},\ \frac{\partial r}{\partial y},\ \frac{\partial^2 r}{\partial x^2},\ \frac{\partial^2 r}{\partial y^2},\ \frac{\partial \varphi}{\partial x},\ \frac{\partial \varphi}{\partial y},\ \frac{\partial^2 \varphi}{\partial x^2},\ \frac{\partial^2 \varphi}{\partial y^2}$$

from equations (2). Adding the right sides of (3), (4) and (5), and equating the sum to zero [since the sum of the left-hand sides of these equations are zero by virtue of (1)], we get

$$\frac{\partial^2 u^*}{\partial r^2} + \frac{1}{r}\frac{\partial u^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u^*}{\partial \varphi^2} + \frac{\partial^2 u^*}{\partial z^2} = 0. \quad (6)$$

This is the *Laplace equation in cylindrical coordinates*.

If the function u is independent of z and is dependent on x and y, then the function u^*, dependent only on r and φ, satisfies the equation

$$\frac{\partial^2 u^*}{\partial r^2} + \frac{1}{r}\frac{\partial u^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u^*}{\partial \varphi^2} = 0, \quad (7)$$

where r and φ are polar coordinates in a plane.

Now let us find the solution to Laplace's equation in the region D (ring) bounded by the circles $C_1: x^2 + y^2 = R_1^2$ and $C_2: x^2 + y^2 = R_2^2$ with the following boundary values imposed:

$$u\,|\,C_1 = u_1, \quad (8)$$

$$u\,|\,C_2 = u_2, \quad (9)$$

where u_1 and u_2 are constants.

We will solve the problem in polar coordinates. Obviously, it is desirable to seek a solution that is independent of φ. Equation (7) in this case takes the form

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=0.$$

Integrating this equation we find

$$u=C_1 \ln r+C_2. \tag{10}$$

We determine C_1 and C_2 from conditions (8) and (9):

$$u_1=C_1 \ln R_1+C_2,$$
$$u_2=C_1 \ln R_2+C_2.$$

Whence we find

$$C_1=\frac{u_2-u_1}{\ln \frac{R_2}{R_1}}, \quad C_2=u_1-(u_2-u_1)\frac{\ln R_1}{\ln \frac{R_2}{R_1}}.$$

Substituting the values of C_1 and C_2 thus found into (10), we finally get

$$u=u_1+\frac{\ln \frac{r}{R_1}}{\ln \frac{R_2}{R_1}}(u_2-u_1). \tag{11}$$

Note. We have actually solved the following problem. To find the function u that satisfies the Laplace equation in a region bounded by the surfaces (in cylindrical coordinates)

$$r=R_1, \; r=R_2, \; z=0, \; z=H,$$

and that satisfies the following boundary conditions:

$$u|_{r=R_1}=u_1, \quad u|_{r=R_2}=u_2,$$
$$\left.\frac{\partial u}{\partial z}\right|_{z=0}=0, \quad \left.\frac{\partial u}{\partial z}\right|_{z=H}=0$$

(the Dirichlet-Neumann problem). It is obvious that the desired solution does not depend either on z or on φ and is given by formula (11).

SEC. 10. THE SOLUTION OF DIRICHLET'S PROBLEM FOR A CIRCLE

In an xy-plane, let there be a circle of radius R with centre at the origin and let there be a certain function $f(\varphi)$, where φ is the polar angle, be given on its circumference. It is required to find the function $u(r, \varphi)$ continuous in the circle (including

the boundary) and satisfying (inside the circle) the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

and, on the circumference, assuming the specified values

$$u|_{r=R}=f(\varphi). \tag{2}$$

We shall solve the problem in polar coordinates. Rewrite equation (1) in these coordinates:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}=0$$

or

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \varphi^2}=0. \tag{1'}$$

We shall seek the solution by the method of separation of variables, placing

$$u=\Phi(\varphi)R(r). \tag{3}$$

Substituting into equation (1′), we get

$$r^2\Phi(\varphi)R''(r)+r\Phi(\varphi)R'(r)+\Phi''(\varphi)R(r)=0$$

or

$$\frac{\Phi''(\varphi)}{\Phi(\varphi)}=-\frac{r^2R''(r)+rR'(r)}{R(r)}=-k^2. \tag{4}$$

Since the left side of this equation is independent of r and the right is independent of φ, it follows that they are equal to a constant which we denote by $-k^2$. Thus, equation (4) yields two equations:

$$\Phi''(\varphi)+k^2\Phi(\varphi)=0, \tag{5}$$

$$r^2R''+rR'-k^2R=0. \tag{5'}$$

The complete integral of (5) will be

$$\Phi=A\cos k\varphi+B\sin k\varphi. \tag{6}$$

We seek the solution of (5′) in the form $R=r^m$. Substituting $R=r^m$ into (5′), we get

$$r^2m(m-1)r^{m-2}+rmr^{m-1}-k^2r^m=0$$

or

$$m^2-k^2=0.$$

We can write two particular linearly independent solutions r^k and r^{-k}. The general solution of equation (5′) is

$$R=Cr^k+Dr^{-k}. \tag{7}$$

We substitute expressions (6) and (7) into (3):

$$u_k=(A_k\cos k\varphi+B_k\sin k\varphi)(C_kr^k+D_kr^{-k}). \tag{8}$$

Function (8) will be the solution of (1′) for any value of k different from zero. If $k=0$, then equations (5) and (5′) take the form

$$\Phi''=0, \quad rR''+R'=0,$$

and, consequently,

$$u_0=(A_0+B_0\varphi)(C_0+D_0\ln r). \tag{8′}$$

The solution must be a periodic function of φ, since for one and the same value of r for φ and $\varphi+2\pi$ we must have the same solution, because one and the same point of the circle is considered. It is therefore obvious that in formula (8′) we must have $B_0=0$. To continue, we seek a solution that is continuous and finite in the circle. Hence, in the centre of the circle the solution must be final for $r=0$, and for that reason we must have $D_0=0$ in (8′) and $D_k=0$ in (8).

Thus, the right side of (8′) becomes the product A_0C_0, which we denote by $A_0/2$. Thus,

$$u_0=\frac{A_0}{2}. \tag{8″}$$

We shall form the solution to our problem as a sum of solutions of the form (8), since a sum of solutions is a solution. The sum must be a periodic function of φ. This will be the case if each term is a periodic function of φ. For this, k must take on integral values. [We note that if we equated the sides of (4) to the number $+k^2$, we would not obtain a periodic solution.] We shall confine ourselves only to positive values:

$$k=1,\ 2,\ \ldots,\ n,\ \ldots$$

because the constants A, B, C, D are arbitrary and therefore the negative values of k do not yield new particular solutions.

Thus,

$$u(r,\ \varphi)=\frac{A_0}{2}+\sum_{n=1}^{\infty}(A_n\cos n\varphi+B_n\sin n\varphi)\,r^n \tag{9}$$

(the constant C_n is included in A_n and B_n). Let us now choose arbitrary constants A_n and B_n so as to satisfy the boundary-value condition (2). Putting into (9) $r=R$, we get, from condition (2),

$$f(\varphi)=\frac{A_0}{2}+\sum_{n=1}^{\infty}(A_n\cos n\varphi+B_n\sin n\varphi)\,R^n. \tag{10}$$

For us to have equality (10), it is necessary that the function should be expandable in a Fourier series in the interval $(-\pi,\ \pi)$ and that A_nR^n and B_nR^n should be its Fourier coefficients. Hence,

A_n and B_n must be defined by the formulas

$$\left.\begin{aligned} A_n &= \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(t)\cos nt\,dt, \\ B_n &= \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(t)\sin nt\,dt. \end{aligned}\right\} \quad (11)$$

Thus, the series (9) with coefficients defined by formulas (11) will be the solution of our problem if it admits termwise iterated differentiation with respect to r and φ (but we have not proved this). Let us transform formula (9). Putting, in place of A_n and B_n, their expressions (11) and performing the trigonometric transformations, we get

$$u(r, \varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(t)\cos n(t-\varphi)\,dt\left(\frac{r}{R}\right)^n =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos n(t-\varphi)\right]dt. \quad (12)$$

Let us transform the expression in the square brackets: *

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos n(t-\varphi) = 1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\left[e^{in(t-\varphi)} + e^{-in(t-\varphi)}\right] =$$

$$= 1 + \sum_{n=1}^{\infty}\left[\left(\frac{r}{R}e^{i(t-\varphi)}\right)^n + \left(\frac{r}{R}e^{-i(t-\varphi)}\right)^n\right] =$$

$$= 1 + \frac{\frac{r}{R}e^{i(t-\varphi)}}{1 - \frac{r}{R}e^{i(t-\varphi)}} + \frac{\frac{r}{R}e^{-i(t-\varphi)}}{1 - \frac{r}{R}e^{-i(t-\varphi)}} =$$

$$= \frac{1 - \left(\frac{r}{R}\right)^2}{1 - 2\frac{r}{R}\cos(t-\varphi) + \left(\frac{r}{R}\right)^2} = \frac{R^2 - r^2}{R^2 - 2Rr\cos(t-\varphi) + r^2}. \quad (13)$$

---

*) In the derivation we determine the sum of an infinite geometric progression whose ratio is a complex number the modulus of which is less than unity. This formula of the sum of a geometric progression is derived in the same way as in the case of real numbers. It is also necessary to take into account the definition of the limit of the complex function of a real argument. Here, the argument is n (see Sec. 4, Ch. VII).

Replacing the expression in square brackets in (12) by expression (13), we get

$$u(r, \varphi)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{R^2-r^2}{R^2-2rR\cos(t-\varphi)+r^2}dt. \tag{14}$$

Formula (14) is called *Poisson's integral*. By an analysis of this formula it is possible to prove that if the function $f(\varphi)$ is continuous, then the function $u(r, \varphi)$ defined by the integral (14) also satisfies equation (1') and $u(r, \varphi)\to f(\varphi)$ as $r\to R$. That is, it is a solution of the Dirichlet problem for a circle.

SEC. 11. SOLUTION OF THE DIRICHLET PROBLEM BY THE METHOD OF FINITE DIFFERENCES

In an xy-plane, let there be given a region D bounded by a contour C. Let there be given a continuous function f on the contour C. It is required to find an approximate solution to the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

that satisfies the boundary condition

$$u|_C=f. \tag{2}$$

We draw two families of straight lines:

$$x=ih \text{ and } y=kh, \tag{3}$$

where h is the given number, and i and k assume successive integral values. We shall say that the region D is covered with a *grid*. We call the points of intersection of the straight lines *nodes of the grid*.

We denote by $u_{i,k}$ the approximate value of the desired function at the point $x=ih$, $y=kh$; that is, $u(ih, kh)=u_{i,k}$. We approximate the region D by the grid region D^*, which consists of all the squares that lie completely in D and of some that are crossed by the boundary C (these may be disregarded). Here, the contour C is approximated by the contour C^*, which consists of segments of straight lines of type (3). In each node lying on the contour C^* we specify the value f^*, which is equal to the value of the function f at the closest point of the contour C (Fig. 377).

The values of the desired function will be considered only at the nodes of the grid. As has already been pointed out in Sec. 6,

the derivatives in this approximate method are replaced by finite differences:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=ih,\ y=kh}=\frac{u_{i+1,\,k}-2u_{i,\,k}+u_{i-1,\,k}}{h^2},$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{x=ih,\ y=kh}=\frac{u_{i,\,k+1}-2u_{i,\,k}+u_{i,\,k-1}}{h^2}.$$

The differential equation (1) is replaced by a *difference equation* (after cancelling out h^2):

$$u_{i+1,\,k}-2u_{i,\,k}+u_{i-1,\,k}+u_{i,\,k+1}-2u_{i,\,k}+u_{i,\,k-1}=0$$

or (Fig. 378)

$$u_{i,\,k}=\frac{1}{4}(u_{i+1,\,k}+u_{i,\,k+1}+u_{i-1,\,k}+u_{i,\,k-1}). \tag{4}$$

For each node of the grid lying inside D^* (and not lying on the boundary C^*), we form an equation (4). If the point $(x=ih,\ y=kh)$ is adjacent to the point of the contour C^*, then the right side of (4) will contain known values of f^*. Thus, we obtain a nonhomogeneous system of N equations in N unknowns, where N is the number of nodes of the grid lying inside the region D^*.

We shall prove that the system (4) has one, and only one, solution. This is a system of N linear equations in N unknowns. It has a unique solution if the determinant of the system is not zero. The determinant of the system is nonzero if the homogeneous system has only a trivial solution. The system will be homogeneous if $f^*=0$ at the nodes on the boundary of the contour C^*. We shall prove that in this case all the values $u_{i,\,k}$ at all interior nodes of the grid are equal to zero. Inside the region, let there be $u_{i,\,k}$ different from zero. For the sake of definiteness, we suppose that the greatest of them is positive. Let us designate it by $\bar{u}_{i,\,k}>0$.

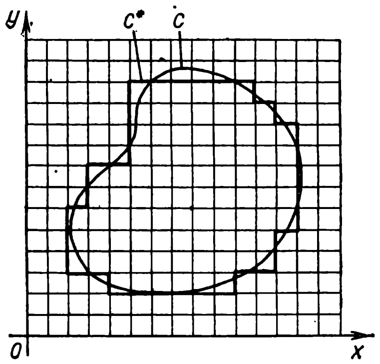


Fig. 377.

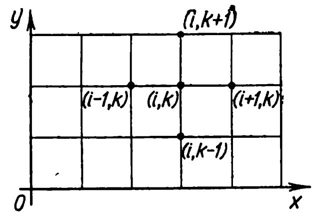


Fig. 378.

By (4) we write

$$\bar{u}_{i,k} = \frac{1}{4}(u_{i+1,k} + u_{i,k+1} + u_{i-1,k} + u_{i,k-1}). \qquad (4')$$

This equation is possible only if all the values of u on the right are equal to the greatest $\bar{u}_{i,k}$. We now have five points at which the values of the desired function are $\bar{u}_{i,k}$. If none of these points is a boundary point, then, taking one of them and writing for it the equation (4), we will prove that at certain other points the value of the desired function will be equal to $\bar{u}_{i,k}$. Continuing in this fashion, we will reach the boundary and will prove that at the boundary point the value of the function will be equal to $\bar{u}_{i,k}$, which is contrary to the fact that $f^* = 0$ at boundary points.

Assuming that inside the region there is a least negative value, we will prove that on the boundary the value of the function is negative, which contradicts the hypothesis.

And so system (4) has a solution which is unique.

The values $u_{i,k}$ *defined from the system (4) are approximate values of the solution of the Dirichlet problem formulated above.* It was proved that if the solution of the Dirichlet problem for a given region D and a given function f exists [we denote it by $u(x, y)$] and if $u_{i,k}$ is the solution of (4), then we have the relation

$$|u(x, y) - u_{i,k}| < Ah^2 \qquad (5)$$

where A is a constant independent of h.

Note. It is sometimes justified (though this has not been rigorously proved) to use the following procedure for evaluating the error of the approximate solution. Let $u_{i,k}^{(2h)}$ be an approximate solution for a step $2h$, $u_{i,k}^{(h)}$ an approximate solution for a step h, and let $E_h(x, y)$ be the error of the solution $u_{i,k}^{(h)}$. Then we have the approximate equality

$$E_h(x, y) \approx \frac{1}{3}(u_{i,k}^{(2h)} - u_{i,k}^{(h)})$$

in the common nodes of the grids. Thus, in order to determine the error of the approximate solution for a step h, it is necessary to find the solution for a step $2h$. One third of the difference of these approximate solutions is the error evaluation of the solution for a step (mesh-length) of h. This remark also refers to the solution of the heat-conduction equation by the finite-difference method.

Exercises on Chapter XVIII

1. Derive an equation of torsional oscillations of a homogeneous cylindrical rod.

Hint. The torque in a cross section of the rod with abscissa x is determined by the formula $M = GI\frac{\partial\theta}{\partial x}$, where $\theta(x, t)$ is the angle of torque of a cross section with abscissa x at time t, G is the shear modulus, and I is the polar moment of inertia of a cross section of the rod.

Ans. $\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2}$, where $a^2 = \frac{GI}{k}$, and k is the moment of inertia of unit length of the rod.

2. Find a solution of the equation $\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2}$ that satisfies the conditions $\theta(0, t) = 0$, $\theta(l, t) = 0$, $\theta(x, 0) = \varphi(x)$, $\frac{\partial\theta(x, 0)}{\partial t} = 0$, where

$$\varphi(x) = \frac{2\theta_0 x}{l} \qquad \text{for } 0 \leqslant x \leqslant \frac{l}{2},$$

$$\varphi(x) = -\frac{2\theta_0 x}{l} + 2\theta_0 \quad \text{for } \frac{l}{2} \leqslant x \leqslant l.$$

Give a mechanical interpretation of the problem.

$$Ans.\ \theta(x, t) = \frac{8\theta_0}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\frac{(2k+1)\pi x}{l}\cos\frac{(2k+1)\pi at}{l}.$$

3. Derive an equation of longitudinal oscillations of a homogeneous cylindrical rod.

Hint. If $u(x, t)$ is the translation of a cross section of rod with abscissa x at time t, then the tensile stress T in a cross section x is defined by the formula $T = ES\frac{\partial u}{\partial x}$, where E is the elasticity modulus of the material and S is the cross-sectional area of the rod.

Ans. $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$, where $a^2 = \frac{E}{\varrho}$, and ϱ is the density of the rod material.

4. A homogeneous rod of length $2l$ was shorted by 2λ under the action of forces applied to its ends. At $t = 0$ it is free of forces acting externally. Determine the displacement $u(x, t)$ of a cross section of the rod with abscissa x at time t (the mid-point of the axis of the rod has abscissa $x = 0$).

$$Ans.\ u(x, t) = \frac{8\lambda}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)^2}\sin\frac{(2k+1)\pi x}{2l}\cos\frac{(2k+1)\pi at}{2l}.$$

5. One end of a rod of length l is fixed, the other end is acted upon by a tensile force P. Find the longitudinal oscillations of the rod if the force P does not operate when $t = 0$. *Ans.* $\frac{8Pl}{ES\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2l}\cos\frac{(2n+1)\pi at}{2l}$ (E and S as in Problem 3).

6. Find a solution to the equation $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$ that satisfies the conditions

$$u(0,\ t)=0,\quad u(l,\ t)=A\sin\omega t,$$
$$u(x,\ 0)=0,\quad \frac{\partial u(x,\ 0)}{\partial t}=0.$$

Give a mechanical interpretation of the problem.

Ans. $u(x,\ t)=\dfrac{A\sin\frac{\omega}{a}x\sin\omega t}{\sin\frac{\omega}{a}l}+\dfrac{2A\omega a}{l}\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{\omega^2-\left(\frac{n\pi a}{l}\right)^2}\sin\dfrac{n\pi at}{l}\sin\dfrac{n\pi x}{l}$.

Hint. Seek the solution in the form of a sum of two solutions:

$$u=v+w,\qquad \text{where } w=\frac{A\sin\frac{\omega}{a}x\sin\omega t}{\sin\frac{\omega}{a}l}$$

is the solution that satisfies the conditions

$$v(0,\ t)=0,\quad v(l,\ t)=0$$
$$v(x,\ 0)=-w(x,\ 0),\quad \frac{\partial v(x,\ 0)}{\partial t}=-\frac{\partial w(x,\ 0)}{\partial t}.$$

$\left(\text{It is assumed that } \sin\frac{\omega}{a}l\neq 0.\right)$

7. Find a solution to the equation $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ that satisfies the conditions

$$u(0,\ t)=0,\quad u(l,\ t)=0,\quad t>0,$$
$$u(x,\ 0)=\begin{cases} x & \text{when } 0\leqslant x\leqslant\frac{l}{2}\\ l-x & \text{when } \frac{l}{2}<x<l.\end{cases}$$

Ans. $h(x,\ t)=\dfrac{4l}{\pi^2}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)^2}e^{-\frac{(2n+1)^2\pi^2a^2}{l^2}t}\sin\dfrac{(2n+1)\pi x}{l}$.

Hint. Solve the problem by the method of separation of variables.

8. Find a solution to the equation $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ that satisfies the conditions

$$u(0,\ t)=u(0,\ l)=0,\ u(x,\ 0)=\frac{x(l-x)}{l^2}.$$

Ans. $u(x,\ t)=\dfrac{8}{\pi^3}\sum_{n=0}^{8}\dfrac{1}{(2n+1)^3}e^{-\frac{(2n+1)^2\pi^2a^2t}{l^2}}\sin\dfrac{(2n+1)\pi x}{l}$.

9. Find a solution to the equation $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ that satisfies the conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0,\quad u(l,\ t)=u_0,\quad u(x,\ 0)=\varphi(x).$$

Point out the physical meaning of the problem.

Ans. $u(x,\ t)=u_0+\sum_{n=0}^{\infty}A_n e^{-a^2\lambda_n^2 t}\cos\frac{(2n+1)\pi}{2l}x,$

where $A_n=\frac{2}{l}\int_0^l \varphi(x)\cos\frac{(2n+1)\pi x}{2l}dx-\frac{(-1)^n 4u_0}{\pi(2n+1)}.$

Hint. Seek the solution in the form $u=u_0+v(x,\ t)$.

10. Find a solution to the equation $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$ that satisfies the conditions

$$u(0,\ t)=0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=l}=-\left.Hu\right|_{x=l},\quad u(x,\ 0)=\varphi(x).$$

Point out the physical meaning of the problem.

Ans. $u(x,\ t)=\sum_{n=1}^{\infty}A_n\frac{p^2+\mu_n^2}{p(p+1)+\mu_n^2}e^{-\frac{\mu_n^2 a^2 t^2}{l^2}}\sin\frac{\mu_n x}{l},$

where $A_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{\mu_n x}{l}dx,\quad p=Hl,\quad \mu_1,\ \mu_2,\ \ldots,\ \mu_n$ are positive roots of the equation $\tan\mu=-\frac{\mu}{p}$.

Hint. At the end of the rod (when $x=l$) a heat exchange occurs with the environment, which has a temperature of zero.

11. Find [by formula (10), Sec. 6, putting $h=0.2$] an approximate solution to the equation $\frac{\partial u}{\partial t}=2\frac{\partial^2 u}{\partial x^2}$ that satisfies the conditions

$$u(x,\ 0)=x\left(\frac{3}{2}-x\right),\quad u(0,\ t)=0,\quad u(1,\ t)=\frac{1}{2},\quad 0\leqslant t\leqslant 4l.$$

12. Find a solution to the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, in a strip $0\leqslant x\leqslant a,\ 0\leqslant y<\infty$ that satisfies the conditions

$$u(0,\ y)=0,\quad u(a,\ y)=0,\quad u(x,\ 0)=A\left(1-\frac{x}{a}\right),\quad u(x,\ \infty)=0.$$

Ans. $u(x,\ t)=\frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\frac{n\pi}{a}y}\sin\frac{n\pi x}{a}.$

Hint. Use the method of the separation of variables.

13. Find a solution to the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ in the rectangle $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$ that satisfies the conditions

$$u(x,\ 0)=0,\quad u(x,\ b)=0,\quad u(0,\ y)=Ay(b-y),\quad u(a,\ y)=0.$$

Ans. $u(x,\ t)=\frac{8Ab^2}{\pi^3}\sum_{n=0}^{\infty}\frac{\sin h\frac{(2n+1)\pi(a-x)}{b}}{(2n+1)^3}\frac{\sin\frac{(2n+1)\pi y}{b}}{\sin h\frac{(2n+1)\pi a}{b}}$.

14. Find a solution to the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ inside a ring bounded by the circles $x^2+y^2=R_1^2$, $x^2+y^2=R_2^2$ that satisfies the conditions

$$\left.\frac{\partial u}{\partial r}\right|_{r=R_1}=+\frac{Q}{\lambda 2\pi R_1},\quad \left.u\right|_{r=R_2}=u_2.$$

Give a hydrodynamic interpretation of the problem.

Hint. Solve the problem in polar coordinates.

Ans. $u=u_2-\frac{Q}{2\lambda\pi}\ln\frac{R_2}{r}$.

15. The function $u(x,\ y)=e^{-y}\sin x$ is a solution of the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ in the square $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$ that satisfies the conditions

$$u(0,\ y)=0,\quad u(1,\ y)=e^{-y}\sin 1,\quad u(x,\ 0)=\sin x,\quad u(x,\ 1)=e^{-1}\sin x.$$

In Problems 12-15 solve the Laplace equations for given boundary conditions by the finite-difference method for $h=0.25$. Compare the approximate solution with the exact solution.

CHAPTER XIX

OPERATIONAL CALCULUS AND CERTAIN OF ITS APPLICATIONS

Operational calculus is an important branch of mathematical analysis. The methods of operational calculus are used in physics, mechanics, electrical engineering and elsewhere. Operational calculus finds especially broad applications in automation and telemechanics. In this chapter we give (on the basis of the foregoing material of this text) the fundamental concepts of operational calculus and operational methods of solving ordinary differential equations.

SEC. 1. THE INITIAL FUNCTION AND ITS TRANSFORM

Let there be given the function of a real variable t defined for $t \geqslant 0$ [we shall sometimes consider that the function $f(t)$ is defined on an infinite interval $-\infty < t < \infty$, but $f(t)=0$ when $t<0$]. We shall assume that the function $f(t)$ is piecewise continuous, that is, such that in any finite interval it has a finite number of discontinuities of the first kind (see Sec. 9, Ch. II). To ensure the existence of certain integrals in the infinite interval $0 \leqslant t < \infty$ we impose an additional restriction on the function $f(t)$: namely, we suppose that there exist constant positive numbers M and s_0 such that

$$|f(t)| < Me^{s_0 t} \tag{1}$$

for any value t in the interval $0 \leqslant t < \infty$.

Let us consider the product of the function $f(t)$ by the complex function e^{-pt} of a real variable *) t, where $p=a+ib$ is some complex number:

$$e^{-pt} f(t). \tag{2}$$

Function (2) is also a complex function of a real variable t:

$$e^{-pt} f(t) = e^{-(a+ib)t} f(t) = e^{-at} f(t) e^{-ibt} = e^{-at} f(t) \cos bt - ie^{-at} f(t) \sin bt.$$

Let us further consider the improper integral

$$\int_0^\infty e^{-pt} f(t)\, dt = \int_0^\infty e^{-at} f(t) \cos bt\, dt - i \int_0^\infty e^{-at} f(t) \sin bt\, dt. \tag{3}$$

We shall show that if the function $f(t)$ satisfies condition (1) and $a > s_0$, then the integrals on the right of (3) exist and the convergence of the integrals is absolute. Let us begin by evaluating the first of these integrals:

$$\left|\int_0^\infty e^{-at} f(t) \cos bt\, dt\right| \leqslant \int_0^\infty \left|e^{-at} f(t) \cos bt\right| dt <$$

$$< M \int_0^\infty e^{-at} e^{s_0 t}\, dt < M \int_0^\infty e^{-(a-s_0)t}\, dt = \frac{M}{a-s_0}.$$

*) See Sec. 4, Ch. VII, concerning complex functions of a real variable.

In similar fashion we evaluate the second integral. Thus, the integral $\int_0^\infty e^{-pt} f(t)\,dt$ exists. It defines a certain function of p, which we denote *) by $F(p)$:

$$F(p)=\int_0^\infty e^{-pt} f(t)\,dt. \qquad (4)$$

The function $F(p)$ is called the *Laplace transform*, or the *L-transform*, or simply the *transform* of the function $f(t)$. The function $f(t)$ is known as the *initial function*, or the *original*. If $F(p)$ is the transform of $f(t)$, then we write

$$F(p) \div f(t), \qquad (5)$$

or

$$f(t) \div F(p), \qquad (6)$$

or

$$L\{f(t)\}=F(p). \qquad (7)$$

As we shall presently see, the meaning of transforms consists in the fact that with their help it is possible to simplify the solution of many problems, for instance, to reduce the solution of differential equations to simple algebraic operations in finding a transform. Knowing the transform one can find the original either from specially prepared "original-transform" tables or by methods that will be given below. Certain natural questions arise.

Let there be given a certain function $F(p)$. Does there exist a function $f(t)$ for which $F(p)$ is a transform? If there does, then is this function the only one? The answer is yes to both questions, given certain definite assumptions with respect to $F(p)$ and $f(t)$. For example, the following theorem, which we give without proof, establishes that the transform is unique:

Uniqueness Theorem. *If two continuous functions $\varphi(t)$ and $\psi(t)$ have one and the same L-transform $F(p)$, then these functions are identically equal.*

This theorem will play an important role throughout the subsequent text. Indeed, if in the solution of some practical problem we have determined, in some way, the transform of a desired function, and from the transform the original function, then on the basis of the foregoing theorem we conclude that the function we have found is the solution of the given problem and that no other solutions exist.

SEC. 2. TRANSFORMS OF THE FUNCTIONS $\sigma_0(t)$, SIN t, COS t

1. The function $f(t)$, defined as

$$f(t)=1 \text{ for } t\geqslant 0,$$
$$f(t)=0 \text{ for } t<0,$$

is called the *Heaviside unit function* and is denoted by $\sigma_0(t)$. The graph of this function is given in Fig. 379. Let us find the L-transform of the Heaviside function:

$$L\{\sigma_0(t)\}=\int_0^\infty e^{-pt}dt=-\frac{e^{-pt}}{p}\Big|_0^\infty=\frac{1}{p}.$$

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*) The function $F(p)$, for $p\neq 0$, is the function of a complex variable (for example, see V. I. Smirnov's "Course of Higher Mathematics", Vol. III, Part 2) (Russian edition).

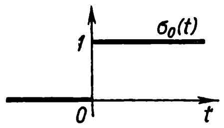


Fig. 379.

Thus, *)

$$1 \div \frac{1}{p} \tag{8}$$

or, more precisely,

$$\sigma_0(t) \div \frac{1}{p}.$$

In some books on operational calculus the following expression is called the transform of the function $f(t)$:

$$F^*(p) = p\int_0^\infty e^{-pt} f(t)\,dt.$$

With this definition we have $\sigma_0(t) \div 1$ and, consequently, $C \div C$, more exactly, $C\sigma_0(t) \div C$.

II. Let $f(t) = \sin t$; then

$$L\{\sin t\} = \int_0^\infty e^{-pt} \sin t\,dt = \frac{e^{-pt}(-p\sin x - \cos x)}{p^2+1}\Bigg|_0^\infty = \frac{1}{p^2+1}.$$

And so

$$\sin t \div \frac{1}{p^2+1}. \tag{9}$$

III. Let $f(t) = \cos t$; then

$$L\{\cos t\} = \int_0^\infty e^{-pt} \cos t\,dt = \frac{e^{-pt}(t\sin t - p\cos t)}{p^2+1}\Bigg|_0^\infty = \frac{p}{p^2+1}.$$

And so

$$\cos t \div \frac{p}{p^2+1}. \tag{10}$$

SEC. 3. THE TRANSFORM OF A FUNCTION WITH CHANGED SCALE OF THE INDEPENDENT VARIABLE. TRANSFORMS OF THE FUNCTIONS SIN*at*, COS *at*

Let us consider the transform of the function $f(at)$, where $a > 0$:

$$L\{f(at)\} = \int_0^\infty e^{-pt} f(at)\,dt.$$

We change the variable in the latter integral, putting $z = at$; hence, $dz = a\,dt$; then we get

$$L\{f(at)\} = \frac{1}{a}\int_0^\infty e^{-\frac{p}{a}z} f(z)\,dz$$

---

*) In computing the integral $\int_0^\infty e^{-pt}dt$ one might represent it as the sum o integrals of real functions; the same result would be obtained. This also holds for the two subsequent integrals.

or

$$L\{f(at)\}=\frac{1}{a}F\left(\frac{p}{a}\right).$$

Thus, *if*

$$F(p) \risingdotseq f(t)$$

then

$$\frac{1}{a}F\left(\frac{p}{a}\right) \risingdotseq f(at). \tag{11}$$

Example 1. From (9), by (11), we straightway get

$$\sin at \risingdotseq \frac{1}{a}\,\frac{1}{\left(\frac{p}{a}\right)^2+1}$$

or

$$\sin at \risingdotseq \frac{a}{p^2+a^2}. \tag{12}$$

Example 2. From (10), by (11), we obtain

$$\cos at \risingdotseq \frac{1}{a}\,\frac{\frac{p}{a}}{\left(\frac{p}{a}\right)^2+1}$$

or

$$\cos at \risingdotseq \frac{p}{p^2+a^2}. \tag{13}$$

SEC. 4. THE LINEARITY PROPERTY OF A TRANSFORM

Theorem. *The transform of a sum of several functions multiplied by constants is equal to the sum of the transforms of these functions multiplied by the corresponding constants, that is, if*

$$f(t)=\sum_{i=1}^{n} C_i f_i(t) \tag{14}$$

(C_i are constants) *and*

$$F(p) \risingdotseq f(t), \quad F_i(p) \risingdotseq f_i(t),$$

then

$$F(p)=\sum_{i=1}^{n} C_i F_i(p). \tag{14'}$$

Proof. Multiplying all the terms of (14) by e^{-pt} and integrating with respect to t from 0 to ∞ (taking the factors C_i outside the integral sign), we get (14').

Example 1. Find the transform of the function

$$f(t)=3\sin 4t-2\cos 5t.$$

Solution. Applying formulas (12), (13), and (15), we have

$$L\{f(t)\}=3\frac{4}{p^2+16}-2\frac{p}{p^2+25}=\frac{12}{p^2+16}-\frac{2p}{p^2+25}.$$

Example 2. Find the original function whose transform is expressed by the formula

$$F(p)=\frac{5}{p^2+4}-\frac{20p}{p^2+9}.$$

Solution. We represent $F(p)$ as

$$F(p)=\frac{5}{2}\,\frac{2}{p^2+(2)^2}+20\,\frac{p}{p^2+(3)^2}.$$

Hence, by (12), (13), and (14′) we have

$$f(t)=\frac{5}{2}\sin 2t+20\cos 3t.$$

From the uniqueness theorem, Sec. 1, it follows that this is the only original function that corresponds to the given $F(p)$.

SEC. 5. THE SHIFT THEOREM

Theorem. *If $F(p)$ is the transform of the function $f(t)$, then $F(p+\alpha)$ is the transform of the function $e^{-\alpha t}f(t)$, that is,*

$$\left.\begin{array}{l} \textit{if } F(p) \risingdotseq f(t) \\ \textit{then } F(p+\alpha)\risingdotseq e^{-\alpha t}f(t). \end{array}\right\} \tag{15}$$

[It is assumed here that $Re\,(p+\alpha)>s_0$.]

Proof. Find the transform of the function $e^{-\alpha t}f(t)$:

$$L\left\{e^{-\alpha t}f(t)\right\}=\int\limits_0^\infty e^{-pt-\alpha t}f(t)\,dt=\int\limits_0^\infty e^{-(p+\alpha)\,t}f(t)\,dt.$$

Thus,

$$L\left\{e^{-\alpha t}f(t)\right\}=F(p+\alpha).$$

This theorem makes it possible to expand considerably the class of transforms for which it is easy to find the original functions.

SEC. 6. TRANSFORMS OF THE FUNCTIONS $e^{-\alpha t}$, SINH αt, COSH αt, $e^{-\alpha t}$ SIN at, $e^{-\alpha t}$ COS at

From (8), on the basis of (15), we straightway get

$$\frac{1}{p+\alpha}\risingdotseq e^{-\alpha t}. \tag{16}$$

Similarly,

$$\frac{1}{p-\alpha}\risingdotseq e^{\alpha t}. \tag{16′}$$

Substracting from the terms of (16′) the corresponding terms of (16) and dividing the results by two, we get

$$\frac{1}{2}\left(\frac{1}{p-\alpha}-\frac{1}{p+\alpha}\right)\risingdotseq\frac{1}{2}\left(e^{\alpha t}-e^{-\alpha t}\right)$$

or

$$\frac{\alpha}{p^2-\alpha^2} \dot{\rightarrow} \sinh \alpha t. \tag{17}$$

Similarly, by adding (16) and (16′), we obtain

$$\frac{p}{p^2-\alpha^2} \dot{\rightarrow} \cosh \alpha t. \tag{18}$$

From (12), by (15), we have

$$\frac{a}{(p+\alpha)^2+a^2} \dot{\rightarrow} e^{-\alpha t} \sin at. \tag{19}$$

Using formula (15) we get from (13)

$$\frac{p+\alpha}{(p+a)^2+a^2} \dot{\rightarrow} e^{-\alpha t} \cos at. \tag{20}$$

Example 1. Find the original function whose transform is given by the formula

$$F(p)=\frac{7}{p^2+10p+20}.$$

Solution. Transform $F(p)$ to the form of expression on the left-hand side of (19):

$$\frac{7}{p^2+10p+41}=\frac{7}{(p+5)^2+16}=\frac{7}{4}\,\frac{4}{(p+5)^2+4^2}.$$

Thus

$$F(p)=\frac{7}{4}\,\frac{4}{(p+5)^2+4^2}.$$

Hence, by formula (19) we will have

$$F(p) \dot{\rightarrow} \frac{7}{4} e^{-5t} \sin 4t.$$

Example 2. Find the original function whose transform is given by the formula

$$F(p)=\frac{p+3}{p^2+2p+10}.$$

Solution. Transform the function $F(p)$:

$$\frac{p+3}{p^2+2p+10}=\frac{(p+1)+2}{(p+1)^2+9}=\frac{p+1}{(p+1)^2+3^2}+\frac{2}{(p+1)^2+3^2}=$$

$$=\frac{p+1}{(p+1)^2+3^2}+\frac{2}{3}\,\frac{3}{(p+1)^2+3^2},$$

using formulas (19) and (20) we find the original function:

$$F(p) \dot{\rightarrow} e^{-t}\cos 3t+\frac{2}{3}e^{-t}\sin 3t.$$

SEC. 7. DIFFERENTIATION OF TRANSFORMS

Theorem. *If* $F(p) \div f(t)$, *then*

$$(-1)^n \frac{d^n}{dp^n} F(p) \div t^n f(t). \tag{21}$$

Proof. We first prove that if $f(t)$ satisfies condition (1), then the integral

$$\int_0^\infty e^{-pt} (-t)^n f(t)\, dt \tag{22}$$

exists.

By hypothesis $|f(t)| < Me^{s_0 t}$, $p = a + ib$, $a > s_0$; and $a > 0$; $s_0 > 0$. Obviously, there will be an $\varepsilon > 0$ such that the inequality $a < s_0 + \varepsilon$ will be fulfilled. As in Sec. 1, it is proved that the following integral exists:

$$\int_0^\infty e^{-(p-\varepsilon)t} |f(t)|\, dt.$$

We then evaluate the integral (22)

$$\int_0^\infty \left| e^{-pt} t^n f(t) \right| dt = \int_0^\infty \left| e^{-(p-\varepsilon)t} e^{-\varepsilon t} t^n f(t) \right| dt.$$

Since the function $e^{-\varepsilon t} t^n$ is bounded and, in absolute value, is less than some number N for any value $t > 0$, we can write

$$\int_0^\infty \left| e^{-pt} t^n f(t) \right| dt < N \int_0^\infty \left| e^{-(p-\varepsilon)t} f(t) \right| dt = N \int_0^\infty e^{-(p-\varepsilon)t} \left| f(t) \right| dt < \infty.$$

It is thus proved that the integral (22) exists. But this integral may be regarded as an nth-order derivative with respect to the parameter *) p of the integral

$$\int_0^\infty e^{-pt} f(t)\, dt.$$

And so, from formula

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt$$

we get the formula

$$\int_0^\infty e^{-pt} (-t)^n f(t)\, dt = \frac{d^n}{dp^n} \int_0^\infty e^{-pt} f(t)\, dt.$$

---

*) Earlier we found a formula for differentiating a definite integral with respect to a real parameter (see Sec. 10, Ch. XI). Here, the parameter p is a complex number, but the differentiation formula holds true.

From these two equations we have

$$(-1)^n \frac{d^n}{dp^n} F(p) = \int_0^\infty e^{-pt} t^n f(t),$$

which is formula (21).

Let us use (22) to find the transform of a power function. We write the formula (8):

$$\frac{1}{p} \doteqdot 1.$$

Using formula (21), from this formula we get

$$(-1) \frac{d}{dp} \left(\frac{1}{p} \right) \doteqdot t$$

or

$$\frac{1}{p^2} \doteqdot t.$$

Similarly

$$\frac{2}{p^3} \doteqdot t^2.$$

For any n we have

$$\frac{n!}{p^{n+1}} \doteqdot t^n. \tag{23}$$

Example 1. From the formula [see (12)]

$$\frac{a}{p^2+a^2} = \int_0^\infty e^{-pt} \sin at \, dt,$$

by differentiating the left and right sides with respect to the parameter p, we get

$$\frac{2pa}{(p^2+a^2)^2} \doteqdot t \sin at. \tag{24}$$

Example 2. From (13), on the basis of (21), we have

$$-\frac{a^2-p^2}{(p^2+a^2)^2} \doteqdot t \cos at. \tag{25}$$

Example 3. From (16), by (12), we have

$$\frac{1}{(p+\alpha)^2} \doteqdot te^{-\alpha t}. \tag{26}$$

SEC. 8. THE TRANSFORMS OF DERIVATIVES

Theorem. *If* $F(p) \doteqdot f(t)$, *then*

$$pF(p) - f(0) \doteqdot f'(t). \tag{27}$$

Proof. From the definition of a transform we can write

$$L\{f'(t)\} = \int_0^\infty e^{-pt} f'(t) \, dt. \tag{28}$$

We shall assume that all the derivatives $f'(t)$, $f''(t)$, ..., $f^{(n)}(t)$ which we encounter satisfy the condition (1), and, consequently, the integral (28) and similar integrals for subsequent derivatives exist. Computing by parts the integral on the right of (28), we find

$$L\{f'(t)\}=\int_0^\infty e^{-pt}f'(t)\,dt=e^{-pt}f(t)\Big|_0^\infty+p\int_0^\infty e^{-pt}f(t)\,dt.$$

But by condition (1)

$$\lim_{t\to\infty} e^{-pt}f(t)=0$$

and

$$\int_0^\infty e^{-pt}f(t)\,dt=F(p).$$

Therefore

$$L\{f'(t)\}=-f(0)+pF(p).$$

The theorem is proved. Let us now consider the transforms of derivatives of any order. Substituting into (27) the expression $pF(p)-f(0)$ in place of $F(p)$ and the expression $f'(t)$ in place of $f(t)$, we get

$$p[pF(p)-f(0)]-f'(0)\,\dot{\div}\,f''(t)$$

or, removing brackets,

$$p^2F(p)-pf(0)-f'(0)\,\dot{\div}\,f''(t). \qquad (29)$$

The transform for a derivative of order n will be

$$p^nF(p)-[p^{n-1}f(0)+p^{n-2}f'(0)+\ldots+pf^{(n-2)}(0)+f^{(n-1)}(0)]\,\dot{\div}\,f^{(n)}(t). \qquad (30)$$

Note. Formulas (27), (29), and (30) are simplified if $f(0)=f'(0)=\ldots=\ldots=f^{(n-1)}(0)=0$. In this case we get

$$F(p)\,\dot{\div}\,f(t),$$
$$pF(p)\,\dot{\div}\,f'(t),$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$p^nF(p)\,\dot{\div}\,f^{(n)}(t).$$

SEC. 9. TABLE OF TRANSFORMS

For convenience, the transforms which we obtained are here given in the form of a table.

Note. Formulas 13 and 15 of this table will be derived later on.

Note. If for the transform of the function $f(t)$ we take

$$F^*(p)=p\int_0^\infty e^{-pt}f(t)\,dt,$$

then in the formulas 1-13 of the table the expressions in the first column must be multiplied by p, and formulas 14 and 15 will take on the following form. Since $F^*(p)=pF(p)$, it follows that by substituting into the left side

Table 1

No.	$F(p) = \int_0^\infty e^{-pt} f(t)\,dt$	$f(t)$
1	$\frac{1}{p}$	1
2	$\frac{a}{p^2+a^2}$	$\sin at$
3	$\frac{p}{p^2+a^2}$	$\cos at$
4	$\frac{1}{p+\alpha}$	$e^{-\alpha t}$
5	$\frac{\alpha}{p^2-\alpha^2}$	$\sin h\, \alpha t$
6	$\frac{p}{p^2-\alpha^2}$	$\cos h\, \alpha t$
7	$\frac{a}{(p+\alpha)^2+a^2}$	$e^{-\alpha t} \sin at$
8	$\frac{p+\alpha}{(p+\alpha)^2+a^2}$	$e^{-\alpha t} \cos at$
9	$\frac{n!}{p^{n+1}}$	t^n
10	$\frac{2pa}{(p^2+a^2)^2}$	$t \sin at$
11	$-\frac{a^2-p^2}{(p^2+a^2)^2}$	$t \cos at$
12	$\frac{1}{(p+\alpha)^2}$	$te^{-\alpha t}$
13	$\frac{1}{(p^2+a^2)^2}$	$\frac{1}{2a^3}(\sin at - at \cos at)$
14	$(-1)^n \frac{d^n}{dp^n} F(p)$	$t^n f(t)$
15	$F_1(p) F_2(p)$	$\int_0^t f_1(\tau) f_2(t-\tau)\,d\tau$

of 14 the expression $\frac{F^*(p)}{p}$ in place of $F(p)$ and multiplying by p, we get

14'. $$(-1)^n p \frac{d^n}{dp^n}\left(\frac{F^*(p)}{p}\right) \div t^n f(t).$$

Substituting into the left side of 15

$$F_1(p)=\frac{F_1^*(p)}{p}, \quad F_2(p)=\frac{F_2^*(p)}{p}$$

and multiplying this product by p, we have

15′. $$\frac{1}{p}F_1^*(p)F_2^*(p)\risingdotseq\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau.$$

SEC. 10. AN AUXILIARY EQUATION FOR A GIVEN DIFFERENTIAL EQUATION

Suppose we have a linear differential equation of order n with constant coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$:

$$a_0\frac{d^nx}{dt^n}+a_1\frac{d^{n-1}x}{dt^{n-1}}+\ldots+a_{n-1}\frac{dx}{dt}+a_nx(t)=f(t). \tag{31}$$

It is required to find a solution of this equation $x=x(t)$ for $t\geqslant 0$ that satisfies the initial conditions

$$x(0)=x_0, \quad x'(0)=x_0', \ \ldots, \ x^{(n-1)}(0)=x_0^{(n-1)}. \tag{32}$$

Before, we used to solve this problem as follows: we found the general solution of equation (31) containing n arbitrary constants; then we determined the constants so that they should satisfy the initial conditions (32).

Here we give a simpler method of solving this problem using operational calculus. We seek the L-transform of the solution $x(t)$ of (31) satisfying the conditions (32). We designate this L-transform by $\bar{x}(p)$; thus, $\bar{x}(p)\risingdotseq x(t)$.

Let us suppose that there exist transforms of the solution of (31) and of its derivatives of order n (after finding the solution we can test the truth of this assumption). We multiply all terms of (31) by e^{-pt}, where $p=a+ib$, and integrate with respect to t from 0 to ∞:

$$a_0\int_0^\infty e^{-pt}\frac{d^nx}{dt^n}\,dt+a_1\int_0^\infty e^{-pt}\frac{d^{n-1}x}{dt^{n-1}}\,dt+\ldots+a_n\int_0^\infty e^{-pt}x(t)\,dt=\int_0^\infty e^{-pt}f(t)\,dt. \tag{33}$$

On the left-hand side of the equation are the L-transforms of the function $x(t)$ and its derivatives, on the right, the L-transform of the function $f(t)$, which we denote by $F(p)$. Hence, equation (33) may be rewritten as

$$a_0L\left\{\frac{d^nx}{dt^n}\right\}+a_1L\left\{\frac{d^{n-1}x}{dt^{n-1}}\right\}+\ldots+a_nL\{x(t)\}=L\{f(t)\}.$$

Substituting into this equation the expressions (27), (29), and (30) in place of the transforms of the function and of its derivatives, we get

$$\begin{aligned}a_0\{p^n\bar{x}(p)&-[p^{n-1}x_0+p^{n-2}x_0'+p^{n-3}x_0''+\ldots+x_0^{(n-1)}]\}+\\&+a_1\{p^{n-1}\bar{x}(p)-[p^{n-2}x_0+p^{n-3}x_0'+\ldots+x_0^{(n-2)}]\}+\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&+a_{n-1}\{p\bar{x}(p)-[x_0]\}+a_n\bar{x}(p)=F(p).\end{aligned} \tag{34}$$

Equation (34) is known as the *auxiliary equation*, or the *transform equation*. The unknown in this equation is the transform $\bar{x}(p)$, which is determined from it. Transform it leaving on the left the terms that contain $\bar{x}(p)$:

$$\begin{aligned}\bar{x}(p)[a_0p^n+a_1p^{n-1}+\ldots+a_{n-1}p+a_n]= \\ &=a_0[p^{n-1}x_0+p^{n-2}x_0'+\ldots+x_0^{(n-1)}]+ \\ &+a_1[p^{n-2}x_0+p^{n-3}x_0'+\ldots+x_0^{(n-2)}]+ \\ &\ldots\ldots\ldots\ldots\ldots \\ &+a_{n-2}[px_0+x_0']+ \\ &+a_{n-1}[x_0]+F(p).\end{aligned} \qquad (34')$$

The coefficient of $\bar{x}(p)$ on the left of (34′) is an nth-degree polynomial in p, which results when in place of the derivatives we put the corresponding degrees of p into the left-hand member of equation (31). We denote the polynomial by $\varphi_n(p)$:

$$\varphi_n(p)=a_0p^n+a_1p^{n-1}+\ldots+a_{n+1}p+a_n. \qquad (35)$$

The right-hand side of (34′) is formed as follows:
the coefficient a_{n-1} is multiplied by x_0,
the coefficient a_{n-2} is multiplied by px_0+x_0',
. . . .
the coefficient a_1 is multiplied by $p^{n-2}x_0+p^{n-3}x_0'+\ldots+x_0^{(n-2)}$,
the coefficient a_0 is multiplied by $p^{n-1}x_0+p^{n-2}x_0'+\ldots+x_0^{(n-1)}$.

All these products are combined. To this is also added the transform of the right side of the differential equation $F(p)$. All terms of the right side of (34′), with the exception of $F(p)$, form, after collecting like terms, a polynomial in p of degree $n-1$ with known coefficients. We denote it by $\psi_{n-1}(p)$. And so equation (34′) can be written as follows:

$$\bar{x}(p)\varphi_n(p)=\psi_{n-1}(p)+F(p).$$

From this equation we determine $\bar{x}(p)$:

$$\bar{x}(p)=\frac{\psi_{n-1}(p)}{\varphi_n(p)}+\frac{F(p)}{\varphi_n(p)}. \qquad (36)$$

Determined in this way, $\bar{x}(p)$ is the transform of the solution $x(t)$ of the equation (31), which solution satisfies the initial conditions (32). If we now find the function $x^*(t)$ whose transform is the function $\bar{x}(p)$ determined by equation (36), then by the uniqueness theorem formulated in Sec. 1 it will follow that $x^*(t)$ is the solution of equation (31) that satisfies the conditions (32), that is,

$$x^*(t)=x(t).$$

If we seek the solution of (31) for zero initial conditions: $x_0=x_0'=x_0''\ldots=x_0^{(n-1)}=0$, then in (36) we will have $\psi_{n-1}(p)=0$ and the equation will take the form

$$\bar{x}(p)=\frac{F(p)}{\varphi_n(p)}$$

or

$$\bar{x}(p)=\frac{F(p)}{a_0p^n+a_1p^{n-1}+\ldots+a_n}. \qquad (36')$$

Example 1. Find the solution of the equation

$$\frac{dx}{dt}+x=1$$

satisfying the conditions $x=0$ for $t=0$.

Solution. Form the auxiliary equation

$$\bar{x}(p)\ (p+1)-0=\frac{1}{p} \text{ or } \bar{x}(p)=\frac{1}{(p+1)\,p}.$$

Decomposing the fraction on the right into partial fractions, we get

$$\bar{x}(p)=\frac{1}{p}-\frac{1}{p+1}.$$

Using formulas 1 and 4 of Table 1, we find the solution:

$$x(t)=1-e^{-t}.$$

Example 2. Find the solution of the equation

$$\frac{d^2x}{dt^2}+9x=1$$

that satisfies the initial conditions: $x_0=x_0'=0$ for $t=0$.

Solution. Write the auxiliary equation (34′)

$$\bar{x}(p)\ (p^2+9)=\frac{1}{p} \text{ or } \bar{x}(p)=\frac{1}{p(p^2+9)}.$$

Decomposing this fraction into partial fractions, we get

$$\bar{x}(p)=\frac{-\frac{1}{9}p}{p^2+9}+\frac{\frac{1}{9}}{p}.$$

Using formulas 1 and 3 of Table 1 we find the solution:

$$x(t)=-\frac{1}{9}\cos 3t+\frac{1}{9}.$$

Example 3. Find the solution of the equation

$$\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=t$$

that satisfies the initial conditions $x_0=x_0'=0$ for $t=0$.

Solution. Write the auxiliary equation (34′)

$$\bar{x}(p)\,(p^2+3p+2)=\frac{1}{p^2}$$

or

$$\bar{x}(p)=\frac{1}{p^2}\,\frac{1}{(p^2+3p+2)}=\frac{1}{p^2(p+1)(p+2)}.$$

Decomposing this fraction into partial fractions by the method of undetermined coefficients, we obtain

$$\bar{x}(p)=\frac{1}{2}\frac{1}{p^2}-\frac{3}{4}\frac{1}{p}+\frac{1}{p+1}-\frac{1}{4(p+2)}.$$

From formulas 9, 1, and 4 of Table 1 we find the solution:

$$x(t)=\frac{1}{2}t-\frac{3}{4}+e^{-t}-\frac{1}{4}e^{-2t}.$$

Example 4. Find the solution of the equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=\sin t$$

satisfying the conditions $x_0=1$, $x_0'=2$ for $t=0$.

Solution. Write the auxiliary equation (34′):

$$\bar{x}(p)(p^2+2p+5)=p\cdot 1+2+2\cdot 1+L\{\sin t\}$$

or

$$\bar{x}(p)(p^2+2p+5)=p+4+\frac{1}{p^2+1},$$

whence we find $\bar{x}(p)$:

$$\bar{x}(p)=\frac{p+4}{p^2+2p+5}+\frac{1}{(p^2+1)(p^2+2p+5)}.$$

Decomposing the latter fraction on the right into partial fractions, we can write

$$\bar{x}(p)=\frac{\frac{11}{10}p+4}{p^2+2p+5}+\frac{-\frac{1}{10}p+\frac{1}{5}}{p^2+1}$$

or

$$\bar{x}(p)=\frac{11}{10}\cdot\frac{p+1}{(p+1)^2+2^2}+\frac{29}{10\cdot 2}\cdot\frac{2}{(p+1)^2+2^2}-\frac{1}{10}\cdot\frac{p}{p^2+1}+\frac{1}{5}\cdot\frac{1}{p^2+1}.$$

Applying formulas 8, 7, 3, and 2 of Table 1, we get the solution

$$x(t)=\frac{11}{10}e^{-t}\cos 2t+\frac{29}{20}e^{-t}\sin 2t-\frac{1}{10}\cos t+\frac{1}{5}\sin t$$

or, finally,

$$x(t)=e^{-t}\left(\frac{11}{10}\cos 2t+\frac{29}{20}\sin 2t\right)-\frac{1}{10}\cos t+\frac{1}{5}\sin t.$$

SEC. 11. DECOMPOSITION THEOREM

From formula (36) of the previous section it follows that the transform of the solution of a linear differential equation consists of two terms: the first term is a proper rational fraction in p, the second term is a fraction whose numerator is the transform of the right side of the equation $F(p)$, while the denominator is the polynomial $\varphi_n(p)$. If $F(p)$ is a rational fraction, then the second term will also be a rational fraction. It is thus necessary to be able to find the original function whose transform is the proper rational

fraction. We shall deal with this question in the present section. Let the L-transform of some function be a proper rational fraction in p:

$$\frac{\psi_{n-1}(p)}{\varphi_n(p)}.$$

It is required to find the original function. In Sec. 7, Ch. X, it was shown that any proper rational fraction may be represented in the form of a sum of elementary fractions of four types:

I. $\frac{A}{p-a}$,

II. $\frac{A}{(p-a)^k}$,

III. $\frac{Ap+B}{p^2+a_1p+a_2}$, where the roots in the denominator are complex, that is, $\frac{a_1^2}{4}-a_2<0$,

IV. $\frac{Ap+B}{(p^2+a_1p+a_2)^k}$ where $k \geqslant 2$, the roots of the denominator are complex.

Let us find the original functions for these elementary fractions. For fraction type I we get (on the basis of formula 4 of Table 1)

$$\frac{A}{p-a} \risingdotseq Ae^{\alpha t}.$$

For a type II fraction, by formulas 9 and 4 of Table 1, we have

$$\frac{A}{(p-a)^k} \risingdotseq A\frac{1}{(k-1)!}t^{k-1}e^{\alpha t}. \tag{37}$$

Let us now consider the type III fraction. We perform identical transformations:

$$\frac{Ap+B}{p^2+a_1p+a_2}=\frac{Ap+B}{\left(p+\frac{a_1}{2}\right)^2+\left(\sqrt{a_2-\frac{a_1^2}{4}}\right)^2}=$$

$$=\frac{A\left(p+\frac{a_1}{2}\right)+\left(B-\frac{Aa_1}{2}\right)}{\left(p+\frac{a_1}{2}\right)^2+\left(\sqrt{a_2-\frac{a_1^2}{A}}\right)^2}=A\frac{p+\frac{a_1}{2}}{\left(p+\frac{a_1}{2}\right)^2+\left(\sqrt{a_2-\frac{a_1^2}{4}}\right)^2}+$$

$$+\left(B-\frac{Aa_1}{2}\right)\frac{1}{\left(p+\frac{a_1}{2}\right)^2+\left(\sqrt{a_2-\frac{a_1^2}{4}}\right)^2}.$$

Denoting the first and second terms by M and N respectively, we get (from formulas 8 and 7 of Table 1)

$$M \risingdotseq Ae^{-\frac{a_1}{2}t}\cos t\sqrt{a_2-\frac{a_1^2}{4}},$$

$$N \mathrel{\dot{\to}} \left(B-\frac{Aa_1}{2}\right)\frac{1}{\sqrt{a_2-\frac{a_1^2}{4}}}\, e^{-\frac{a_1}{2}t}\sin t\sqrt{a_2-\frac{a_1^2}{4}}\,.$$

And, finally,

$$\frac{Ap+B}{p^2+a_1p+a_2} \mathrel{\dot{\to}}$$

$$\to e^{-\frac{a_1}{2}t}\left[A\cos t\sqrt{a_2-\frac{a_1^2}{4}}+\frac{B-\frac{Aa_1}{2}}{\sqrt{a_2-\frac{a_1^2}{4}}}\sin t\sqrt{a_2-\frac{a_1^2}{4}}\right]. \tag{38}$$

We shall not consider the case of elementary fraction IV, since it would involve considerable calculations. We shall consider certain special cases below.

SEC. 12. EXAMPLES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS AND SYSTEMS OF DIFFERENTIAL EQUATIONS BY THE OPERATIONAL METHOD

Example 1. Find the solution of the equation

$$\frac{d^2x}{dt^2}+4x=\sin 3x$$

that satisfies the initial conditions $x_0=0$, $x_0'=0$ when $t=0$.

Solution. Form the auxiliary equation (34′)

$$\bar{x}(p)(p^2+4)=p\cdot 0+0+\frac{3}{p^2+9}, \quad \bar{x}(p)=\frac{3}{(p^2+9)(p^2+4)}$$

$$\bar{x}(p)=\frac{-\frac{3}{5}}{p^2+9}+\frac{\frac{3}{5}}{p^2+4}=-\frac{1}{5}\cdot\frac{3}{p^2+9}+\frac{3}{10}\cdot\frac{2}{p^2+4},$$

whence we get the solution

$$x(t)=\frac{3}{10}\sin 2t-\frac{1}{5}\sin 3t.$$

Example 2. Find the solution of the equation

$$\frac{d^3x}{dt^3}+x=0$$

that satisfies the initial conditions $x_0=1$, $x_0'=3$, $x_0''=8$ when $t=0$.

Solution. Form the auxiliary equation (34′)

$$\bar{x}(p)(p^3+1)=p^2\cdot 1+p\cdot 3+8,$$

we find

$$\bar{x}(p)=\frac{p^2+3p+8}{p^3+1}=\frac{p^2+3p+8}{(p+1)(p^2-p+1)}\,.$$

Decomposing the rational fraction obtained into partial fractions, we get

$$\frac{p^2+3p+8}{(p+1)(p^2-p+1)}=\frac{2}{p+1}+\frac{-p+6}{p^2-p+1}=$$

$$=2\cdot\frac{1}{p+1}-\frac{p-\frac{1}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\sqrt{\frac{3}{4}}\right)^2}+\frac{11}{\sqrt{3}}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}.$$

Using Table 1, we write the solution:

$$x(t)=2e^{-t}+e^{\frac{1}{2}t}\left(-\cos\frac{\sqrt{3}}{2}t+\frac{11}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right).$$

Example 3. Find the solution of the equation

$$\frac{d^2x}{dt^2}+x=t\cos 2t$$

that satisfies the initial conditions $x=0$, $x_0'=0$ when $t=0$.

Solution. Write the auxiliary equation (34′)

$$\overline{x}(p)(p^2+1)=\frac{1}{p^2+4}-\frac{8}{(p^2+4)^2},$$

whence

$$\overline{x}(p)=-\frac{5}{9}\frac{1}{p^2+1}+\frac{5}{9}\frac{1}{p^2+4}+\frac{8}{3}\frac{1}{(p^2+4)^2}.$$

Consequently,

$$x(t)=-\frac{5}{9}\sin t+\frac{5}{18}\sin 2t+\frac{1}{3}\left(\frac{1}{2}\sin 2t-t\cos 2t\right).$$

Obviously, the operational method may also be used to solve systems of linear differential equations. The following is an illustration.

Example 4. Find the solutions of the set of equations

$$3\frac{dx}{dt}+2x+\frac{dy}{dt}=1,$$

$$\frac{dx}{dt}+4\frac{dy}{dt}+3y=0$$

that satisfy the initial conditions $x=0$, $y=0$ when $t=0$.

Solution. We denote $x(t)\leftarrow\!\!\!\!\div\,\overline{x}(p)$, $y(t)\leftarrow\!\!\!\!\div\,\overline{y}(p)$ and write the system of auxiliary equations:

$$(3p+2)\,\overline{x}(p)+p\overline{y}(p)=\frac{1}{p},$$

$$p\overline{x}(p)+(4p+3)\,\overline{y}(p)=0.$$

Solving this system, we find

$$\overline{x}(p)=\frac{4p+3}{p(p+1)(11p+6)}=\frac{1}{2p}-\frac{1}{5(p+1)}-\frac{1}{10(11p+6)},$$

$$\overline{y}(p)=-\frac{1}{(11p+6)(p+1)}=\frac{1}{5}\left(\frac{1}{p+1}-\frac{11}{11p+6}\right).$$

From the transforms we find the original functions—the sought-for solutions of the system:

$$x(t)=\frac{1}{2}-\frac{1}{5}e^{-t}-\frac{3}{10}e^{-\frac{6}{11}t},$$

$$y(t)=\frac{1}{5}\left(e^{-t}-e^{-\frac{6}{11}t}\right).$$

Linear systems of higher orders are solved in similar fashion.

SEC 13. THE CONVOLUTION THEOREM

The following convolution theorem is frequently useful when solving differential equations by the operational method.

Convolution Theorem. *If $F_1(p)$ and $F_2(p)$ are the transforms of the functions $f_1(t)$ and $f_2(t)$, that is,*

$$F_1(p) \xrightarrow{\cdot} f_1(t) \text{ and } F_2(p) \xrightarrow{\cdot} f_2(t),$$

then $F_1(p)\cdot F_2(p)$ is the transform of the function

$$\int_0^t f_1(\tau) f_2(t-\tau)\,d\tau,$$

that is,

$$F_1(p)\,F_2(p) \xrightarrow{\cdot} \int_0^t f_1(\tau) f_2(t-\tau)\,d\tau. \tag{39}$$

Proof. We find the transform of the function

$$\int_0^t f_1(\tau) f_2(t-\tau)\,d\tau$$

from the definition of a transform:

$$L\left\{\int_0^t f_1(\tau) f_2(t-\tau)\,d\tau\right\}=\int_0^\infty e^{-pt}\left[\int_0^t f_1(\tau) f_2(t-\tau)\,d\tau\right]dt.$$

The integral on the right is a double integral of the form $\iint\limits_D \Phi(\tau, t)\,dt\,d\tau$, which is taken over a region bounded by the straight lines $\tau=0$, $\tau=t$ (Fig. 380). Changing the order of integration in this double integral, we get

$$L\left\{\int_0^t f_1(\tau) f_2(t-\tau)\,d\tau\right\}=\int_0^\infty\left[f_1(\tau)\int_\tau^\infty e^{-pt} f_2(t-\tau)\,dt\right]d\tau.$$

Changing the variable $t-\tau=z$ in the inner integral, we obtain

$$\int_\tau^\infty e^{-pt} f_2(t-\tau)\,dt=\int_0^\infty e^{-p(z+\tau)} f_2(z)\,dz=e^{-p\tau}\int_0^\infty e^{-pz} f_2(z)\,dz=e^{-p\tau}F_2(p).$$

Hence,

$$L\left\{\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right\}=\int_0^\infty f_1(\tau)e^{-p\tau}F_2(p)\,d\tau=F_2(p)\int_0^\infty e^{-p\tau}f_1(\tau)\,d\tau=$$
$$=F_2(p)F_1(p).$$

And so

$$\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau \mathrel{\dot{\leftarrow}} F_1(p)F_2(p).$$

This is formula 15 in Table 1.

Note 1. The expression $\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$ is called the *convolution* (*Faltung, resultant*) of two functions $f_1(t)$ and $f_2(t)$. The operation of obtaining it is also known as the *convolution* of two functions; here

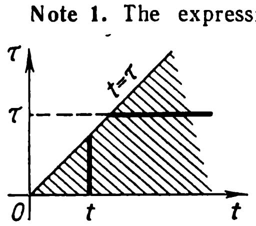


Fig. 380.

$$\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau=\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau.$$

That this equation is true is evident if we change the variable $t-\tau=z$ in the right-hand integral.

Example. Find the solution to the equation

$$\frac{d^2x}{dt^2}+x=f(t)$$

that satisfies the initial conditions $x_0=x_0'=0$ for $t=0$.

Solution. Write the auxiliary equation (34′)

$$\bar{x}(p)(p^2+1)=F(p),$$

where $F(p)$ is the transform of the function $f(t)$. Hence, $\bar{x}(p)=\frac{1}{p^2+1}F(p)$, but $\frac{1}{p^2+1}\mathrel{\dot{\rightarrow}}\sin t$ and $F(p)\mathrel{\dot{\rightarrow}}f(t)$. Applying the convolution formula (39) and denoting $\frac{1}{p^2+1}=F_2(p)$, $F(p)=F_1(p)$,

we get

$$x(t)=\int_0^t f(\tau)\sin(t-\tau)\,d\tau. \tag{40}$$

Note 2. On the basis of the convolution theorem it is easy to find the transform of the integral of the given function if we know the transform of this function; namely, if $F(p)\mathrel{\dot{\rightarrow}}f(t)$, then

$$\frac{1}{p}F(p)\mathrel{\dot{\rightarrow}}\int_0^t f(\tau)\,d\tau. \tag{41}$$

Indeed, if we denote

$$f_1(t)=f(t),\ f_2(t)=1,\ \text{then}\ F_1(p)=F(p),\ F_2(p)=\frac{1}{p}.$$

Putting these functions into (39), we get formula (41).

SEC. 14. THE DIFFERENTIAL EQUATIONS OF MECHANICAL OSCILLATIONS. THE DIFFERENTIAL EQUATIONS OF ELECTRIC-CIRCUIT THEORY

From mechanics we know that the oscillations of a material point of mass m are described by the equation *)

$$\frac{d^2x}{dt^2}+\frac{\lambda}{m}\frac{dx}{dt}+\frac{k}{m}x=\frac{1}{m}f_1(t), \tag{42}$$

where x is the deflection of the point from a certain position and k is the rigidity of the elastic system, for instance, a spring (a car spring), the force of resistance to motion is proportional (the proportionality constant is λ) to the first power of the velocity, and $f_1(t)$ is the outer (or disturbing) force.

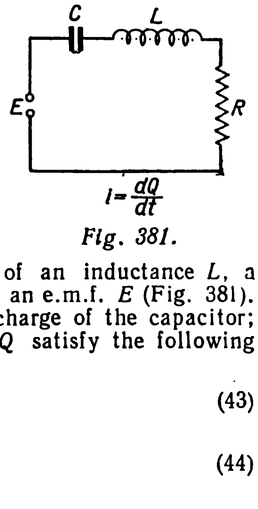


Fig. 381.

Equations of type (42) describe small vibrations of other mechanical systems with one degree of freedom, for example, the torsional oscillations of a flywheel on an elastic shaft, if x is the angle of rotation of the flywheel, m is the moment of inertia of the flywheel, k is the torsional rigidity of the shaft, and $mf_1(t)$ is the moment of the outer forces relative to the axis of rotation. Equations of type (42) describe not only mechanical vibrations but also phenomena that occur in electric circuits.

Suppose we have an electric circuit consisting of an inductance L, a resistance R and a capacitance C, to which is applied an e.m.f. E (Fig. 381). We denote by i the current in the circuit, by Q the charge of the capacitor; then, as we know from electrical engineering, i and Q satisfy the following equations:

$$L\frac{di}{dt}+Ri+\frac{Q}{C}=E, \tag{43}$$

$$\frac{dQ}{dt}=i. \tag{44}$$

From (44) we get

$$\frac{d^2Q}{dt^2}=\frac{di}{dt}. \tag{44'}$$

Substituting (44) and (44′) into (43), we get for Q an equation of type (42):

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E. \tag{45}$$

*) See, for example, Ch. XIII, Sec. 26, where such an equation is derived in considering the oscillation of a weight on a car spring.

Differentiating both sides of (43) and utilising (44), we obtain an equation for determining the current i:

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=\frac{dE}{dt}. \tag{46}$$

Equations (45) and (46) are type (42) equations.

SEC. 15. SOLUTION OF THE DIFFERENTIAL OSCILLATION EQUATION

Let us write the oscillation equation in the form

$$\frac{d^2x}{dt^2}+a_1\frac{dx}{dt}+a_2x=f(t), \tag{47}$$

where the mechanical and physical meaning of the desired function x, of the coefficients a_1, a_2, and of the function $f(t)$ is readily established by comparing this equation with equations (42), (45), (46). Let us find the solution to equation (47) that satisfies the initial conditions $x=x_0$, $x'=x_0'$ when $t=0$.

We form the auxiliary equation for equation (47):

$$\overline{x}(p)(p^2+a_1p+a_2)=x_0p+x_0'+a_1x_0+F(p), \tag{48}$$

where $F(p)$ is the transform of the function $f(t)$. From (48) we find

$$\overline{x}(p)=\frac{x_0p+x_0'+a_1x_0}{p^2+a_1p+a_2}+\frac{F(p)}{p^2+a_1p+a_2}. \tag{49}$$

Thus, for a solution $Q(t)$ of equation (45) that satisfies the initial conditions $Q=Q_0$, $Q'=Q_0'$ when $t=0$, the transform will have the form

$$\overline{Q}(p)=\frac{L(Q_0p+Q_0')+RQ_0}{Lp^2+Rp+\frac{1}{C}}+\frac{\overline{E}(p)}{Lp^2+Rp+\frac{1}{C}}.$$

The type of solution is significantly dependent on whether the roots of the trinomial $p^2+a_1p+a_2$ are complex, or real and distinct, or real and equal. Let us examine in detail the case when the roots of the trinomial are complex, that is, when $\left(\frac{a_1}{2}\right)^2-a_2<0$. The other cases are considered in similar fashion.

Since the transform of a sum of two functions is equal to the sum of their transforms, it follows from formula (38) that the original function for the first fraction on the right of (49) will have the form

$$\frac{x_0p+x_0'+a_1x_0}{p^2+a_1p+a_2}\doteq e^{-\frac{a_1}{2}t}\left[x_0\cos t\sqrt{a_2-\frac{a_1^2}{4}}+\right.$$
$$\left.+\frac{x_0'+\frac{x_0a_1}{2}}{\sqrt{a_2-\frac{a_1^2}{4}}}\sin t\sqrt{a_2-\frac{a_1^2}{4}}\right]. \tag{50}$$

Let us then find the original function corresponding to the fraction

$$\frac{F(p)}{p^2+a_1p+a_2}.$$

Here, we take advantage of the convolution theorem, first noting that

$$\frac{1}{p^2+a_1p+a_2} \risingdotseq \frac{e^{-\frac{a_1}{2}t}}{\sqrt{a_2-\frac{a_1^2}{4}}} \sin t \sqrt{a_2-\frac{a_1^2}{4}}, \quad F(p) \risingdotseq f(t).$$

Hence, from (39) we get

$$\frac{F(p)}{p^2+a_1p+a_2} \risingdotseq \frac{1}{\sqrt{a_2-\frac{a_1^2}{4}}} \int_0^t f(\tau)\, e^{-\frac{a_1}{2}(t-\tau)} \sin(t-\tau)\sqrt{a_2-\frac{a_1^2}{4}}\, d\tau. \qquad (51)$$

And so, from (49), taking into account (50) and (51), we get

$$x(t) = e^{-\frac{a_1}{2}t}\left[x_0 \cos t\sqrt{a_2-\frac{a_1^2}{4}} + \frac{x_0' + \frac{x_0a_1}{2}}{\sqrt{a_2-\frac{a_1^2}{4}}} \sin t\sqrt{a_2-\frac{a_1^2}{4}}\right] +$$

$$+\frac{1}{\sqrt{a_2-\frac{a_1^2}{4}}} \int_0^t f(\tau)\, e^{-\frac{a_1}{2}(t-\tau)} \sin(t-\tau)\sqrt{a_2-\frac{a_1^2}{4}}\, d\tau. \qquad (52)$$

If the external force $f(t) \equiv 0$, which means that if we have free mechanical or electrical oscillations, then the solution is given by the first term on the right-hand side of expression (52). If the initial data are equal to zero, i.e., if $x_0 = x_0' = 0$, then the solution is given by the second term on the right side of (52). Let us consider these cases in more detail.

SEC. 16. INVESTIGATING FREE OSCILLATIONS

Let equation (47) describe free oscillations, that is, $f(t) \equiv 0$. For convenience in writing we introduce the notation $a_1 = 2n$, $a_2 = k^2$, $k_1^2 = k^2 - n^2$. Then (47) will have the form

$$\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + k^2x = 0. \qquad (53)$$

The solution of this equation x_{fr} that satisfies the initial conditions $x = x_0$, $x' = x_0'$ for $t = 0$ is given by the formula (50) or by the first term of (52):

$$x_{fr}(t) = e^{-nt}\left[x_0 \cos k_1t + \frac{x_0' + x_0n}{k_1} \sin k_1t\right]. \qquad (54)$$

We denote $x_0 = a$, $\dfrac{x_0' + x_0 n}{k_1} = b$. It is obvious that for any a and b we can select M and δ such that the following equalities will be fulfilled:

$$a = M \sin\delta, \quad b = M \cos\delta,$$

here,

$$M^2 = a^2 + b^2, \quad \tan\delta = \frac{a}{b}.$$

We rewrite formula (54) as

$$x_{fr} = e^{-nt}\,[M \cos k_1 t \sin\delta + M \sin k_1 t \cos\delta],$$

or, in final form, the solution may be written thus:

$$x_{fr} = \sqrt{a^2 + b^2}\, e^{-nt} \sin(k_1 t + \delta). \tag{55}$$

Solution (55) corresponds to damped oscillations.

If $2n = a_1 = 0$, that is, if there is no internal friction, then the solution will be of the form

$$x_{fr} = \sqrt{a^2 + b^2} \sin(k_1 t + \delta).$$

In this case harmonic oscillations occur. (In Ch. XIII, Sec. 27, Figs. 270 and 271 give graphs of harmonic and damped oscillations.)

SEC. 17. INVESTIGATING MECHANICAL AND ELECTRICAL OSCILLATIONS IN THE CASE OF A PERIODIC EXTERNAL FORCE

When studying elastic oscillations of mechanical systems and, in particular, when studying electrical oscillations, one has to consider different types of external force $f(t)$. Let us consider in detail the case of a periodic external force. Let equation (47) have the form

$$\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + k^2 x = A \sin\omega t. \tag{56}$$

To determine the nature of the motion it is sufficient to consider the case when $x_0 = x_0' = 0$. One could obtain the solution of the equation by formula (52), but pedagogically speaking, it is more convenient to obtain the solution by carrying out all the intermediate calculations.

Let us write the transform equation

$$\bar{x}(p)\,(p^2 + 2np + k^2) = A\frac{\omega}{p^2 + \omega^2},$$

from which we get

$$\bar{x}(p) = \frac{A\omega}{(p^2 + 2np + k^2)(p^2 + \omega^2)}. \tag{57}$$

We consider the case when $2n \neq 0$ $(n^2 < k^2)$. Decompose the fraction on the right into partial fractions:

$$\frac{A\omega}{(p^2 + 2np + k^2)(p^2 + \omega^2)} = \frac{Np + B}{p^2 + 2np + k^2} + \frac{Cp + D}{p^2 + \omega^2}. \tag{58}$$

We determine the constants B, C, D by the method of undetermined coefficients. Using formula (38), we find the original function from its

L-transform (57):

$$x(t)=\frac{A}{(k^2-\omega^2)^2+4n^2\omega^2}\left\{(k^2-\omega^2)\sin\omega t-2n\omega\cos\omega t+\right.$$
$$\left.+e^{-nt}\left[(2n^2-k^2+\omega^2)\frac{\omega}{k_1}\sin k_1t+2n\omega\cos k_1t\right]\right\}; \tag{59}$$

here again, $k_1=\sqrt{k^2-n^2}$. This is the solution of equation (56) that satisfies the initial conditions $x_0=x_0'=0$ when $t=0$.

Let us consider a special case when $2n=0$. This corresponds to a mechanical system with no internal resistance, or to an electric circuit where $R=0$ (no internal resistance in the circuit). Equation (56) then takes the form

$$\frac{d^2x}{dt^2}+k^2x=A\sin\omega t, \tag{60}$$

and we get the solution of this equation satisfying the conditions $x_0=x_0'=0$ for $t=0$ if in (59) we put $n=0$:

$$x(t)=\frac{A}{(k^2-\omega^2)k}[-\omega\sin kt+k\sin\omega t]. \tag{61}$$

Here we have the sum of two harmonic oscillations: natural oscillations with frequency k:

$$x_{nat}t=-\frac{A}{k^2-\omega^2}\frac{\omega}{k}\sin kt,$$

and forced oscillations with frequency ω:

$$x_{for}(t)=\frac{A}{k^2-\omega^2}\sin\omega t.$$

The type of oscillations for the case $k\gg\omega$ is shown in Fig. 382.

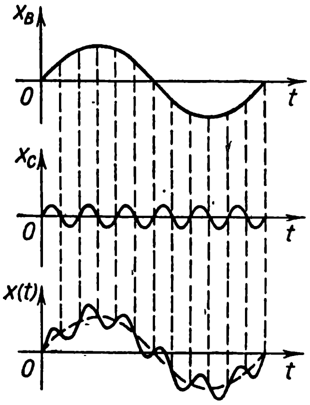


Fig. 382.

Let us again return to formula (59). If $2n>0$ (which occurs in the mechanical and electrical forces under consideration), then the term containing the factor e^{-nt}, which represents damped natural oscillations for increasing t, rapidly decreases. For t sufficiently large, the character of the oscillations will be determined by the term that does not contain the factor e^{-nt}; that is, by the term

$$x(t)=\frac{A}{(k^2-\omega^2)+4n^2\omega^2}\{(k^2-\omega^2)\sin\omega t-2n\omega\cos\omega t\}. \tag{62}$$

We introduce the notations

$$\frac{A(k^2-\omega^2)}{(k^2-\omega^2)^2+4n^2\omega^2}=M\cos\delta;\quad -\frac{A\cdot 2n\omega}{(k^2-\omega^2)^2+4n^2\omega^2}=M\sin\delta, \tag{63}$$

where

$$M=\frac{A}{\sqrt{(k^2-\omega^2)^2+4n^2\omega^2}}.$$

The solution (62) may be rewritten as follows:

$$x(t)=\frac{A}{k^2\sqrt{\left(1-\frac{\omega^2}{k^2}\right)^2+4n^2\frac{\omega^2}{k^4}}}\sin(\omega t+\delta). \tag{64}$$

From formula (64) it follows that the frequency of forced oscillations coincides with that of the external force. If the internal resistance, characterised by the number n, is small and the frequency of the external force ω is not very different from that of the natural oscillations k, then the amplitude of oscillations may be made as great as one pleases, since the denominator may be arbitrarily small. For $n=0$, $\omega^2=k^2$, the solution is not expressed by formula (64).

SEC. 18. SOLVING THE OSCILLATION EQUATION IN THE CASE OF RESONANCE

Let us consider the special case when $a_1=2n=0$, that is, when there is no resistance and the frequency of the external force coincides with that of the natural oscillations $k=\omega$. The equation then takes the form

$$\frac{d^2x}{dt^2}+k^2x=A\sin kt. \tag{65}$$

We shall seek the solution that satisfies the initial conditions $x_0=0$, $x_0'=0$ for $t=0$. The auxiliary equation will be

$$\bar{x}(p)(p^2+k^2)=A\frac{k}{p^2+k^2},$$

whence

$$\bar{x}(p)=\frac{Ak}{(p^2+k^2)^2}. \tag{66}$$

We have a proper rational fraction of type IV, which we have not considered in the general form. To find the original function for the transform of (66), we take advantage of the following procedure. We write the identity (formula 2 of Table 1)

$$\frac{k}{p^2+k^2}=\int_0^\infty e^{-pt}\sin kt\,dt. \tag{67}$$

We differentiate both sides of this equation with respect to k (the integral on the right may be represented in the form of a sum of two integrals of a real variable, each of which depends on the parameter k):

$$\frac{1}{p^2+k^2}-\frac{2k^2}{(p^2+k^2)^2}=\int_0^\infty e^{-pt}\,t\cos kt\,dt.$$

Utilising (67) we can rewrite this equation as

$$-\frac{2k^2}{(p^2+k^2)^2}=\int_0^\infty e^{-pt}\left[t\cos kt-\frac{1}{k}\sin kt\right]dt.$$

Whence it follows directly that

$$\frac{Ak}{(p^2+k^2)^2} \doteq \frac{A}{2k}\left(\frac{1}{k}\sin kt - t\cos kt\right)$$

(from this formula we have formula 13, Table 1). Thus, the solution of equation (65) satisfying the initial conditions $x_0 = x_0' = 0$ for $t = 0$ will be

$$x(t) = \frac{A}{2k}\left(\frac{1}{k}\sin kt - t\cos kt\right). \tag{68}$$

Let us study the second term of this equation:

$$x_2(t) = -\frac{A}{2k}\, t\cos kt; \tag{68'}$$

this quantity is not bounded as t increases. The amplitude of oscillations that correspond to formula (68') increases without bound as t increases without bound. Hence, the amplitude of oscillations corresponding to formula (68) also increases without bound. This is *resonance*; it occurs when the frequency of the natural oscillations coincides with that of the external force (see also Ch. XIII, Sec. 29, Fig. 273).

SEC. 19. THE DELAY THEOREM

Let the function $f(t)$, for $t < 0$, be identically equal to zero (Fig. 383, *a*). Then the function $f(t-t_0)$ will be identically zero for $t < t_0$ (Fig. 383, *b*). We shall prove a theorem which is known as the delay theorem.

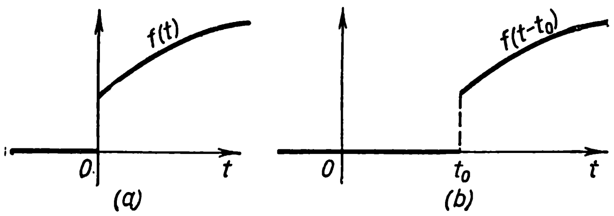


Fig. 383.

Theorem. *If $F(p)$ is the transform of the function $f(t)$, then $e^{-pt_0}F(p)$ is the transform of the function $f(t-t_0)$; that is, if $f(t) \doteq F(p)$, then*

$$f(t-t_0) \doteq e^{-pt_0}F(p).$$

Proof. By the definition of a transform we have

$$L\{f(t-t_0)\} = \int_0^\infty e^{-pt} f(t-t_0)\,dt = \int_0^{t_0} e^{-pt} f(t-t_0)\,dt + \int_{t_0}^\infty e^{-pt} f(t-t_0)\,dt.$$

The first integral on the right of the equation is zero since $f(t-t_0)=0$ for $t<t_0$. In the last integral we change the variable, putting $t-t_0=z$:

$$L\{f(t-t_0)\}=\int_0^{\infty} e^{-p(z+t_0)} f(z)\,dz=e^{-pt_0}\int_0^{\infty} e^{-pz} f(z)\,dz=e^{-pt_0}F(p).$$

Thus, $f(t-t_0) \doteq e^{-pt_0}F(p)$.

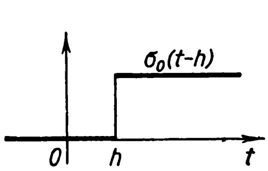


Fig. 348.

Example. In Sec. 2 it was established for the Heaviside unit function that

$$\sigma_0(t) \doteq \frac{1}{p}.$$

It follows, from the theorem that has just been proved, that for the function $\sigma_0(t-h)$ depicted in Fig. 384, the L-transform is

$$\frac{1}{p}e^{-ph},$$

that is,

$$\sigma_0(t-h) \doteq \frac{1}{p}e^{-ph}.$$

Exercises on Chapter XIX

Find solutions to the following equations for the indicated initial conditions:

1. $\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=0$, $x=1$, $x'=2$ for $t=0$. *Ans.* $x=4e^{-t}-3e^{-2t}$.

2. $\frac{d^3x}{dt^3}-\frac{d^2x}{dt^2}=0$, $x=2$, $x'=0$, $x''=1$ for $t=0$. *Ans.* $x=1-t+e^t$.

3. $\frac{d^2x}{dt^2}-2a\frac{dx}{dt}+(a^2+b^2)x=0$, $x=x_0$, $x'=x_0'$ for $t=0$. *Ans.* $x=$ $=\frac{e^{at}}{b}[x_0b\cos bt+(x_0'-x_0a)\sin bt]$.

4. $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=e^{5x}$, $x=1$, $x'=2$ for $t=0$. *Ans.* $x=\frac{1}{12}e^{5t}+$ $+\frac{1}{4}e^t-\frac{4}{3}e^{2t}$.

5. $\frac{d^2x}{dt^2}+m^2x=a\cos nt$, $x=x_0$, $x'=x_0'$ for $t=0$. *Ans.* $x=\frac{a}{m^2-n^2}\times$ $\times(\cos nt-\cos mt)+x_0\cos nt+\frac{x_0}{m}\cos mt$.

6. $\frac{d^2x}{dt^2}-\frac{dx}{dt}=t^2$, $x=0$, $x'=0$ for $t=0$. *Ans.* $x=3e^t-\frac{1}{3}t^3-t^2-2t-3$.

7. $\frac{d^3x}{dt^3}+x=\frac{1}{2}t^2e^t$, $x=x'=x''=0$ for $t=0$. *Ans.* $x=\frac{1}{4}\left(t^2-3t+\right.$ $\left.+\frac{3}{2}\right)e^t-\frac{1}{24}e^{-t}-\frac{1}{3}\left\{\cos\left(\frac{1}{2}\sqrt{3}t\right)-\sqrt{3}\sin\left(\frac{1}{2}\sqrt{3}t\right)\right\}e^{\frac{1}{2}t}$.

8. $\frac{d^3x}{dt^3}+x=1$, $x_0=x_0'=x_0''=0$ for $t=0$. *Ans.* $x=1-\frac{1}{3}e^{-t}-$ $-\frac{2}{3}e^{\frac{t}{2}}\cos\frac{t\sqrt{3}}{2}$.

9. $\frac{d^4x}{dt^4}-2\frac{d^2x}{dt^2}+x=\sin t$, $x_0=x_0'=x_0''=x_0'''=0$ for $t=0$. *Ans.* $x=$ $=\frac{1}{8}(3-t^2)\sin t-\frac{3}{8}t\cos t$.

10. Find solutions to the system of differential equations

$$\frac{d^2x}{dt^2}+y=1, \quad \frac{d^2y}{dt^2}+x=0,$$

that satisfy the initial conditions $x_0=y_0=x_0'=y_0'=0$ for $t=0$. *Ans.* $x(t)=$ $=-\frac{1}{2}\cos t+\frac{1}{4}e^t+\frac{1}{4}e^{-t}$, $y(t)=-\cos t+e^t+e^{-t}-1$.

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